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Schwarzian derivative as a proof of the chaotic behaviour

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Abstract. In recent years, a sufficient condition for determining chaotic behaviours of the nonlinear systems has been characterized by the negative Schwarzian derivative (Hacıbekiroğlu *et al*, *Nonlinear Anal.: Real World Appl.* **10**, 1270 (2009)). In this work, the Schwarzian derivative has been calculated for investigating the quantum chaotic transition points in the high-temperature superconducting frame of reference, which is known as a nonlinear dynamical system that displays some macroscopic quantum effects. In our previous works, two quantum chaotic transition points of the critical transition temperature, T_c, and paramagnetic Meissner transition temperature, T_{PME}, have been phenomenologically predicted for the mercury-based high-temperature superconductors (Onbaşlı *et al*, *Chaos, Solitons and Fractals* **42**, 1980 (2009); Aslan *et al*, *J. Phys.: Conf. Ser.* **153**, 012002 (2009); Çataltepe, *Superconductor* (Sciyo Company, India, 2010)). The T_c, at which the one-dimensional global gauge symmetry is spontaneously broken, refers to the second-order phase transition, whereas the T_{PME}, at which time reversal symmetry is broken, indicates the change in the direction of orbital current in the system (Onbaşlı *et al*, *Chaos, Solitons and Fractals* **42**, 1980 (2009)). In this context, the chaotic behaviour of the mercury-based high-temperature superconductors has been investigated by means of the Schwarzian derivative of the magnetic moment versus temperature. In all calculations, the Schwarzian derivatives have been found to be negative at both T_c and T_{PME} which are in agreement with the chaotic behaviour of the system.

Keywords. Mercury cuprate superconductors; nonlinear dynamics and chaotic behaviour; Schwarzian derivative; paramagnetic Meissner effect.

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1. Introduction

By displaying some macroscopic quantum effects such as Josephson effect, magnetic flux quantization and time dilatation effect, the superconductivity plays an important role in establishing nonlinear quantum theory. Superconductors as nonlinear dynamical systems also constitute a natural frame of reference to observe the quantum chaotic transitions which emanate themselves as spontaneous symmetry breakings due to nonlinear interactions within the system [1,2]. The concept of symmetry breakings in superconductors, especially in mercury-based high-temperature superconductors, was discussed by means of chaotic behaviour (i.e. chaotic transitions) in detail in our previous works [3–5]. Moreover, solitons, which are accepted as a universal concept in nonlinear science, had also been previously predicted in the mercury-based high-temperature superconductors [3]. As is known, solitons can be found in a Josephson junction, where two superconductors are separated by a thin insulating layer [6]. The copper oxide layered mercury cuprate superconducting system has an intrinsic Josephson junction structure, as well. Furthermore, there have been various studies about dynamical chaos in long Josephson junctions since 1982. The role of the phase of quantum wave function in long Josephson junction is considered as dynamical chaos that is described by nonlinear sine-Gordon equation [7–12]. For these reasons, one has the right to consider a high-temperature mercury cuprate superconductor, for which electron–phonon interactions are strong and nonlinear [13] and which displays intrinsic Josephson effect, as a nonlinear dynamical system. In this point of view, determining the Schwarzian derivative is a convenient mathematical method to investigate the chaotic behaviours in such nonlinear dynamical condensed matter systems.

According to our previous works [3–5], mercury-based high-temperature superconductors exhibit two quantum chaotic transitions: at the critical transition temperature (T_c), and at paramagnetic Meissner transition temperature (T_{PME}). Hence, this work is devoted to establish the mathematical determination of the chaotic behaviours in mercury cuprate high-temperature superconductors by determining the Schwarzian derivative. For this purpose, the following sections are focussed on the main features of the superconducting state by means of chaotic transitions and the Schwarzian derivative for deciding the chaotic behaviours in superconductors.

2. Chaos and superconductivity

The combination of the concepts of chaos and superconductivity produces an interesting phenomenon for almost 30 years. In this section, we shall be focussing on both chaos and superconductivity phenomena and their interesting relation.

The concept of chaos in science is something different from the concept in common usage. The centre of chaos theory is the discovery of the unpredictable behaviour of systems in long terms that have strong relation between long-range order and structure. In this context, ‘chaos’ refers not to true randomness but to the orderly disorder characteristics of these systems. In other words, the main aim of the science of chaos is to comprehend this unusual complex attitude of such systems via mathematics [14].

Superconductivity itself is a natural consequence of ‘macroscopic quantum effect’. As is known, macroscopic quantum effects result from the collective behaviour of quantum particles in such a way that all entities condense into the same state of being. In this context, superconductivity emanates from the collective motion of a huge number of harmonized electron pairs with zero spin and momentum (i.e. bosons) at low temperatures resulting in a coherent and highly long-range ordered state. In other words, in the superconducting state, fermionic electrons are unexpectedly paired by rearrangement of their spins via phonons that results in bosonic quasiparticles. As was previously stated by some scientists as well as Protogenov and Ryndyk from International Centre for Theoretical Physics in Trieste, ‘The transition from the superconducting state to the normal state is accompanied by the propagation of a quantum chaos from one cell to the whole space and corresponds to the transition into the confinement phase of spin-wave excitation sources. The formation of this rigid coherent system of quantum braids manifests itself in the character of the spectrum of quasiparticle's collective motions. It has a finite gap in the long wave limit and is almost dispersionless' [15]. Moreover, the microscopic origin of chaos is related to the appearance of an excitation gap in the superconducting state. The occurrence of an excitation gap in superconductors is accepted as a signature of quantum chaos [16,17].

The condensation of electron pairs into a single quantum state in the superconducting phase, the so called as Bose–Einstein condensation (BEC), occurs at a specific temperature, known as critical transition temperature (T_c), at which the gap appears above the Fermi level. The critical transition temperature also marks a second-order phase transition which manifests itself as a discontinuity of the third derivative of thermodynamic function of the system [1]. The experimental output of this discontinuity is clearly observed in both resistance vs. temperature and specific heat vs. temperature data. In the picture of chaos, the superconducting transition is considered as chaotic transition because of all these reasons previously mentioned in this section.

Furthermore, the essence of chaos is directly related to the transition from one state to another where the probability density of the system changes with temperature [18]. In other words, one of the most characteristic features of the chaotic behaviour of the systems is the strong dependence of system's states on the changes in its parameters. The state of the system changes with only small variations in some parameters of the system such as temperature [19]. In this context, the probability distribution function of the superconducting system experiences a transition from Fermi–Dirac (FD) to Bose–Einstein (BE) due to small variation in temperature at T_c [5]. Moreover, the one-dimensional global gauge symmetry is broken together with the order parameter at T_c due to the off-diagonal long-range order [20]. This unexpected transition at T_c is detected on the magnetic data of the system. Based on the relevant literature, the transition from zero magnetic moment state to diamagnetic state at T_c, which is the main feature of superconductivity, is named as 'chaotic'. The mathematical interpretation of this chaotic behaviour at T_c is represented by the concept of change of probability density. Although the temperature parameter is the main driving force for this kind of chaotic transition, it has been understood that various parameters and the initial conditions such as crystallographic structure, oxygen content, purity (lack of impurities) etc. have crucial role in the transition from FD to BE. This means that the dynamical chaos can be controlled precisely by some advanced laboratory workings.

In this context, the critical transition temperature (T_c) is considered as the first chaotic point that is experimentally determined by resistance vs. temperature, magnetic moment vs. temperature and specific heat vs. temperature curves. In this work, the critical transition temperature was determined by the real component of the magnetic moment vs. temperature data. As is known, at T_c, the real component of the magnetic moment goes to zero.

The second quantum chaotic transition point is the paramagnetic Meissner transition temperature (T_{PME}) at which the orbital current of the system changes its direction. Moreover, it has been previously determined that the concept of time reversal symmetry breaking becomes detectable on paramagnetic Meissner effect (PME) at T_{PME} by observing the change of direction of orbital current [3–5]. In the PME, in contrast to the fundamental diamagnetic response of superconductors to external magnetic field, superconductors acquire a net positive magnetic moment when they are cooled in very weak magnetic fields of the order of 1 G. The PME can be observed on both DC (direct current) and AC (alternative current) magnetic moment vs. temperature data of some very cleanly prepared high-temperature superconductors such as mercury cuprate superconductors. In this work, we shall focus on the AC magnetic moment vs. temperature data of $HgBa_2Ca_2Cu_3O_{8+x}$ (Hg-1223) superconductors.

Hg-1223 superconductors are very special because these have the highest critical transition temperature at normal atmospheric pressure [3,21]. The new world record of critical transition temperature of $T_c = 140$ K has also been observed in the optimally oxygen-doped Hg-1223 superconductors [3]. Mercury cuprate superconductors also exhibit PME as shown in figures 1 and 2. PME manifests itself as the maximum positive magnetic moment signal at T_{PME} on the imaginary component of the magnetic moment vs. temperature data illustrated in figures 1 and 2. The AC magnetic moment vs. temperature curves was obtained by Quantun Design MPMS-5S model superconducting quantum interference device (SQUID).

The magnetic moment vs. temperature curves shown in figures 1 and 2 belong to the same batch of optimally oxygen-doped mercury cuprate superconductors. The main difference between the samples is their geometry. Figure 1 refers to the uncut, i.e. random shaped sample, whereas figure 2 corresponds to the cut, i.e. rectangular shaped optimally oxygen-doped $HgBa_2Ca_2Cu_3O_{8+x}$ superconductors. Since the PME is the intrinsic feature

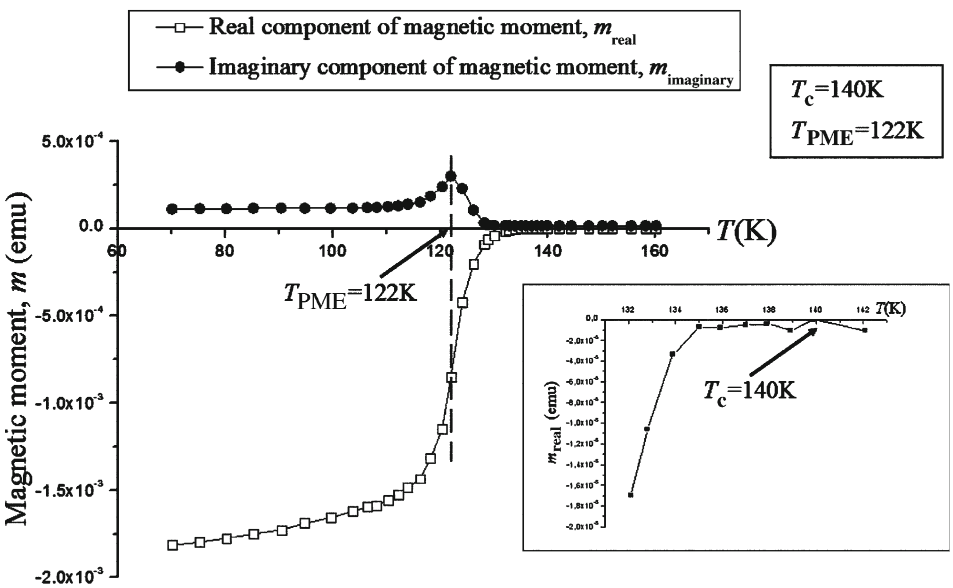

Figure 1. The real and imaginary components of the magnetic moment of the optimally oxygen-doped mercury cuprate superconductor, $HgBa_2Ca_2Cu_3O_{8+x}$. The imaginary part of the magnetic moment has a maximum at $T_{PME} = 122$ K. The inset shows critical transition temperature, $T_c = 140$ K via real component of the magnetic moment of the system.

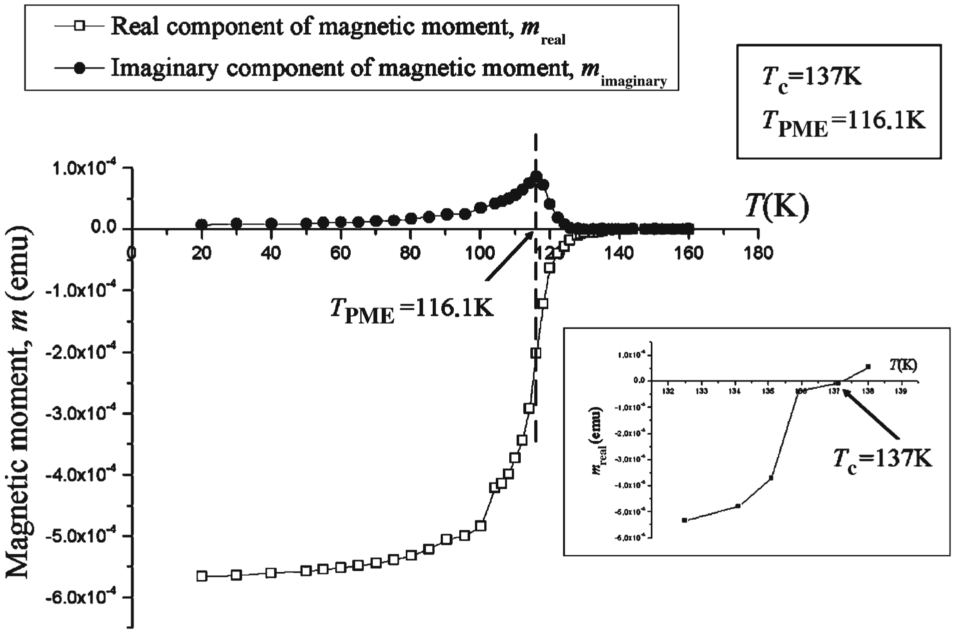

Figure 2. The real and imaginary components of the magnetic moment of the optimally oxygen-doped and cut mercury cuprate superconductor, $HgBa_2Ca_2Cu_3O_{8+x}$. The imaginary part of the magnetic moment has a maximum at $T_{PME} = 116.1$ K. The inset shows critical transition temperature, $T_c = 137$ K via real component of the magnetic moment of the system.

of the mercury cuprates, PME has been observed on both samples but it has been observed that cutting process lowers both T_c and T_{PME}.

According to figures 1 and 2, the imaginary component of the magnetic moment ($m_{imaginary}(T)$) has a critical point at T_{PME} and $m_{imaginary}(T)$ increases for $T < T_{PME}$, whereas $m_{imaginary}(T)$ decreases for $T > T_{PME}$.

3. The Schwarzian derivative for determining the chaotic behaviours in superconductors

One of the most crucial intrinsic properties of the chaotic systems is that the state of the system can be affected by small variations of its parameters. For instance, small changes in temperature parameter at T_c in the superconducting chaotic system manifest itself as an occurrence of the transition from Fermi–Dirac to Bose–Einstein distribution function. From this point of view, small variations in parameters in the condensed matter media, which display chaotic behaviour, can be investigated by means of derivative process, since the derivative of a function represents an infinitesimal change in the function with respect to one of its variables. So the derivative process is the most appropriate mathematical tool to detect chaotic behaviours. In this context, the Schwarzian derivative method has been utilized to find a sufficient condition for the chaotic behaviours of nonlinear dynamical systems [22].

The Schwarzian derivative $Sf(x)$ of a locally univalent analytic function f at point x, is defined by

$$Sf\,(x) = \left(\frac{f''\,(x)}{f'\,(x)}\right)' - \frac{1}{2}\left(\frac{f''\,(x)}{f'\,(x)}\right)^2, \tag{1}$$

where $f(x)$ is a function with one variable and $f'(x)$ and $f''(x)$ are its first and second continuous derivatives, respectively. The Schwarzian derivative is named after the German mathematician Hermann Schwarz who studied complex valued functions, but it was used for hypergeometric differential equations by Kummer in 1836 [23]. Moreover, Schwarzian derivative is invariant under linear fractional transformations (T), i.e. Möbius transformation given by eq. (2).

$$S\,(T \circ f) = Sf. \tag{2}$$

This means that Schwarzian derivative of any linear fractional transformation is zero [24,25]. In particular, Möbius transformation is the only function that satisfies the condition defined by

$$S\,(f) = 0. \tag{3}$$

Although Möbius transformations are utilized for mapping in geometry, they have many applications that cover different disciplines in science from neuroscience to relativity theory. The fundamental Möbius transformations cover translation, inversion, and rotation operations as well as dilation effect. In our previous works, it has been determined that the superconducting system is invariant under both translation and inversion operations [3]. The long-range order and the tetragonal P/4 mmm lattice structure of the system enable invariance under these transformations. Moreover, the dilation effect in time manifests itself as the shift of plasma resonance frequency with respect to small variations in temperature [26–28]. From this point of view, superconducting system can be considered as a natural laboratory of mathematics that reflects some general properties of invariant Schwarzian derivative under Möbius transformations, as well.

In 1980s, the Schwarzian derivative was used for limiting the behaviour of dynamical systems [29,30]. According to Katz [31] and Hacıbekiroğlu *et al* [22], when the system behaves chaotically, the Schwarzian derivative of the function is negative. Moreover, the eventual negative Schwarzian derivative has also been utilized for searching chaotic behaviours in neuroscience, especially explaining the electrical activity in neural cells, a behaviour described as 'bursting' [32,33].

To investigate the chaotic behaviours in mercury cuprates, the Schwarzian derivatives of both the real and imaginary components of the magnetic moment of Hg-1223 superconductors, $S[m_{\text{real}}(T)]$ and $S[m_{\text{imaginary}}(T)]$, have been calculated by eqs (4) and (5), respectively.

$$S\,[m_{\text{real}}(T)] = \frac{m_{\text{real}}(T)'''}{m_{\text{real}}(T)'} - \frac{3}{2}\left(\frac{m_{\text{real}}(T)''}{m_{\text{real}}(T)'}\right)^2, \tag{4}$$

$$S\left[m_{\text{imaginary}}(T)\right] = \frac{m_{\text{imaginary}}(T)'''}{m_{\text{imaginary}}(T)'} - \frac{3}{2}\left(\frac{m_{\text{imaginary}}(T)''}{m_{\text{imaginary}}(T)'}\right)^2, \tag{5}$$

where $m'(T)$, $m''(T)$ and $m'''(T)$ represent the first-, second- and third-order derivatives of the magnetic moment with respect to temperature.

To calculate $S[m_{\text{real}}(T)]$ and $S[m_{\text{imaginary}}(T)]$, the first-, second- and third-order derivatives of the components of the magnetic moments have been taken. The variations of the related derivatives with temperature for the optimally oxygen-doped sample are shown in figures 3a and 3b.

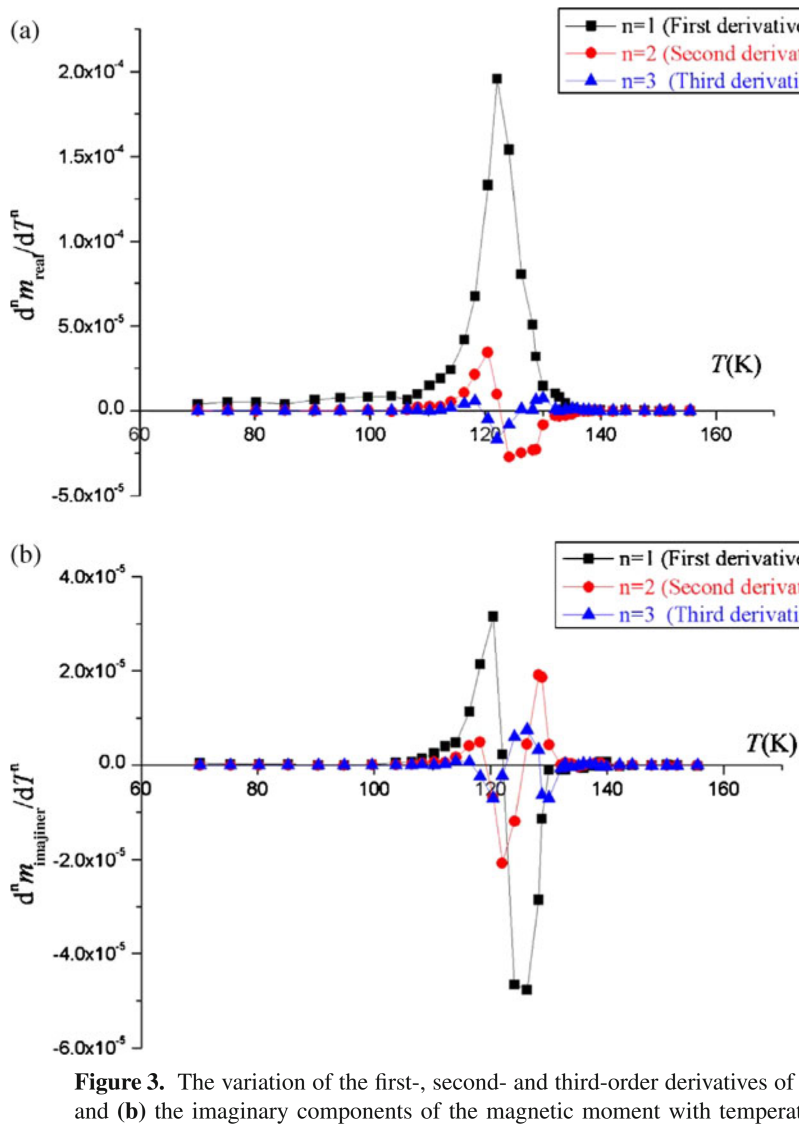

Figure 3. The variation of the first-, second- and third-order derivatives of **(a)** the real and **(b)** the imaginary components of the magnetic moment with temperature for the optimally oxygen-doped sample.

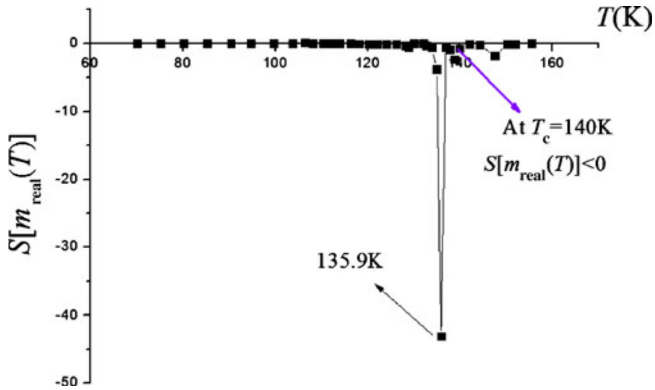

Figure 4. The Schwarzian derivative of the real component of the magnetic moment of the optimally oxygen-doped mercury cuprate superconductor, $HgBa_2Ca_2Cu_3O_{8+x}$.

The Schwarzian derivatives of the real and imaginary components of the magnetic moment of the optimally oxygen-doped mercury cuprates with $T_c = 140$ K and $T_{PME} = 122$ K are given in figures 4 and 5, respectively.

The Schwarzian derivatives of the real and imaginary components of the magnetic moment of the optimally oxygen-doped and cut mercury cuprates with $T_c = 137$ K and $T_{PME} = 116$ K are given in figures 6 and 7, respectively.

4. Discussions

In this work, the chaotic behaviours of the mercury cuprate superconducting system have been investigated by calculating the Schwarzian derivatives. The investigation was realized

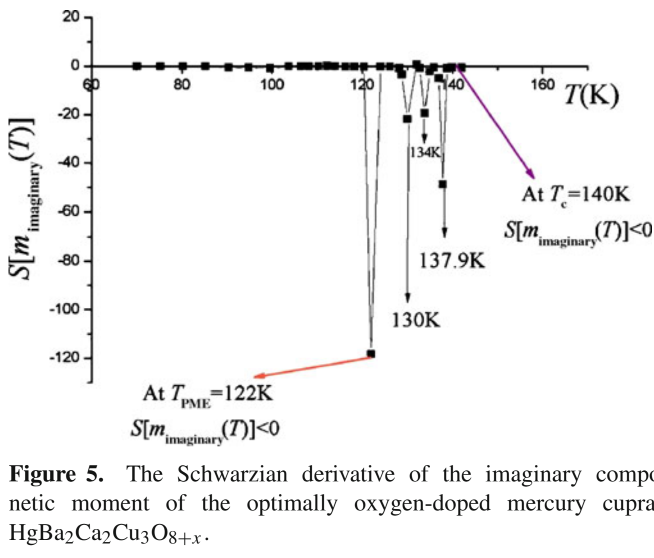

Figure 5. The Schwarzian derivative of the imaginary component of the magnetic moment of the optimally oxygen-doped mercury cuprate superconductor, $HgBa_2Ca_2Cu_3O_{8+x}$.

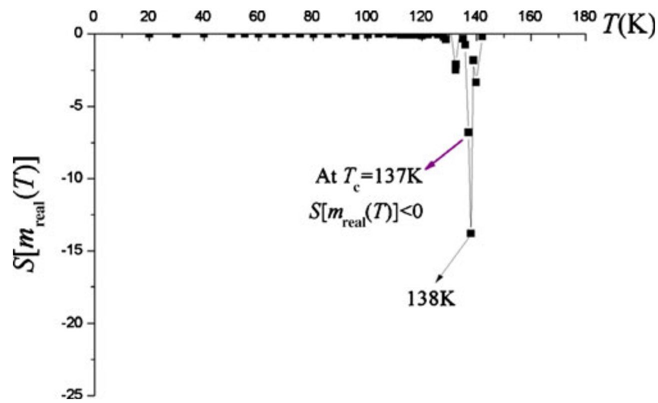

Figure 6. The Schwarzian derivative of the real component of the magnetic moment of the optimally oxygen-doped and cut mercury cuprate superconductor, $HgBa_2Ca_2Cu_3O_{8+x}$.

on two mercury cuprate superconducting systems which have different geometric shapes. Regardless of the geometry of the superconducting samples, the chaotic behaviours at both T_c and T_{PME} manifest themselves as the emergence of negativity in the Schwarzian derivatives of the real and imaginary parts of the magnetic moment data. Moreover, the second chaotic point (T_{PME}) always reveals itself as the more sharp and distinct negative Schwarzian derivative peak than the first chaotic point (T_c) as shown in figures 5 and 7. Furthermore, a remarkable point in figures 5 and 7 is that some distinct negative Schwarzian derivative peaks have been determined between T_c and T_{PME} on both samples. While three distinct negative Schwarzian derivative peaks can be observed at 130 K, 134 K and approximately 138 K for the optimally oxygen-doped sample, only one negative and sharp peak can be seen at about 130 K for the optimally oxygen-doped and cut sample (figures 5 and 7). In this point of view, it can be deduced that the cutting process, that enables us to rearrange

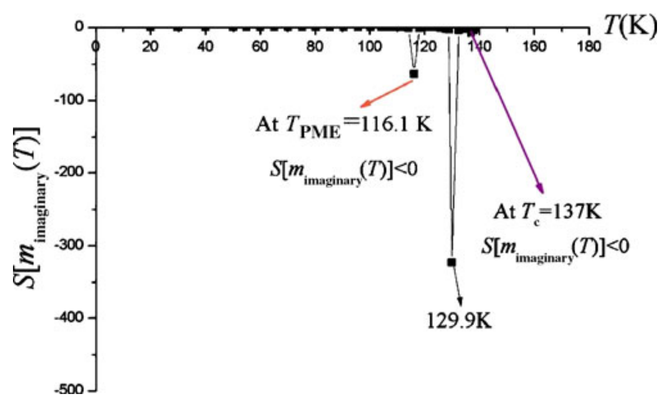

Figure 7. The Schwarzian derivative of the imaginary component of the magnetic moment of the optimally oxygen-doped and cut mercury cuprate superconductor, $HgBa_2Ca_2Cu_3O_{8+x}$.

the geometry of the sample as a parameter, also decreases the chaotic fluctuations between T_c and T_{PME} in the system. On the other hand, it is understood that no matter what parameters such as geometry, oxygen doping process etc., are changed, the chaotic behaviour always appears in the system which is proved by negative Schwarzian derivative peaks. Furthermore, a considerably large number of temperature points that satisfy the conditions are given below.

$$S\left[m_{\text{real}}\left(T\right)\right]=0$$

$$S\left[m_{\text{imaginary}}\left(T\right)\right]=0$$

From this point of view, it is concluded that the system is also invariant under linear fractional transformations, namely Möbius transformations.

5. Conclusions

In this paper, the Schwarzian derivative method, which is utilized for predicting the chaotic behaviours mathematically in nonlinear dynamical systems, was applied to a high-temperature superconducting condensed matter system for determining the chaotic points which were previously predicted phenomenologically. Moreover, it is proved that the system is invariant under translation and inversion operations and exhibits the time dilation effect by determining the points at which the Schwarzian derivative is zero. Hence, in this work the fact that chaos represents the order of disorder in long range has been determined by Schwarzian derivative method.

As result of the mathematical study, it has been proposed that the Schwarzian derivative is a convenient mathematical method for precise prediction of chaotic points in nonlinear superconducting condensed matter systems as well.

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