PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 78, No. 1
January 2012
pp. 155–164

Noncommutative geometry-inspired rotating black hole in three dimensions

JUAN MANUEL TEJEIRO and ALEXIS LARRAÑAGA*
National Astronomical Observatory, National University of Colombia, Bogotá, Colombia
*Corresponding author. E-mail: ealarranaga@unal.edu.co

MS received 22 March 2011; revised 23 May 2011; accepted 8 June 2011

Abstract. We find a new rotating black hole in three-dimensional anti-de Sitter space using an anisotropic perfect fluid inspired by the noncommutative black hole. We deduce the thermodynamical quantities of this black hole and compare them with those of a rotating BTZ solution and give corrections to the area law to get the exact nature of the Bekenstein–Hawking entropy.

Keywords. Quantum aspects of black holes; thermodynamics of black holes.

PACS Nos 97.60.Lf; 04.70.Dy

1. Introduction

The theoretical discovery of radiating black holes [1] opened the first physically relevant window on the mysteries of quantum gravity. After many years of intensive research in this field, various aspects of the problem still remain under debate. For instance, a fully satisfactory description of the late stage of black hole evaporation is still missing. The string/black hole correspondence principle [2] suggests that in this extreme regime stringy effects cannot be neglected. This is just one of the many examples of how the development of string theory has affected various aspects of theoretical physics. Among different outcomes of string theory, we focus on the result that target space-time coordinates become noncommuting operators on a D-brane [3]. Thus, string–brane coupling shows the necessity of space-time quantization.

The noncommutativity of space-time can be encoded in the commutator

$$[x^{\mu}, x^{\nu}] = i\theta^{\mu\nu}, \tag{1.1}$$

where $\theta^{\mu\nu}$ is an antisymmetric matrix which determines the fundamental cell discretization of space-time much in the same way as the Planck constant $\hbar$ discretizes the phase space. This noncommutativity provides a black hole, with a minimum scale $\sqrt{\theta}$, known as the noncommutative black hole [4–9], whose commutative limit is the Schwarzschild metric.

DOI: 10.1007/s12043-011-0206-0; *e*Publication: 16 December 2011

Myung *et al* [10] have studied the thermodynamics and evaporation process of this noncommutative black hole while the entropy issue of this black hole was discussed in [11,12] and Hawking radiation was considered in [13].

In this paper, we construct a new rotating black hole in AdS3 space-time using an anisotropic perfect fluid inspired by the 4D noncommutative black hole, resulting in a solution with two horizons. We compare the thermodynamics of this black hole with that of a rotating BTZ black hole [14,15].

2. Derivation of the rotating solution

It has been shown [4,5] that the noncommutativity eliminates point-like structures in favour of smeared objects in flat space-time. Following the analysis in [16] based on the formulational and interpretational aspects of noncommutative quantum mechanics, it is important to highlight the important part played by the Voros star product in writing down the mass of a static, spherically symmetric, smeared, particle-like gravitational source required to get the noncommutative geometry-inspired black hole. In the absence of a resolution beyond the minimal length $\sqrt{\theta}$, a point-like object is no longer meaningful in a noncommutative background. Therefore, the sharpest distribution which faithfully described a localized object is no longer a Dirac delta, but a Gaussian, whose width coincides with $\sqrt{\theta}$,

$$\rho^{(n+1)\mathrm{D}}(r) = \frac{M}{(4\pi\theta)^{n/2}}\mathrm{e}^{-r^2/4\theta}, \tag{2.1}$$

where n is the number of spatial dimensions. In four-dimensional (4D) space-times, the Dirac delta function $\delta^{4\mathrm{D}}(r)$ is replaced by [4–9]

$$\rho^{4\mathrm{D}}(r) = \frac{M}{(4\pi\theta)^{3/2}}\mathrm{e}^{-r^2/4\theta} \tag{2.2}$$

and the corresponding mass distribution is given by

$$m^{4\mathrm{D}}(r) = 4\pi\int_0^r r'^2\rho^{4\mathrm{D}}(r')\mathrm{d}r' = \frac{2M}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{r^2}{4\theta}\right), \tag{2.3}$$

where $\gamma\left(3/2, r^2/4\theta\right)$ is the lower incomplete gamma function defined as

$$\gamma(a,z) = \int_0^z t^{a-1}\mathrm{e}^{-t}\mathrm{d}t. \tag{2.4}$$

In three-dimensional space-times, the Dirac delta function $\delta^{3\mathrm{D}}(r)$ is replaced by a Gaussian distribution with minimal width $\sqrt{\theta}$, [10]

$$\rho^{3\mathrm{D}}(r) = \frac{M}{4\pi\theta}\mathrm{e}^{-r^2/4\theta} \tag{2.5}$$

and the corresponding mass distribution is now

$$m^{3D}(r) = 2\pi \int_0^r r' \rho^{3D}(r') \mathrm{d}r' = M\gamma\left(1, \frac{r^2}{4\theta}\right) \tag{2.6}$$

$$= M(1 - \mathrm{e}^{-r^2/4\theta}). \tag{2.7}$$

To find a black hole solution in AdS_3 space-time, we introduce the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} + \frac{1}{\ell^2} g_{\mu\nu}, \tag{2.8}$$

where ℓ is related to the cosmological constant by

$$\Lambda = -\frac{1}{\ell^2}. \tag{2.9}$$

The energy–momentum tensor will take the anisotropic form

$$T^{\mu}_{\nu} = \mathrm{diag}(-\rho, p_r, p_\perp). \tag{2.10}$$

In order to completely define this tensor, we rely on the covariant conservation condition ${T^{\mu\nu}}_{;\nu} = 0$. This gives the source as an anisotropic fluid of density ρ, radial pressure

$$p_r = -\rho \tag{2.11}$$

and tangential pressure

$$p_\perp = -\rho - r\partial_r \rho. \tag{2.12}$$

Solving the above equations, we find the line element

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f^{-1}(r)\,\mathrm{d}r^2 + r^2(\mathrm{d}\varphi + N^\varphi \mathrm{d}t)^2, \tag{2.13}$$

where

$$f(r) = -8M(1 - \mathrm{e}^{-r^2/4\theta}) + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \tag{2.14}$$

$$N^\varphi = -\frac{J}{2r^2}. \tag{2.15}$$

Note that when $(r^2/4\theta) \to \infty$, either for considering a large black hole ($r \to \infty$) or for considering the commutative limit ($\theta \to 0$), we obtain the well-known BTZ rotating solution with angular momentum J and total mass M,

$$f^{\mathrm{BTZ}}(r) = -8M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}. \tag{2.16}$$

The line element (2.13) describes the geometry of a noncommutative black hole with event horizons given by the condition

$$f(r_\pm) = -8M(1 - \mathrm{e}^{-r_\pm^2/4\theta}) + \frac{r_\pm^2}{\ell^2} + \frac{J^2}{4r_\pm^2} = 0. \tag{2.17}$$

This equation cannot be solved in closed form. However, by plotting $f(r)$ one can read intersections with the r-axis and determine numerically the existence of horizon(s) and their radii. Figure 1 shows that the existence of angular momentum introduces new behaviour with respect to the noncommutative black hole studied by Myung and Yoon [17] and others reported before [18,19]. Instead of a single-event horizon, there are different possibilities:

(1) Two distinct horizons for $M > M_0$
(2) One degenerate horizon (extremal black hole) for $M = M_0$
(3) No horizon for $M < M_0$.

In view of these results, there can be no black hole if the original mass is less than the lower limit mass M_0. The horizon of the extremal black hole is determined by the conditions $f = 0$ and $\partial_r f = 0$, which gives

$$r_0^4 \left[\frac{1 - \left(1 + \frac{r_0^2}{4\theta}\right) \mathrm{e}^{-r_0^2/4\theta}}{1 - \left(1 - \frac{r_0^2}{4\theta}\right) \mathrm{e}^{-r_0^2/4\theta}} \right] = \frac{J^2\ell^2}{4} \tag{2.18}$$

and then, the mass of the extremal black hole can be written as

$$M_0 = \frac{\left(\frac{r_0^2}{\ell^2} + \frac{J^2}{4r_0^2}\right)}{8(1 - \mathrm{e}^{-r_0^2/4\theta})}. \tag{2.19}$$

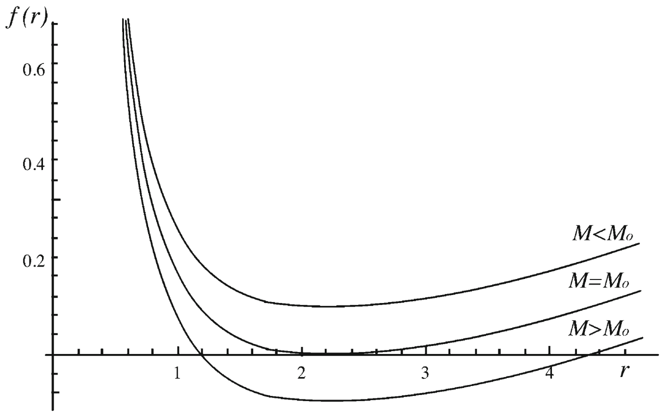

Figure 1. Metric function f as a function of r. We have taken the values $\theta = 0.1$, $\ell = 10$ and $J = 1$. The minimum mass $M_0 \approx 0.0125$.

In the commutative limit, $\theta \to 0$, the extreme black hole has the horizon at

$$r_0^{\mathrm{BTZ}} = \sqrt{\frac{J\ell}{2}} \tag{2.20}$$

and its mass is

$$M_0^{\mathrm{BTZ}} = \frac{1}{8}\frac{J}{\ell}.$$

3. Thermodynamics

The black hole temperature is given by

$$T_{\mathrm{H}} = \frac{1}{4\pi}\,\partial_r f|_{r_+} \tag{3.1}$$

$$T_{\mathrm{H}} = \frac{r_+}{2\pi\ell^2}\left[1 - \frac{J^2\ell^2}{4r_+^4} + \frac{\left(r_+^2 + \frac{J^2\ell^2}{4r_+^2}\right)}{4\theta\left(1 - \mathrm{e}^{r_+^2/4\theta}\right)}\right]. \tag{3.2}$$

For large black holes, i.e. $(r_+^2/4\theta) \gg 0$, one recovers the temperature of the rotating BTZ black hole,

$$T_{\mathrm{H}}^{\mathrm{BTZ}} = \frac{r_+}{2\pi\ell^2}\left[1 - \frac{J^2\ell^2}{4r_+^4}\right]. \tag{3.3}$$

As shown in figure 2, the temperature is a monotonically increasing function of the horizon radius for large black holes and the temperature of the extreme black hole is zero.

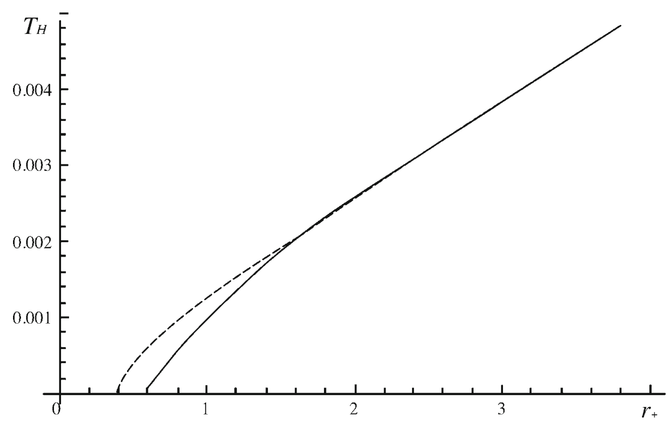

Figure 2. Black hole temperature T_{H} vs. r_+. The solid line represents the temperature for the noncommutative black hole with $\theta = 0.1$. The dashed line represents the temperature for the rotating BTZ black hole. In both cases $\ell = 10$ and $J = 0.03$.

3.1 *Entropy*

The first law of thermodynamics for a rotating black hole reads as

$$\mathrm{d}M = T_{\mathrm{H}}\mathrm{d}S + \Omega \mathrm{d}J, \tag{3.4}$$

where the angular velocity of the black hole is given by

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{r_+} = \frac{J}{2r_+^2}, \tag{3.5}$$

that is exactly the same as the rotating BTZ solution. Equation (3.4) can be written in the Gibbs form as

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}M - \frac{\Omega}{T}\mathrm{d}J. \tag{3.6}$$

At the outer horizon, $r = r_+$, we have

$$M = M(r_+, J)\,, \tag{3.7}$$

and

$$\mathrm{d}M = \frac{\partial M}{\partial r_+}\mathrm{d}r_+ + \frac{\partial M}{\partial J}\mathrm{d}J = \frac{\partial M}{\partial r_+}\mathrm{d}r_+ + \Omega \mathrm{d}J. \tag{3.8}$$

Using eq. (3.6) we get

$$\mathrm{d}S = \frac{1}{T}\frac{\partial M}{\partial r_+}\mathrm{d}r_+, \tag{3.9}$$

where S is the entropy of the noncommutative black hole. From eq. (2.17) we finally obtain

$$S = \frac{\pi}{2}\int_{r_0}^{r_+}\left(\frac{1}{1 - \mathrm{e}^{-\xi^2/4\theta}}\right)\mathrm{d}\xi. \tag{3.10}$$

The entropy as a function of r_+ is depicted in figure 3. Note that, in the large black hole limit, the entropy function corresponds to the Bekenstein–Hawking entropy (area law), $S_{\mathrm{BH}} = \pi r_+/2$, for the rotating BTZ geometry.

3.2 *Entropy corrections from noncommutativity*

Following [12], we shall estimate the entropy corrections due to noncommutativity space-time using a graphical analysis. For this, we shall compare the relation for the noncommutative black hole

$$\frac{\mathrm{d}S}{\mathrm{d}r_+} = \frac{\pi}{2}\frac{1}{1 - \mathrm{e}^{-r_+^2/4\theta}} \tag{3.11}$$

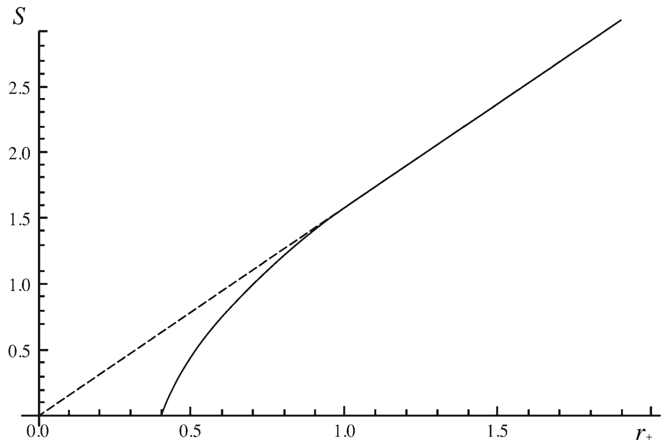

Figure 3. Entropy vs. r_+. The solid line represents the entropy of the noncommutative black hole with $\theta = 0.1$. The dashed line represents the entropy of the rotating BTZ black hole. In both cases $\ell = 10$.

with the corresponding relation for the commutative BTZ black hole,

$$\frac{\mathrm{d}S_{\mathrm{BTZ}}}{\mathrm{d}r_+} = \frac{\pi}{2}. \tag{3.12}$$

The plot of $\mathrm{d}S/\mathrm{d}r_+$ as a function of r_+ for both equations are given in figure 4 and it can be seen that the two curves coincide asymptotically, i.e. for $r \to \infty$, or for $\theta \to 0$.

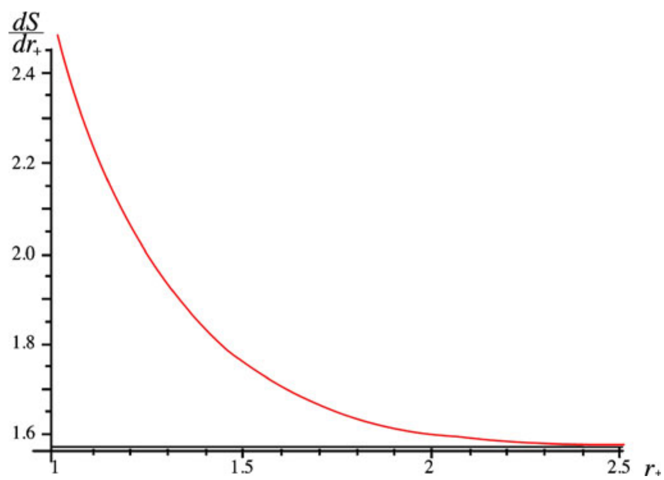

Figure 4. $(\mathrm{d}S/\mathrm{d}r_+)$ (red line) and $\mathrm{d}S_{\mathrm{BTZ}}/\mathrm{d}r_+$ (black line) vs. r_+ from eqs (3.11) and (3.12). In both cases we have used units of $2\sqrt{\theta}$.

This behaviour shows that there is a deviation from the usual area law and therefore we shall find the corrections to the Bekenstein–Hawking entropy. For this, note that for $|e^{-r_+^2/4\theta}| < 1$, we can expand as

$$\frac{dS}{dr_+} = \frac{\pi}{2}(1 - e^{-r_+^2/4\theta})^{-1} \tag{3.13}$$

$$\frac{dS}{dr_+} = \frac{\pi}{2}(1 + e^{-r_+^2/4\theta} + e^{-r_+^2/2\theta} + \cdots). \tag{3.14}$$

The first term in (3.14) corresponds to the usual area law and the other terms can be interpreted as corrections to the entropy. Taking only the first-order correction, we have

$$\frac{dS}{dr_+} = \frac{\pi}{2}(1 + e^{-r_+^2/4\theta}) \tag{3.15}$$

that is plotted as a function of r_+ as shown in figure 5. It is observed that this curve has the behaviour of the area law for great r_+ and agrees with the red curve almost to the point $r_+ = 2.6\sqrt{\theta}$. To improve this situation, the second-order correction can be included as

$$\frac{dS}{dr_+} = \frac{\pi}{2}(1 + e^{-r_+^2/4\theta} + e^{-r_+^2/2\theta}). \tag{3.16}$$

The plot of this function in figure 6 shows that now the blue curve coincides with the red curve for $r_+ \geq 2.4\sqrt{\theta}$. From this analysis one can expect that including more terms in the expansion gives a greater coincidence of the curves.

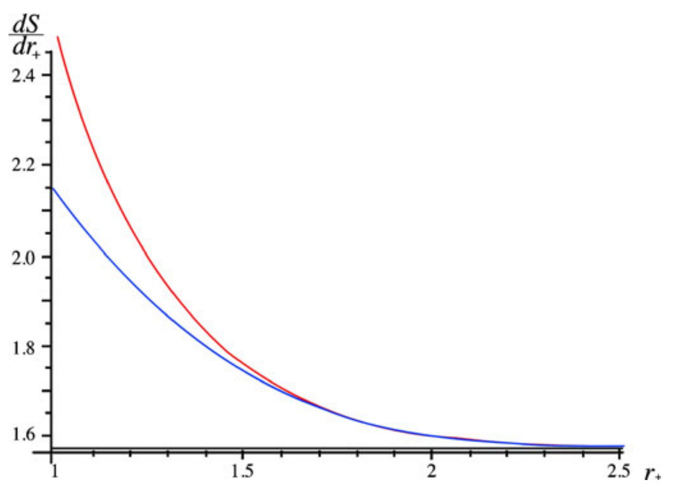

Figure 5. dS/dr_+ to the first order (blue line) vs. r_+ from eq. (3.15). We have used units of $2\sqrt{\theta}$.

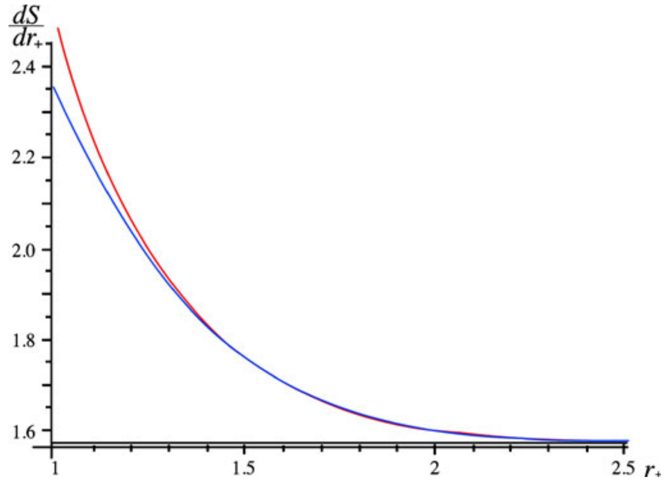

Figure 6. $\mathrm{d}S/\mathrm{d}r_+$ to the second order (blue line) vs. r_+ from eq. (3.16). We have used units of $2\sqrt{\theta}$.

Integrating (3.16) over r_+ yields the corrected entropy

$$S = \frac{\pi r_+}{2} + \frac{\sqrt{\pi\theta}}{2}\mathrm{Erf}\left(\frac{r_+}{2\sqrt{\theta}}\right) + \frac{\sqrt{2\pi\theta}}{2}\mathrm{Erf}\left(\frac{r_+}{\sqrt{2\theta}}\right). \tag{3.17}$$

Note that the first term corresponds to the Bekenstein–Hawking semiclassical entropy and the other terms are the corrections due to noncommutativity. It is interesting to note that these terms do not involve logarithmic terms as in [20,21], but include only error functions.

4. Conclusion

We construct a noncommutative rotating black hole in AdS_3 space-time using an anisotropic perfect fluid inspired by the 4D noncommutative black hole. We have investigated the Hawking temperature, entropy and the area law for a BTZ black hole whose metric is modified by the effects of noncommutative space-time. The noncommutative version of the semiclassical Bekenstein–Hawking area law holds for large horizon radius. From a graphical analysis we find the correction terms to the area law that involve only error functions.

Acknowledgement

This work was supported by the Universidad Nacional de Colombia. Hermes Project Code 13038.

References

[1] S W Hawking, *Commun. Math. Phys*. **43**, 199 (1975)
[2] L Susskind, *Phys. Rev. Lett*. **71**, 2367 (1993)
[3] E Witten, *Nucl. Phys*. **B460**, 335 (1996)
N Seiberg and E Witten, *J. High Energy Phys*. **032**, 9909 (1999)
[4] A Smailagic and E Spallucci, *J. Phys*. **A36**, L467 (2003)
[5] T G Rizzo, *J. High Energy Phys*. **09**, 021 (2006)
[6] P Nicolini, A Smailagic and E Spallucci, *Phys. Lett*. **B632**, 547 (2006)
[7] S Ansoldi, P Nicolini, A Smailagic and E Spallucci, *Phys. Lett*. **B645**, 261 (2007)
[8] E Spallucci, A Smailagic and P Nicolini, *Phys. Lett*. **B670**, 449 (2009), arXiv:0801.3519 [hep-th]
[9] P Nicolini, arXiv:0807.1939[hep-th]
[10] Y S Myung, Y W Kim and Y J Park, *J. High Energy Phys*. **0702**, 012 (2007)
[11] R Banerjee, B R Majhi and S Samanta, *Phys. Rev*. **D77**, 124035 (2008)
[12] R Banerjee, B R Majhi and S K Modak, *Class. Quant. Grav*. **26**, 085010 (2009)
[13] K Nozari and S H Mehdipour, *Class. Quant. Grav*. **25**, 175015 (2008)
[14] M Banados, C Teitelboim and J Zanelli, *J. Phys. Rev. Lett*. **69**, 1849 (1992)
[15] M Banados, M Henneaux, C Teitelboim and J Zanelli, *J. Phys. Rev*. **D48**, 1506 (1993)
[16] R Banerjee, S Gangopadhyay and S Kumar Modak, *Phys. Lett*. **B686**, 181 (2010)
[17] Y S Myung and M Yoon, *Eur. Phys. J*. **C62**, 405 (2009)
[18] J Sadeghi and M R Setare, *Int. J. Theor. Phys*. **46**, 817 (2007)
[19] Y W Kim, Y J Park, C Rim and J H Yee, *J. High Energy Phys*. **10**, 060 (2008)
[20] S K Modak, *Phys. Lett*. **B671**, 167 (2009)
[21] T R Govindarajan, R K Kaul and V Suneeta, *Class. Quant. Grav*. **18**, 2877 (2001)