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Canonical form of Nambu–Poisson bracket: A pedestrian approach

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Abstract. In the seventies, Nambu (*Phys. Rev.* **D7**, 2405 (1973)) proposed a new approach to classical dynamics based on an N-dimensional Nambu–Poisson (NP) manifold replacing the primitive even-dimensional Poisson manifold and on N–1 Hamiltonians in place of a single Hamiltonian. This approach has had many promoters including Bayen and Flato (*Phys. Rev.* **D11**, 3049 (1975)), Mukunda and Sudarshan (*Phys. Rev.* **D13**, 2846 (1976)), and Takhtajan (*Comm. Math. Phys.* **160**, 295 (1994)) among others. While Nambu had originally considered $N = 3$, the illustration of his ideas for $N = 4$ and 6 was given by Chatterjee (*Lett. Math. Phys.* **36**, 117 (1996)) who observed that the classical description of dynamical systems having dynamical symmetries is described elegantly by Nambu's formalism of mechanics. However, his considerations do not quite yield the beautiful canonical form conjectured by Nambu himself for the N-ary NP bracket. By making a judicious choice for the 'extra constant of motion' of namely, α and β, which are the orientation angles in Kepler problem and isotropic harmonic oscillator (HO) respectively, we show that the dynamical systems with dynamical symmetries can be recast in the beautiful form suggested by Nambu. We believe that the techniques used and the theorems suggested by us in this work are of general interest because of their involvement in the transition from Hamiltonian mechanics to Nambu mechanics.

Keywords. Hamiltonian mechanics; Nambu mechanics.

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1. Introduction

The Hamiltonian formulation of classical mechanics which has paved the way for quantum mechanics is acclaimed worldwide. The state of a dynamical system having N degrees of freedom is classically given by specifying N coordinates $q^1, q^2, ..., q^N$ and N momenta $p^1, p^2, ..., p^N$, collectively referred to as the canonical variables x^I, $I = 1, 2, \ldots, 2N$ of the dynamical system. An observable $F(x^I)$ such as the energy and momentum of the system is a function of the canonical variables. The equations of flow for the canonical variables

$$\dot{q}^i = \frac{\partial H(q, p)}{\partial p^i}, \tag{1}$$

$$\dot{p}^i = -\frac{\partial H(q, p)}{\partial q^i}, \tag{2}$$

are the Hamilton's canonical equations of motion where the Hamiltonian function $H(q, p)$ is the total energy of the system. The observables form a Lie algebra with respect to the Poisson bracket (PB),

$$[F, G] = \sum_{i=1}^{N} \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial G}{\partial q^i} = \sum_{I=1}^{2N} \alpha^{IJ} \frac{\partial F \partial G}{\partial x^I \partial x^J}, \tag{3}$$

where the $2N \times 2N$ matrix which defines the PB algebra

$$\alpha^{IJ} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Here I denotes an $N \times N$unit matrix, and 0 an $N \times N$ zero matrix.

The Hamilton's equations can be written in terms of the PB as

$$\dot{q}^i = [q^i, H], \tag{4}$$

$$\dot{p}^i = [p^i, H]. \tag{5}$$

In general, the evolution of an observable $F(x^I)$ becomes

$$\frac{\mathrm{d}F}{\mathrm{d}t} = [F, H] \equiv X_H(F). \tag{6}$$

Thus, the flow in phase space is generated by a Hamiltonian vector field X_H which involves a single function H of the canonical variables. The flow generated by X_H is a one-dimensional group of canonical transformations, the latter being by definition automorphisms of the PB algebra.

In the last quarter of the last century, a new formulation of classical mechanics was sought which has had many promoters – starting with Nambu [1] and that alone should be ample recommendation. Motivated by Liouville theorem and the form of Euler equations for the motion of a rigid body, Nambu replaced the usual pair of canonical variables by a triplet of canonical variables (x^1, x^2, x^3), the components of position vector. Introducing two functions $H(x^1, x^2, x^3)$ and $G(x^1, x^2, x^3)$ which serve as a pair of 'Hamiltonians' determining the flow in phase space, Nambu proposed that the flow is generated by a Hamiltonian vector field involving the H, G pair according to the following rule:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_{ijk=1}^{3} \epsilon_{ijk} \frac{\partial F}{\partial x^i}\frac{\partial H}{\partial x^j}\frac{\partial G}{\partial x^k} \equiv X_{H,G}(F). \tag{7}$$

Now, if one defines a NP bracket involving three functions of the canonical variables as the Jacobian $J(F_1, F_2, F_3)$

$$[F_1, F_2, F_3] = \frac{\partial(F_1, F_2, F_3)}{\partial(x^1, x^2, x^3)} = J(F_1, F_2, F_3), \tag{8}$$

the time evolution of an observable is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = [F, H, G] = X_{H,G}(F). \tag{9}$$

Apart from the work of Bayen and Flato [2] and of Mukunda and Sudarshan [3] which sheds light on the connection between the Hamiltonian and Nambu formalisms, Nambu's work was largely unnoticed until Takhtajan [4] made attempts to formulate the basic principles of Nambu mechanics in the invariant geometric form similar to that of Hamiltonian mechanics, thereby displaying that the Nambu formalism is a profound generalization of the Hamiltonian formalism of classical mechanics. (For a review of Takhtajan's work which is at once intuitive, geometric, application-oriented and mathematically rigorous, see Chatterjee [5].)

In the classical description of a dynamical system, the NP manifold plays the same role in Nambu mechanics that the Poisson manifold plays in Hamiltonian dynamics. Dynamics on the NP manifold with coordinates $x^1, x^2, ..., x^N$ is determined by a Hamiltonian vector field X involving $(N-1)$ functions called 'Hamiltonians' $H_1, ..., H_{N-1}$ which are first integrals of motion (IOM). The dynamics generated by $X_{H_1,H_2,...,H_{N-1}}$ is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = [F, H_1, ..., H_{N-1}] = X_{H_1,H_2...,H_{N-1}}(F). \tag{10}$$

By putting Nambu's ideas on a solid footing, Takhtajan emphasized the role that new mathematical structures play in passing from Hamilton's to Nambu's dynamical picture. Nambu himself had conjectured the following beautiful formula for the NP bracket:

$$[F_1, F_2, ..., F_N] = \frac{\partial(F_1, F_2, ..., F_N)}{\partial(x^1, x^2, ..., x^N)} = J(F_1, F_2, ..., F_N), \tag{11}$$

where on the right-hand side is the Jacobian operation defined on an N-dimensional manifold. This canonical form (which may be called the canonical Nambu bracket for short) for the abstract NP bracket is the driving force behind our work.

Following Takhtajan there was a spate of papers on Nambu mechanics. In furtherance of Nambu ideas, it is remarkable that Chatterjee [5] revisited dynamical symmetries. He showed that the familiar dynamical systems, namely the Kepler problem and the isotropic HO problem, are described elegantly in the framework of Nambu's proposed generalizations. As symmetries are signalled by conserved quantities, the existence of dynamical symmetries in these special systems results in extra IOM beyond those needed for complete integrability [5–7]. As the choice of the IOM is not unique, in these systems the abstract NP bracket instead takes the form [5]

$$[F_1, ..., F_n] = \frac{1}{C}\frac{\partial(F_1, ..., F_n)}{\partial(x^1, ..., x^n)} = \frac{1}{C}J(F_1, F_2, ..., F_N), \tag{12}$$

where C is an integral of motion. It may be appreciated that while the above formula is a suitable candidate for the abstract NP bracket, the canonical Nambu bracket is the one that appears in (11).

In this work we apply the most efficient method of evaluating a numerical (or literal) determinant. This is done, in general, by pivotal condensation, according to the technique given, for instance, in [8] in which the evaluation proceeds by means of a second-order determinant. These techniques become cumbersome when the order of the determinant is larger than four or five. The techniques suggested are elegant and powerful for small order of the determinant, and we believe that, owing to their involvement in exposing the connection between the Hamiltonian and Nambu formalisms in a pedestrian manner, they are of interest generally in the study of mechanics. Our work which complements the formal considerations of Mukunda and Sudarshan [3] shows the equivalence of the two approaches for a general audience. What we have shown is the following:

1. We review the argument of Mukunda and Sudarshan [3] showing the connection between the Nambu and Hamiltonian ideas.
2. The application of our technique to a third-order determinant shows that the Nahm's system of equations [9] fit in well into the formalism of Nambu mechanics.
3. We apply our technique to fourth-order determinant showing that for two degrees of freedom, the canonical equations of motion may be written in terms of the NP bracket. This result helps us to cast the abstract NP bracket in canonical form, i.e. the constant in front of the Jacobian goes to unity. A careful choice of IOM yields the beautiful canonical form for the abstract NP bracket.
4. We revisit the Kepler problem and isotropic HO in the framework of Nambu mechanics. Following O'Connell and Jagannathan [6], a conserved dynamical variable α that characterizes the orientation of the orbit in two-dimensional configuration space (or four-dimensional phase space) for the Kepler problem (and an analogous variable β for the HO) is defined. The orbit orientation variable α and the analogous variable β are separately found to be canonically conjugate to the angular momentum component normal to the plane of motion. It is this observation which establishes the canonical form of the NP bracket. Our conclusions are discussed at the end of this work.

2. Transition from Hamiltonian to Nambu framework

One can ask whether a general Hamiltonian system can be described in such a way that its equation of motion appear in the Nambu form.

Let us begin with the example of Nambu characterized by a primitive Nambu triplet. Let x^1, x^2, x^3 be the independent members of a Nambu triplet. The equation of motion for x^j involves two algebraically independent functions $F(x^1, x^2, x^3)$ and $G(x^1, x^2, x^3)$ and are postulated by Nambu to be

$$\dot{x}^j = \frac{\partial(F, G)}{\partial(x^k, x^\ell)}\,, \tag{13}$$

where $j, k, \ell =$ cyclic permutations of 1, 2, 3.

The Nambu form for an N-tuple of variables $x^1, x^2, ..., x^N$ involves $(N-1)$ independent Hamiltonians $H_1(x), H_2(x), ..., H_{N-1}(x)$ according to the rule

$$\dot{x}^J = \frac{\partial(x^J, H_1, H_2, ..., H_{N-1})}{\partial(x^1, x^2, ..., x^N)}, \quad J = 1, 2, ..., N. \tag{14}$$

Given a mechanical system with n degrees of freedom, let us show how to pass from a conventional Hamiltonian system of equations governed by a single Hamiltonian to the above Nambu form for $N = 2n$. There exist, at least locally, $2n-1$ functionally independent constants of motion $x^{2'}, ..., x^{2n'}$ which can be used as part of a coordinate system in phase space. Furthermore, it is possible locally to determine a function $x^{1'}$ such that $\dot{x}^{1'} = 1$. By means of a coordinate transformation one can pass from $x^1, ..., x^{2n}$ to $x^{1'}, ..., x^{2n'}$ such that the Hamiltonian becomes say $x^{2'}$:

$$H(x^1, ..., x^n) = x^{2'}. \tag{15}$$

$x^{2'}, ..., x^{n'}, x^{n+1'}, ..., x^{2n'}$ are $(2n-1)$ independent constants of motion, and Hamilton's equations are very simple:

$$\dot{x}^{1'} = 1, \qquad \dot{x}^{2'} = \dot{x}^{3'} = \cdots = x^{2n'} = 0. \tag{16}$$

But these same equations are reproduced if in

$$\dot{F} = \frac{\partial(F, x^{2'}, ..., x^{2n'})}{\partial(x^{1'}, x^{2'}, ..., x^{2n'})}, \tag{17}$$

we set F equal to $x^{1'}, ..., x^{2n'}$ in turn. Therefore, Hamilton's equations for the coordinates, and by the derivation property for a general dynamical variable F, are equivalent to the Nambu form. We have also the liberty of going back to the original canonical coordinates $\{x\}$ because the transformation $\{x\} \leftrightarrow \{x'\}$ has unit Jacobian and we see that the Hamilton's general equation

$$\dot{F}(x) = [F(x), H(x)] \tag{18}$$

is equivalent to

$$\dot{F} = \frac{\partial(F, x^{2'}(x), ..., x^{2n'}(x))}{\partial(x^1, ..., x^{2n})}. \tag{19}$$

It is worth noting that this is true only locally in phase space since in general one cannot find $(2n-2)$ global constants of motion to go with a given H.

3. Three-dimensional NP manifold and Nambu's ideas

Following considerations analogous to Nambu, we apply our technique to yield the Nahm's system of equations [9] in the theory of static $SU(2)$ monopoles,

$$\begin{aligned} \dot{S}_1 &= S_2 S_3, \\ \dot{S}_2 &= S_3 S_1, \\ \dot{S}_3 &= S_1 S_2. \end{aligned} \tag{20}$$

Using pivotal condensation technique, a 3×3 determinant can be expanded to give

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a^{-1}\left\{\begin{vmatrix} a & b \\ d & e \end{vmatrix}\begin{vmatrix} a & c \\ g & i \end{vmatrix} - \begin{vmatrix} a & b \\ g & h \end{vmatrix}\begin{vmatrix} a & c \\ d & f \end{vmatrix}\right\}. \tag{21}$$

Let us define a 3×3 Jacobian, where $F_i = \partial F/\partial S_i$ etc.,

$$\frac{\partial(F,G,H)}{\partial(S_1,S_2,S_3)} = \begin{vmatrix} F_1 & G_1 & H_1 \\ F_2 & G_2 & H_2 \\ F_3 & G_3 & H_3 \end{vmatrix} = \begin{vmatrix} F_1 & S_1 & 0 \\ F_2 & -S_2 & S_2 \\ F_3 & 0 & -S_3 \end{vmatrix}$$

$$= (F_1)^{-1}\left[\begin{vmatrix} F_1 & S_1 \\ F_2 & -S_2 \end{vmatrix}\begin{vmatrix} F_1 & 0 \\ F_3 & -S_3 \end{vmatrix} - \begin{vmatrix} F_1 & S_1 \\ F_3 & 0 \end{vmatrix}\begin{vmatrix} F_1 & 0 \\ F_2 & S_2 \end{vmatrix}\right].$$

Given the definitions of G and H, we have shown that the Jacobian

$$J(F,G,H) = S_2S_3F_1 + S_3S_1F_2 + S_1S_2F_3.$$

So, for the three-dimensional Nahm system, if one defines

$$[F_1,F_2,F_3] = \frac{\partial(F_1,F_2,F_3)}{\partial(S_1,\ S_2,\ S_3)} = J(F_1,F_2,F_3), \tag{22}$$

we see that this dynamical system is described by $X_{G,H}$, that is, $\dot{F} = [F,G,H] = X_{G,H}(F)$. In other words, with the right choice for IOM, we have arrived at the canonical form for NP bracket for the Nahm dynamical system. Originally, Nambu's considerations had yielded a Euler's equation for the rigid body. The foregoing development paves the way for dynamical symmetries and Nambu mechanics considered in the following section.

4. The four-dimensional phase space and Nambu mechanics

The technique used by us enables to reduce a Jacobian operation defined on a four-dimensional phase space (q^1, p^1, q^2, p^2) as a sum of the products of PBs. Application of the pivotal condensation technique in the following form

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = \begin{vmatrix} a & b \\ e & f \end{vmatrix}\begin{vmatrix} k & l \\ o & p \end{vmatrix} + \begin{vmatrix} i & j \\ m & n \end{vmatrix}\begin{vmatrix} c & d \\ g & h \end{vmatrix}$$
$$- \begin{vmatrix} a & b \\ i & j \end{vmatrix}\begin{vmatrix} g & h \\ o & p \end{vmatrix} - \begin{vmatrix} c & d \\ k & l \end{vmatrix}\begin{vmatrix} e & f \\ m & n \end{vmatrix}$$
$$+ \begin{vmatrix} a & b \\ m & n \end{vmatrix}\begin{vmatrix} g & h \\ k & l \end{vmatrix} + \begin{vmatrix} c & d \\ o & p \end{vmatrix}\begin{vmatrix} e & f \\ i & j \end{vmatrix} \tag{23}$$

yields the desired formula expressing a 4×4 Jacobian as sum of products of PBs. Let

$$\frac{\partial(A,B,C,D)}{\partial(q^1,p^1,q^2,p^2)} = \begin{vmatrix} A_{q_1} & A_{p_1} & A_{q_2} & A_{p_2} \\ B_{q_1} & B_{p_1} & B_{q_2} & B_{p_2} \\ C_{q_1} & C_{p_1} & C_{q_2} & C_{p_2} \\ D_{q_1} & D_{p_1} & D_{q_2} & D_{p_2} \end{vmatrix}, \tag{24}$$

where

$$A_{q_1} = \frac{\partial A}{\partial q^1}, \qquad B_{p_2} = \frac{\partial B}{\partial p^2}, \dots .$$

Thus, the 4×4 Jacobian

$$\frac{\partial(A, B, C, D)}{\partial(q^1, p^1, q^2, p^2)} = [A, B]_1[C, D]_2 + [A, B]_2[C, D]_1 - [A, C]_1[B, D]_2 - [A, C]_2[B, D]_1 + [A, D]_1[B, C]_2 + [A, D]_2[B, C]_1,$$

where $[X, Y]_i = \partial(X, Y)/\partial(q^i, p^i)$ is the Jacobian in the (q^i, p^i) subspace, where $i = 1, 2$. It is straightforward to verify that the 4×4 Jacobian

$$J(A, B, C, D) = \frac{\partial(A, B, C, D)}{\partial(q^1, p^1, q^2, p^2)} = [A, B][C, D] - [A, C][B, D] + [A, D][B, C], \tag{25}$$

where the PB is defined in the usual manner as

$$[X, Y] = [X, Y]_1 + [X, Y]_2. \tag{26}$$

In the following considerations, this formula will come on the centre stage. Let us consider a four-dimensional phase space. Let H_1 be the classical Hamiltonian and H_2, H_3 be the two IOM of a dynamical system. For some function F of the canonical variables (x^1, x^2, x^3, x^4), the 4×4 Jacobian,

$$J(F, H_1, H_2, H_3) = [F, H_1][H_2, H_3] - [F, H_2][H_1, H_3] + [F, H_3][H_1, H_2] = [F, H_1][H_2, H_3].$$

The last equality follows because the PB of a constant of motion with Hamiltonian vanishes, i.e. $[H_1, H_3] = [H_1, H_2] = 0$. By Poisson's theorem let $[H_2, H_3] = C$ where C is some constant, then by virtue of the Hamilton's equation of motion

$$\dot{F} = [F, H_1] = \frac{1}{C} J(F, H_1, H_2, H_3) = \frac{1}{C}\frac{\partial(F, H_1, H_2, H_3)}{\partial(x^1, x^2, x^3, x^4)}$$

and we find that if the NP bracket is defined as

$$[F_1, F_2, F_3, F_4] = \frac{1}{C}\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(x^1, x^2, x^3, x^4)}, \tag{27}$$

the flow in phase space is generated by the Hamiltonian vector field X_{H_1,H_2,H_3}.

The examples put forward by Chatterjee [5] are easily verified through considerations analogous to those given above. As an illustration, we consider the isotropic HO which has two degrees of freedom, and is described by variables q^1, p^1 and q^2, p^2 as part of the coordinate system for its phase space. Consider the space of homogeneous quadratic IOM spanned by

$$I_1 = (p^1)^2 + (q^1)^2 = C_1, \qquad I_2 = (p^2)^2 + (q^2)^2 = C_2,$$
$$I_3 = q^1p^2 - q^2p^1 = C_3, \qquad I_4 = p^1p^2 + q^1q^2 = C_4, \tag{28}$$

where C_is are the constant values assumed by the IOM I_is. If one defines the NP bracket [5]

$$[F_1, F_2, F_3, F_4] = \left(\frac{1}{C_4}\right)\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(q^1, p^1, q^2, p^2)}, \tag{29}$$

then using (28), it is easily verified that X_{I_1,I_2,I_3} yields the flow in phase space generated by the classical Hamiltonian $I_1 + I_2$, that is

$$\frac{\mathrm{d}p^1}{\mathrm{d}t} = [p^1, I_1, I_2, I_3] = -2q^1, \qquad \frac{\mathrm{d}q^1}{\mathrm{d}t} = [q^1, I_1, I_2, I_3] = 2p^1,$$
$$\frac{\mathrm{d}p^2}{\mathrm{d}t} = [p^2, I_1, I_2, I_3] = -2q^2, \qquad \frac{\mathrm{d}q^2}{\mathrm{d}t} = [q^2, I_1, I_2, I_3] = 2p^2. \tag{30}$$

5. The Kepler problem

The historic Kepler problem has one dynamical axis of symmetry. Apart from the two constants of motion, the usual Hamiltonian and the angular momentum which fixes the plane of motion, there is an extra constant of motion which we now proceed to discuss. The Kepler problem Hamiltonian has the form (in suitable units)

$$H = \frac{(p^1)^2}{2} + \frac{(p^2)^2}{2} - \frac{1}{r}, \tag{31}$$

where $r = \sqrt{(q^1)^2 + (q^2)^2}$ and p^1 and p^2 are the momentum coordinates. Because H possesses rotational symmetry, the angular momentum vector is conserved which restricts the motion to the q^1–q^2 plane. The angular momentum $L = q^1 p^2 - q^2 p^1$. Also, the constancy of the angular momentum can be expressed as the vanishing of its PB with the Hamiltonian $[L, H] = 0$.

A further time-independent, single-valued conserved quantity $\vec{A}$ will cause the orbit to close. The additional conserved quantity is a vector called the Laplace–Runge–Lenz vector,

$$\vec{A} = \vec{p} \times \vec{L} - \hat{r}. \tag{32}$$

$\vec{A}$ is in the plane of motion and its magnitude is the eccentricity of the conic section that is the orbit, and points from the centre of force to the pericentre of the orbit in the attractive case under consideration.

In the present case the centre of the force does not coincide with the geometric centre of the orbit which has thus only one dynamical axis of symmetry. The Laplace–Runge–Lenz vector has two components:

$$A_1 = Lp^2 - \frac{q^1}{r},$$
$$A_2 = -Lp^1 - \frac{q^2}{r}. \tag{33}$$

Because the components are each constants of motion, so is the angle between the vector and the q^1-axis, defined by

$$\alpha(q^1, p^1, q^2, p^2) = \tan^{-1} \frac{A_2}{A_1}. \tag{34}$$

α specifies the orientation of the orbit and may be called the orientation angle. The canonical variables have been explicitly indicated as the arguments of α to emphasize that α is a dynamical variable and hence is a function of the state of the system just as L is. A straightforward calculation shows that α is canonically conjugate to L, i.e.,

$$[\alpha, L] = 1. \tag{35}$$

Also, as α is a constant of motion, we have

$$[\alpha,\ H] = 0. \tag{36}$$

This fact is special about α. A number of angle variables, such as $\theta = \tan^{-1}\left(q^2/q^1\right)$, have the right PB with L to constitute potentially a canonically conjugate pair, but in general, they would not be conserved. Thus, not only can we make canonical transformation to a new set of coordinates in phase space $x^{2'} = H,\ x^{3'} = \alpha,\ x^{4'} = L$ with

$$[x^{1'},\ x^{2'}] = [x^{3'}, x^{4'}] = 1, \tag{37}$$

but if we do so, all members of this set except $x^{1'}$ are conserved ($\dot{x}^{1'} = 1$ in conformity with (16)).

Coming now to the definition of the NP bracket in this case, let F be some function of phase space coordinates, it is now straightforward to calculate the Jacobian

$$\begin{aligned}\frac{\partial(F, H, \alpha, L)}{\partial(q^1, p^1, q^2, p^2)} &= [F, H][\alpha, L] - [F, \alpha][H, L] + [F, L][H, \alpha] \\ &= [F, H][\alpha, L] = [F, H].\end{aligned}$$

If one defines the NP bracket $[F_1, F_2, F_3, F_4] = J(F_1, F_2, F_3, F_4)$, the Kepler problem is described by the Hamiltonian vector field $X_{H,\alpha,L}$, i.e.,

$$\dot{F} = [F, H] = J(F, H, \alpha, L) = [F, H, \alpha, L] = X_{H,\alpha,L}(F). \tag{38}$$

Thus, a careful choice of the constants of motion can yield the canonical Nambu bracket. It may be recalled that Chatterjee [5] has shown that a Hamiltonian system possessing dynamical symmetry can be realized in the Nambu formalism. We have shown here that the NP bracket can be given by the beautiful formula conjectured by Nambu himself.

6. The two-dimensional isotropic harmonic oscillator

In this case the classical Hamiltonian H in suitable units is one half the sum of the squares of the canonical variables. As the motion is restricted to the q^1–q^2 plane, the angular momentum $L = q^1 p^2 - q^2 p^1$ which points in a direction perpendicular to this plane has a fixed value. As L is a constant of the motion, its PB with H is zero, i.e.,

$$[L, H] = 0. \tag{39}$$

In this case also we have closed elliptical orbits. There are two dynamical axes of symmetry meaning that the centre of force coincides with the geometric centre of the orbit. So it becomes necessary to look for a tensorial constant of motion. The object that best captures the degeneracy arising from the dynamical symmetry is a symmetric second rank tensor

$$A_{ij} = \frac{1}{2}\begin{pmatrix} (p^1)^2 + (q^1)^2 & p^1p^2 + q^1q^2 \\ p^1p^2 + q^1q^2 & (p^2)^2 + (q^2)^2 \end{pmatrix}, \qquad i, j = 1, 2. \tag{40}$$

A_{ij} is conserved in this system. Trace of the tensor A is the energy of the motion, and the determinant of the tensor is $L^2/4$. Thus, given the conservation of energy and angular momentum, there is only one more independent conserved quantity that the tensor A represents which may be considered as the orientation of the elliptical orbit in the q^1–q^2 plane.

It may be noted that since each of the components of A is conserved, it may be evaluated at any point in the orbit. Consider an elliptical orbit as shown in figure 1 whose major axis makes an angle β with the positive q^1-axis. From figure 1, the coordinate and momentum component at the point P, the farthest point from the geometric centre on the right of it, are

$$\begin{aligned} q^1 &= a\cos\beta, \qquad q^2 = a\sin\beta, \\ p^1 &= -b\sin\beta, \qquad p^2 = b\cos\beta, \end{aligned} \tag{41}$$

where a and b are, respectively, the semi-major and semi-minor axes of the ellipse. Hence,

$$A_{11} = \frac{1}{2}(a^2\cos^2\beta + b^2\sin^2\beta),$$

$$A_{12} = A_{21} = \frac{1}{2}(a^2 - b^2)\cos\beta\sin\beta,$$

and

$$A_{22} = \frac{1}{2}(a^2\sin^2\beta + b^2\cos^2\beta). \tag{42}$$

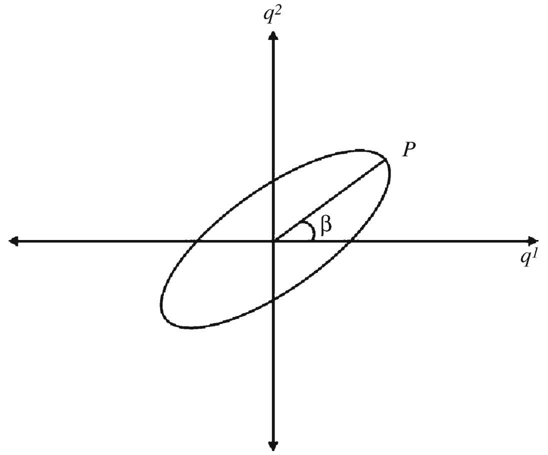

Figure 1. The harmonic oscillator: Major axis of the elliptical orbit makes an angle β with the positive q^1-axis.

Using simple algebra to eliminate a and b yields

$$\beta(q^1, p^1, q^2, p^2) = \frac{1}{2}\tan^{-1}\left[\frac{2A_{12}}{A_{11} - A_{22}}\right]. \tag{43}$$

Evidently β is a constant of motion being expressed as a function only of the other constants of motion. It has a vanishing PB with the Hamiltonian

$$[\beta, H] = 0. \tag{44}$$

Also, a straightforward calculation yields

$$[\beta, L] = 1. \tag{45}$$

Let F be some function of the phase space coordinates. The Jacobian for this system leads to

$$\begin{aligned}\frac{\partial(F, H, \beta, L)}{\partial(q^1, p^1, q^2, p^2)} &= [F, H][\beta, L] - [F, L][\beta, H] + [F, \beta][L, H]\\ &= [F, H][\beta, L] = [F, H] = \dot{F}.\end{aligned} \tag{46}$$

Thus, if one defines the NP bracket $[F_1, F_2, F_3, F_4]$ to be given by the formula

$$[F_1, F_2, F_3, F_4] = \frac{\partial(F_1, F_2, F_3, F_4)}{\partial(q^1, p^1, q^2, p^2)}, \tag{47}$$

we find that the dynamical system under consideration is equivalent to a Nambu mechanical system described by the Hamiltonian vector field $X_{H,\beta,L}$ involving three Hamiltonians H, β, L.

It is worthwhile comparing our formula (47) to eq. (29) which is due to Chatterjee [5]. The bracket in (29) is the Jacobian modified by the addition of a constant while the bracket in (47) is equal to the Jacobian itself.

7. Discussion and conclusions

Thirty seven years ago Nambu proposed a new mechanics aimed at generalizing the elements of classical Hamiltonian mechanics. Nambu's profound generalization of mechanics was based on a three-dimensional 'phase space' spanned by a canonical triplet of dynamical variables and two Hamiltonians. Later, Mukunda *et al* and Bayen *et al* further discussed the Hamiltonian and Nambu pictures. Takhtajan's higher-order extension of Nambu's mechanics based on $N - 1$ Hamiltonians paved the way for further developments in this direction. The first serious study of the important problem of IOM in Nambu's mechanics was initiated by Chatterjee who gave a plethora of illustrative Nambu mechanical examples described by three or more Hamiltonians. It is noteworthy that for a dynamical system which is described by a Hamiltonian vector field $X_{H_1,\dots,H_{N-1}}$ involving $N-1$ Hamiltonians, $\dot{F} = X_{H_1,\dots,H_{N-1}}(F) = [F, H_1, ..., H_{N-1}]$. Nambu himself had suggested the beautiful form for the Nth order bracket $[F_1, F_2, ..., F_N] = J(F_1, F_2, ..., F_N)$ in terms of the Jacobian. In contrast to this, Chatterjee's considerations yield a 'canonical' form for the higher-order bracket $[F_1, F_2, ..., F_N] = C^{-1}J(F_1, F_2, ..., F_N)$, in which modification of the canonical result arises due to the occurrence of a constant. Given the techniques used and the theorems suggested by us in this work, we are now in a position to state a simple formula for C, so that it no longer appears to be 'pulled out of the hat.' For a mechanical system with n degrees of freedom, let $H_1, H_2, ..., H_{2n-1}$ be $2n-1$ functionally independent IOM out of which we choose H_1 as our Hamiltonian, i.e. the PB $[H_1, H_I] = 0$, $I = 2, 3, ..., 2n-1$.

Let us parametrize a point in phase space in terms of $x^{1'}, x^{2'}, ..., x^{2n'}$ in place of $x^1, x^2, ..., x^{2n}$. Here, $[x^1, H_1] = 1, x^{2'} = H_1, x^{3'} = H_2, ..., x^{2n'} = H_{2n-1}$. The flow in phase space $\dot{x}^{1'} = 1, \dot{x}^{2'} = \dot{x}^{3'} = \cdots = \dot{x}^{2n'} = 0$ is described by $X_{H_1,H_2,\dots,H_{2n-1}}$, defined such that

$$\dot{F} = X_{H_1,H_2,\dots,H_{2n-1}}(F) = \frac{\partial(F, H_1, H_2, ..., H_{2n-1})}{\partial(x^{1'}, x^{2'}, ..., x^{2n'})}$$

$$= \frac{J, (F, H_1, H_2, ..., H_{2n-1})}{J(x^{1'}, x^{2'}, ..., x^{2n'})}$$

$$= \frac{J, (F, H_1, H_2, ..., H_{2n-1})}{[x^{1'}, x^{2'}][x^{3'}, x^{4'}]\cdots s[x^{(2n-1)'}, x^{2n'}]}.$$

This means that if we define the NP bracket by the formula $[F_1, F_2, ..., F_{2N}] = C^{-1}J(F_1, F_2, ..., F_{2n})$ with $C = [H_2, H_3]\cdots[H_{2n-2}, H_{2n-1}]$, the mechanical system defined on $2n$-dimensional space is described by $X_{H_1,H_2,\dots,H_{2n-1}}$ in the Nambu picture involving $2n-1$ Hamiltonians. This is our formula for the constant in the 'canonical' NP higher-order bracket which appears in Chatterjee's considerations on dynamical symmetries and Nambu mechanics. The canonical form for the NP bracket can be arrived at through a suitable choice of the IOM. What is germane to arriving at the canonical form is the value of PB involving each pair $[H_2, H_3], ..., [H_{2n-2}, H_{2n-1}]$ which must be chosen to be unity. We have been led to make such a choice of the IOM while describing the Kepler problem and the isotropic HO problem in the Nambu picture. The Kepler problem is described in the canonical Nambu form by $X_{H,\alpha,L}$ and the isotropic HO problem is described in the canonical Nambu form by $X_{H,\beta,L}$. This is so because as a result of dynamical symmetries, a dynamical variable α (and an analogous variable β) characterizing the orientation of the orbit in two-dimensional configuration space for the Kepler problem (isotropic HO) is found to be canonically conjugate to L, the angular momentum component normal to the plane of motion, i.e., $[\alpha, L] = 1$ (analogously $[\beta, L] = 1$). For other choices of IOM, one obtains only a 'canonical' Nambu form for the higher-order bracket.

Although, originally intended for use in numerical analysis, the pivotal condensation techniques have also been shown here to prove useful in treating the problem of higher-order algebraic structures in mechanics. We believe they are of general interest particularly because of their suitability in extending the domain of application of Nambu mechanics.

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