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Origin of inertia in large-amplitude collective motion in finite Fermi systems

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Abstract. We argue that mass parameters appearing in the treatment of large-amplitude collective motion, be it fission or heavy-ion reactions, originate as a consequence of their relation with Lyapunov exponents coming from the classical dynamics, and, fractal dimension associated with diffusive modes coming from hydrodynamic description.

Keywords. Cranking model; mass parameters; large-amplitude collective motion.

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Nuclear fission is one of the most well-studied subjects as an instance of large-amplitude collective motion (LACM) [1]. There is a deep interplay of collective motion and single-particle motion in various kinds of LACM in general. It is well-known that the classical phase space of a fissioning nucleus is highly chaotic [2]. Shell corrections to the liquid drop description of a fissioning nucleus also is a reminder of the important single-particle effects [3]. Since this pioneering work by Strutinsky, and, with the development of periodic orbit theory around the same time by Gutzwiller [4] and, Balian and Bloch [5], there have been many important advances establishing the micro–macro connection in finite Fermi systems. Relatively recently, a simple, beautiful explanation was found for mass asymmetry of the fission fragments in terms of shortest periodic orbits [6,7]. Nonlinear dynamics for the cranking model has been investigated using semiclassical ideas [8]. Within a mean-field description, a connection between the response function of finite Fermi systems and the geometric phase acquired by a single-particle wave function was brought out [9]. Thus, it is indeed important and interesting to obtain micro–macro connections.

A lot of nuclear phenomena involving large-amplitude collective motion (LACM) [10] like fission takes nuclei to highly excited states far from equilibrium. Relaxation of a nucleus occurs via complicated paths through a myriad of non-equilibrium shapes eventually acquiring an equilibrium shape. This process of relaxation occurs in a multi-dimensional space of deformation parameters. Thus, inertia associated with the dynamics is a complicated tensorial object. Here we discuss the origin of 'mass' or inertia tensor in terms of fundamental dynamical and statistical quantities. It is well-known that extraction of mass parameters that enter the inertia tensor is a very difficult and important problem. Here we bring out the fundamental origin of mass encountered in LACM, and do not deal with the important problem of extraction of the parameters. The principal objective of this work is to propose a connection at three levels of description – quantum mechanical, classical and hydrodynamical.

Before we embark upon the details of the relation between macroscopic quantity like inertia tensor, relaxation rate, and microscopic quantifier of single-particle motion, it is important to describe the limitations and the model. One of the most successful models employed to extract mass parameters for LACM is the cranking model and we shall make use of the well-known cranking formula [11,12]. Cranking model has been extensively used to understand many aspects of deformed and superdeformed nuclei [8,13–15].

Since we shall devote our discussion to large surface deformations, we begin with the Lagrangian describing the evolution of deformation parameters under the influence of a potential $V(\boldsymbol{\beta})$:

$$L(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = T(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) - V(\boldsymbol{\beta}), \tag{1}$$

with kinetic energy given by

$$T(\boldsymbol{\beta}, \dot{\boldsymbol{\beta}}) = \frac{1}{2} \sum_{i,j=1}^{N} \dot{\beta}_i B_{ij}(\boldsymbol{\beta}) \dot{\beta}_j. \tag{2}$$

$B_{ij}(\boldsymbol{\beta})$ are the mass parameters making up the inertia tensor, $\mathbf{B}$; since it depends on the deformation variables, the Lagrangian introduces a nonlinear and in general a chaotic dynamics of the system in the $\boldsymbol{\beta}$-space. There is a tacit assumption that the collective variables (shape) determine the internal structure and state of the nucleus. A detailed derivation of eq. (2) based on the principle of least action is given in [16]. During collective motion, the eigenstate does not change leading thereby to adiabatic approximation, and we shall employ the cranking model to force the phenomenological single-particle potential externally. This leads to the dependence of wave functions on deformation variables. Thus, the cranking model describes the quantum evolution of states and then let the deformation variables evolve according to the Lagrangian (1).

In the cranking model, the collective coordinates change very slowly so that nucleus settles down in an adiabatic state at each point. The Hamiltonian of the cranking model for a nucleus with A nucleons is $H(r_1, \ldots, r_A; \boldsymbol{\beta})$ which is parametrically dependent on β. The associated adiabatic eigenfunctions satisfy the eigenvalue problem:

$$H(\{\mathbf{r}\}; \boldsymbol{\beta})\psi_i(\{\mathbf{r}\}; \boldsymbol{\beta}) = E_i(\boldsymbol{\beta})\psi_i(\{\mathbf{r}\}; \boldsymbol{\beta}), \tag{3}$$

where $\boldsymbol{\beta} = \boldsymbol{\beta}(t)$. Assume that the change in deformation is so slow that the single-particle excitations are not generated. This implies that the characteristic time-scale of collective motion is much lager than the single-particle time-scale. The separation of time-scales allows us to employ the adiabatic basis $\{\psi_i\}$:

$$\Psi(\{\mathbf{r}\}, t) = \sum_i c_i(t)\psi_i(\{\mathbf{r}\}; \boldsymbol{\beta}(t)) \mathrm{e}^{-(i/\hbar)\int^t \mathrm{d}t' E_i(\beta(t'))}. \tag{4}$$

Substituting this into the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = H(\{\mathbf{r}\};\, \boldsymbol{\beta}(t))\Psi, \tag{5}$$

we get the following coupled equations for the coefficients:

$$i\hbar\dot{c}_j = -i\hbar\sum_i c_i\langle\psi_j|\dot{\boldsymbol{\beta}}\cdot\nabla_{\boldsymbol{\beta}}|\psi_i\rangle \mathrm{e}^{-(i/\hbar)\int^t \mathrm{d}t'[E_i(\beta(t'))-E_j(\beta(t'))]}. \tag{6}$$

As the matrix element and the eigenenergies in the phase factor depend on the collective coordinates, they do not change much. Thus the contribution to the sum is dominated by the ground state ($i = 0$). Hence the solution for c_j for $j \neq 0$ is:

$$c_j(t) = \frac{\langle\psi_j|\dot{\boldsymbol{\beta}}\cdot\nabla_{\boldsymbol{\beta}}|\psi_0\rangle}{E_j - E_0}\mathrm{e}^{-(i/\hbar)\int^t \mathrm{d}t'[E_0(\beta(t'))-E_j(\beta(t'))]}. \tag{7}$$

The expectation value of $H(\boldsymbol{\beta}(t))$ can be shown to be [10,11]

$$\langle\Psi(t)|H(\boldsymbol{\beta}(t))|\Psi(t)\rangle = \sum_i |c_i|^2 E_i(t) = E_0(\boldsymbol{\beta}(t)) + \frac{1}{2}\dot{\boldsymbol{\beta}}\cdot\mathbf{B}\cdot\dot{\boldsymbol{\beta}}. \tag{8}$$

The mass parameters identified with the elements B_{kl} of the tensor $\mathbf{B}$ are given by [11,12]

$$B_{kl} = 2\hbar^2\sum_{j\neq 0}\frac{\langle\psi_0|\mathcal{A}_k|\psi_j\rangle\langle\psi_j|\mathcal{A}_l|\psi_0\rangle}{E_j - E_0}, \tag{9}$$

where $\mathcal{A}_j = \partial/\partial\beta_j$. To understand physically, let us quote the instance where mass parameter is the moment of inertia and $\mathcal{A}$ is an angular momentum operator conjugate to the angular variable. A strong variation in the parameters results in the presence of avoided crossing among the energy levels, or, when the wave functions change to effect a change in deformation. Quantitatively, we can interpret it as a two-time correlation function, $C_{kl}(t)$, as follows:

$$B_{kl} = 2\hbar\left|\int_0^\infty \mathrm{d}t\, C_{kl}(t)\right|, \tag{10}$$

where

$$C_{kl}(t) = \left\langle\psi_0\middle|\mathrm{e}^{(iHt/\hbar)}\mathcal{A}_k\mathrm{e}^{(-iHt/\hbar)}\mathcal{A}_l\middle|\psi_0\right\rangle. \tag{11}$$

Notice that the quantity appearing in absolute sign in (10) is the spectral density – the Fourier transform of time correlation function – evaluated at zero frequency. It is well-known that this is related to diffusion coefficient [17], suggesting a random walk interpretation in $\boldsymbol{\beta}$-space as the system evolves [18]. Diffusive mode is a hydrodynamic mode which is connected to classically chaotic dynamics also at the microscopic level

[19]. The relaxation rate of a hydrodynamic mode of 'wave number' k is given by the decay rate of the van Hove intermediate incoherent scattering function [20] as

$$s_k = \lim_{t\to\infty} \frac{1}{t} \log\left\langle \exp\left[i\mathbf{k}\cdot(\beta(t)-\beta(0))\right]\right\rangle = -Dk^2, \tag{12}$$

where angular brackets denote an average over an ensemble of initial conditions and D denotes the diffusion coefficient. The second equality in eq. (12) is from the dispersion relation of diffusion [20]. It is worthwhile to note here that $\mathbf{k}$ is the conjugate Fourier partner of $\boldsymbol{\beta}$. Also, recall that the physical situation of (say) fission is described in terms of a penetration factor across the barrier connecting a point A inside the barrier to a point B outside it. Precisely, these end-points define in the $\boldsymbol{\beta}$-space an action that is integrated over a possible trajectory in that space. Thus, there exists an ensemble of trajectories forming a complex collection of paths that are allowed with different probabilities. That underlying the LACM of a nucleus is classically chaotic dynamics has been argued in the past [9,21–23]. In chaotic systems, the hydrodynamic modes of diffusion are given by singular distributions [24] which are best studied in terms of a cumulative distribution function

$$F_{\mathbf{k}}(\theta) = \lim_{t\to\infty} \frac{\int' \mathrm{d}\theta' \exp\{i\mathbf{k}\cdot[\beta(t;\theta') - \beta(0;\theta')]\}}{\int \mathrm{d}\theta' \exp\{i\mathbf{k}\cdot[\beta(t;\theta') - \boldsymbol{\beta}(0;\theta')]\}}. \tag{13}$$

The integration in the numerator is over a certain range of initial conditions over angular variables, $\theta(=\{\theta_i\})$ in the N-dimensional deformation space. As remarked earlier, the classical mechanics on the deformation space is nonlinear. To visualize, let us consider the case of two deformation parameters. Thus, the dimension of the phase space is four and because the dynamics is chaotic, we can imagine that there is at least one unstable and one stable direction. For simplicity, let us assume that two remaining directions are neutral. There is only one angle variable θ which will be transverse to the unstable direction so that fractal structure of modes appears along the stable direction [24]. To understand the role of chaos in non-equilibrium statistical mechanics, perhaps the most accessible reference is the book by Dorfman [19]. Fractal sets play a fundamental role in understanding the systems out of equilibrium when the underlying dynamics is chaotic. This has been explicitly studied for simple systems. In the case of nuclei, for most practically relevant situations, two to three deformation variables are employed to understand the nuclear dynamics [25]. This would imply that there would be more complicated topology of stable and unstable manifolds. Therefore, the set of relevant angle variables corresponding to the integration in (13) is over the manifold that is transverse to the unstable manifold. Having described the singular distribution in terms of a smooth cumulative distribution function, we can borrow the theory developed by Gaspard *et al* [26] where the diffusion coefficient is shown to be given in terms of the Hausdorff dimension of the diffusive mode and the positive Lyapunov exponent, λ, of the classical system. We must remark that although the system we are considering is quantum mechanical, the dynamics on the $\boldsymbol{\beta}$-space can still be fruitfully considered classical as evident in various phenomenological treatments of experimental data (see, for instance, refs [27–29]). To us, this is a beautiful interplay of microscopic dynamics described quantum mechanically and macroscopic dynamics in the deformation space. In the same vein, there is a penetration factor giving the probability of fission quantum mechanically and a hydrodynamic (diffusive) mode giving us the collection of paths that would enable fission to occur.

In chaotic systems, the diffusive modes have fine scale structures which make the graph of imaginary part of $F_{\mathbf{k}}$ vs. its real part singular. This is characterized by the Hausdorff dimension, **d** [30]. For long wavelengths, the relation is

$$D = \lambda \lim_{\mathbf{k}\to 0} \frac{d(\mathbf{k}) - 1}{k^2} \tag{14}$$

for the case of two-dimensional space of deformation parameters. Using this and eq. (10), we can write the mass tensor element as

$$B_{kl} = 2\alpha\lambda\hbar \lim_{\mathbf{k}\to 0} \frac{d_{kl}(\mathbf{k}) - 1}{k^2}, \tag{15}$$

where α is a geometric quantity and d_{kl} is the fractal dimension of the intersection of the set with the (kl)-hypersurface of the stable manifold (somewhat akin to the yz-plane in three-dimensional Euclidean space). For each pair, $(\mathcal{A}_k, \mathcal{A}_l)$, there will be Hausdorff dimension, d_{kl}, of a singular diffusive mode.

In the context of nuclear physics, the relevant number of deformation parameters may be more than two (see, for instance, the typical references [19,31,32]). For more than two parameters, rigorous derivation of a formula like (14) requires the concept of topological pressure and its relation with Lyapunov exponents in higher dimensions. Fortunately, these generalizations exist at the formal, rigorous level and have been illustrated with examples [33].

To summarize, eq. (15) encapsulates three aspects of LACM at once – classical, hydrodynamic and quantal. Whereas the expression for the mass parameters from the cranking formula is purely quantal, it is linked with classically chaotic dynamics in the space of deformation parameters, thereby leading to the appearance of hydrodynamic modes with a fine structure. With special reference to the phenomenon of spontaneous fission, we have been able to make a connection between the notion of tunnelling probability in real space, and, the accompanying dynamics in deformation space leading to an emergence of a hydrodynamic mode in $\boldsymbol{\beta}$-space where the dynamics is nonlinear. Employing the cranking model, we have obtained a relation (15) that defines the connection just mentioned.

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