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Synchronization of indirectly coupled Lorenz oscillators: An experimental study

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Abstract. The dynamics of indirectly coupled Lorenz circuits is investigated experimentally. The in-phase and anti-phase synchronization of indirectly coupled chaotic oscillators reported in *Phys. Rev.* **E 81**, 046216 (2010) is verified by physical experiments with electronic circuits. Two chaotic systems coupled through a common dynamic environment shows the verity of synchronization behaviours such as anti-phase synchronization, in-phase synchronization, identical synchronization, anti-synchronization, etc.

Keywords. Synchronization; indirect coupling; electronic circuit.

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1. Introduction

Synchronization of chaotic oscillators in dynamical systems is one of the phenomena of nonlinear dynamics that is most extensively studied [1]. Huygens observed that a couple of pendulum clocks hanging from a common support synchronizes, i.e. their oscillations coincided perfectly and pendula moved always in opposite direction [2]. Other interesting examples of synchronization phenomena may be seen in fireflies flashing in perfect union [3], chemical reactions [4], semiconductor lasers exhibiting chaotic emission on subnanosecond time-scale [5], human cardiorespiratory systems [6], ecological systems [7], the magnetoencephalographic activity of Parkinsonian patients [8], electrosensitive cells of paddlefish [9] etc. In the context of the coupling strength and the nature of the coupling, different types of synchronizations studied in literature are: complete or identical [10], in-phase [11], anti-phase [12], lag [13], generalized synchronization [14] and anti-synchronization [15].

In many real-world situations, the systems are not directly coupled and synchronization behaviour in such systems can occur due to interaction through a common medium. For instance, synchronization behaviour with self-pulsating periodic and chaotic oscillations occur by controlling field cavity detuning, in an ensemble of cold atoms interacting with coherent electromagnetic field [16]. Spontaneous synchronization takes place through global level of neurotransmitter concentration in coupled circadian oscillators in Suprachiasmatic nucleus (SCN) [17]. The coupling function is dynamically modulated by the system dynamics in all these cases. The effect of linear diffusive and nondiffusive couplings on synchronization through intercellular signalling in a population of genetic oscillators is also studied [18]. Partial synchronization occurs in a population of chemical oscillators coupled through the concentration of chemical in the surrounding solutions [19]. Two nonlinear chaotic systems coupled indirectly through a common dynamic environment synchronize to in-phase or anti-phase state [20]. The early stages of Alzheimer's disease can also be explained with a model of two excitatory synaptically coupled neurons in the presence of amyloid beta ($A\beta$) protein [21].

On the other hand, in the field of communications and electronics, a number of nonlinear circuits, demonstrating chaotic behaviour have been presented in the last decade [22]. Using chaotic synchronization phenomena, Pecora and Carroll [23] realized that the chaos could be controlled and used in secure communication systems. Different synchronization regimes have been observed between two unidirectional or bidirectional coupled chaotic Rössler circuit [24], Chua's circuits [25], van der Pol oscillator, Duffing oscillator [26] and Sprott circuits [27]. Synchronization and phase-flip in delay coupled Chua's circuit [28] and in-phase or anti-phase synchronization by open-loop–closed-loop (OPCL) based coupling in coupled electronic circuits is also observed [29].

In this paper, we present an experimental circuit realization of in-phase and anti-phase synchronizations of two Lorenz-like electronic circuits with chaotic attractor coupled through a dynamic linear circuit. In the next section, the model system and circuit equations of the indirectly coupled Lorenz systems are given. Sections 3 and 4 describe the experimental set-up and experimental results. The phase relationship between the outputs of coupled chaotic circuits and synchronization error of the circuit dynamics is studied experimentally in a wide range of control parameters. The conclusion is given in §5.

2. The model system

The two Lorenz systems coupled through dynamic environment are given by [20]

$$\begin{aligned}
\dot{x_{i1}} &= \sigma(x_{i2} - x_{i1}) + \varepsilon_1 \mu_i y,\\
\dot{x_{i2}} &= (r - x_{i3})x_{i1} - x_{i2},\\
\dot{x_{i3}} &= x_{i1}x_{i2} - b x_{i3},\\
\dot{y} &= -\kappa y - \frac{\varepsilon_2}{2}\sum_{i=1,2}\mu_i x_{i1},
\end{aligned} \tag{1}$$

where x_{ij} ($i = 1, 2$ and $j = 1, 2, 3$) are variables of two Lorenz oscillators and y is the dynamic environment variable through which the two oscillators are coupled. The Lorenz system is studied for standard values of parameters $\sigma = 10$, $r = 28$ and $b = 8/3$ in the chaotic region. When μ_1 and μ_2 are of the same sign, $(\mu_1, \mu_2) = (1, 1)$, the coupling is repulsive and can drive the systems to anti-phase synchronization. When μ_1 and μ_2 are of different signs, $(\mu_1, \mu_2) = (1, -1)$, the coupling is attractive and can drive the systems to in-phase synchronization [20].

An approximate stability analysis and transition to different states of synchronous behaviour in the parameter plane of the coupling strength have been studied extensively using various indices as correlation, average phase difference and Lyapunov exponents [20].

The circuit equations describing the indirectly coupled Lorenz systems for in-phase synchronization are

$$\begin{aligned}
C_1\frac{\mathrm{d}x_{11}}{\mathrm{d}t} &= \left(\frac{1}{R_1}x_{12} - \frac{1}{R_9}x_{11}\right) + \frac{R_{27}}{R_1 R_{28}}y,\\
C_2\frac{\mathrm{d}x_{12}}{\mathrm{d}t} &= \frac{1}{R_{10}}x_{11} - \frac{1}{R_3}x_{12} - \frac{R_7}{R_4 R_8}x_{11}x_{13},\\
C_3\frac{\mathrm{d}x_{13}}{\mathrm{d}t} &= \frac{1}{R_5}x_{11}x_{12} - \frac{1}{R_6}x_{13},\\
C_4\frac{\mathrm{d}x_{21}}{\mathrm{d}t} &= \left(\frac{1}{R_{12}}x_{22} - \frac{1}{R_{19}}x_{21}\right) - \frac{1}{R_{11}}y,\\
C_5\frac{\mathrm{d}x_{22}}{\mathrm{d}t} &= \frac{1}{R_{20}}x_{21} - \frac{1}{R_{13}}x_{22} - \frac{R_{17}}{R_{14} R_{18}}x_{21}x_{23},\\
C_6\frac{\mathrm{d}x_{23}}{\mathrm{d}t} &= \frac{1}{R_{15}}x_{21}x_{22} - \frac{1}{R_{16}}x_{23},\\
C_7\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{1}{R_{21}}y - \frac{1}{R_{22}}x_{11} + \frac{R_{25}}{R_{23} R_{26}}x_{21}.
\end{aligned} \tag{2}$$

Similarly, for anti-phase synchronization, the circuit equations are

$$\begin{aligned}
C_1\frac{\mathrm{d}x_{11}}{\mathrm{d}t} &= \left(\frac{1}{R_1}x_{12} - \frac{1}{R_9}x_{11}\right) + \frac{R_{27}}{R_1 R_{28}}y,\\
C_2\frac{\mathrm{d}x_{12}}{\mathrm{d}t} &= \frac{1}{R_{10}}x_{11} - \frac{1}{R_3}x_{12} - \frac{R_7}{R_4 R_8}x_{11}x_{13},\\
C_3\frac{\mathrm{d}x_{13}}{\mathrm{d}t} &= \frac{1}{R_5}x_{11}x_{12} - \frac{1}{R_6}x_{13},\\
C_4\frac{\mathrm{d}x_{21}}{\mathrm{d}t} &= \left(\frac{1}{R_{12}}x_{22} - \frac{1}{R_{19}}x_{21}\right) + \frac{R_{25}}{R_{11} R_{24}}y,\\
C_5\frac{\mathrm{d}x_{22}}{\mathrm{d}t} &= \frac{1}{R_{20}}x_{21} - \frac{1}{R_{13}}x_{22} - \frac{R_{17}}{R_{14} R_{18}}x_{21}x_{23},\\
C_6\frac{\mathrm{d}x_{23}}{\mathrm{d}t} &= \frac{1}{R_{15}}x_{21}x_{22} - \frac{1}{R_{16}}x_{23},\\
C_7\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{1}{R_{21}}y - \frac{1}{R_{22}}x_{11} - \frac{1}{R_{23}}x_{21}.
\end{aligned} \tag{3}$$

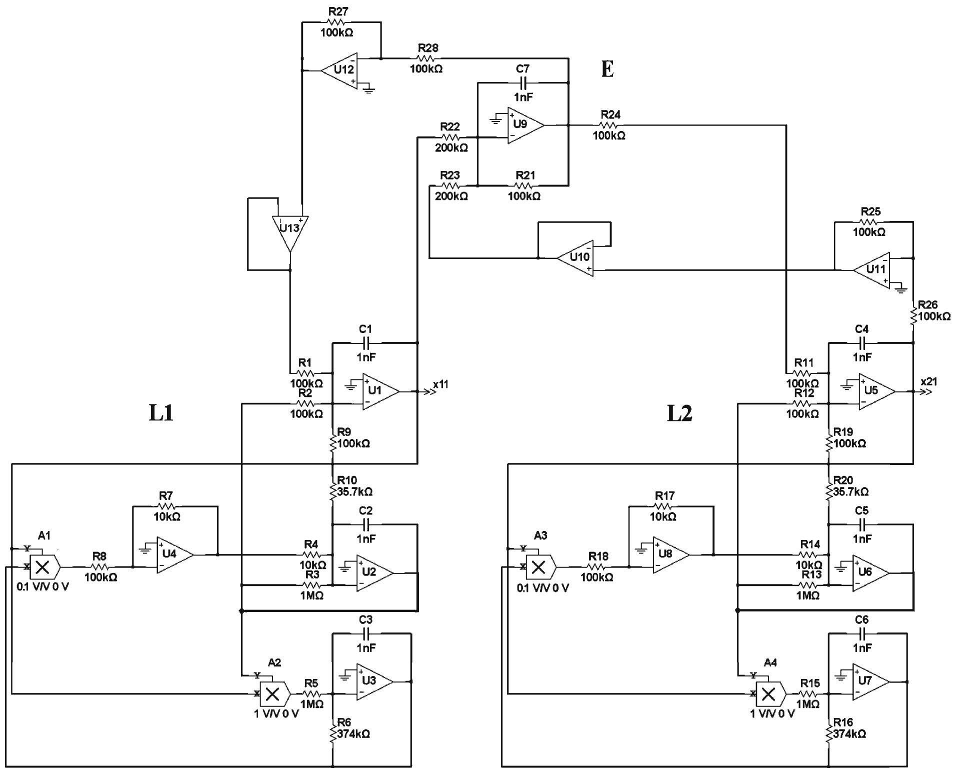

Figure 1. Schematic diagram of two indirectly coupled Lorenz electronic circuits for in-phase synchronization corresponding to eq. (2). All resistances (with 5% tolerance) are metallic and capacitors (with 1% tolerance) are ceramic. The circuit is run by ±12 V.

Here the inverse of the resistances $1/R_1$, $1/R_{11}$ and $1/R_{22}$, $1/R_{23}$ represent the coupling strengths ε_1 and ε_2 respectively (here $1/R_1 = 1/R_{11} = \varepsilon_1$ and $1/R_{22} = 1/R_{23} = 2\varepsilon_2$).

An analog circuit implementation of the circuits in eqs (2) and (3) are shown in figures 1 and 2 respectively. An average synchronization error corresponding to in-phase and anti-phase synchronizations is described by the difference or sum of x variables of the two systems averaged over time respectively, i.e. $\langle |x_{11} \mp x_{21}| \rangle$.

3. Experimental set-up

We construct a pair of electronic oscillators whose dynamics mimic that of the chaotic Lorenz oscillators (L_1 and L_2) [30], connected to linear system (E). Linear circuit was constructed by simple integrator circuit. Both Lorenz circuits and linear circuit consist of passive components like resistors R_{1-28}, capacitors C_{1-7}, multipliers AD633 A_{1-4} and operational amplifier 741 U_{1-13}. We use the simple linear feedback scheme for coupling between the two Lorenz systems and the linear system via OPAMP U_9. Feedback scheme is applied by the two inverting amplifiers and the buffer amplifier for unidirectional flow. Both Lorenz systems are connected to linear system by two resistors R_{22}, R_{23}, and output of the linear system with input of the first OPAMP's (U_1 and U_5) of both Lorenz systems by resistors R_1, R_{11}, characterizes the coupling strength ε_2 and ε_1 respectively. The OPAMP U_{11} and U_{12} are inverting amplifiers for inverting the voltage. We study two different cases of in-phase and anti-phase synchronizations for chaotic Lorenz oscillators coupled indirectly via linear system.

Figures 1 and 2 show the experimental set-ups of in-phase synchronization and anti-phase synchronization respectively. For in-phase synchronization, the output of U_1 and U_5 followed by the inverting amplifier with buffer (U_{11} and U_{13}) is connected to the resistances R_{22} and R_{23}, and the output of the linear system (U_9) is connected to R_1 followed by the inverting amplifier with buffer (U_{12} and U_{13}) and with R_{11} at the input of U_5 (as shown in figure 1).

Similarly, in the case of anti-phase synchronization, the outputs of both U_1 and U_5 are directly connected to the linear circuit via R_{22} and R_{23}, and the output of the linear circuit

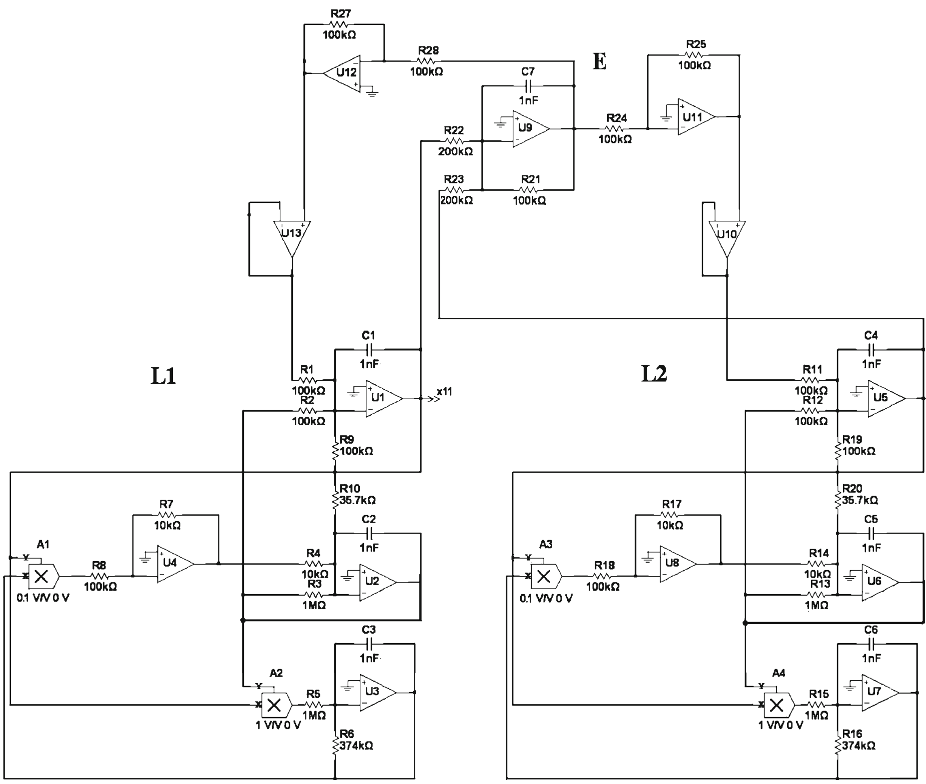

Figure 2. Schematic diagram of two indirectly coupled Lorenz electronic circuits for anti-phase synchronization corresponding to eq. (3). All resistances (with 5% tolerance) are metallic and capacitors (with 1% tolerance) are ceramic. The circuit is run by ±12 V.

(U_9) followed by the inverting amplifier with buffer is connected to the resistances R_1 and R_{11} (as shown in figure 2).

The electronic components in each circuit are carefully chosen and values are given in the diagram (figures 1 and 2). The typical oscillating frequencies of the circuits are in audio range. Both circuits are operated by a low-ripple and low noise power supply (12 V). The output voltages of both the indirectly coupled oscillators are monitored using digital oscilloscope 100 MHz 2 channel (Agilent DSO1012A) with a maximum sampling rate of 2 GSa/s.

4. Experimental results

When the coupling is very weak (at higher resistance R), the dynamics of both the oscillators are uncorrelated so that there is no synchronization. As the coupling strength increases

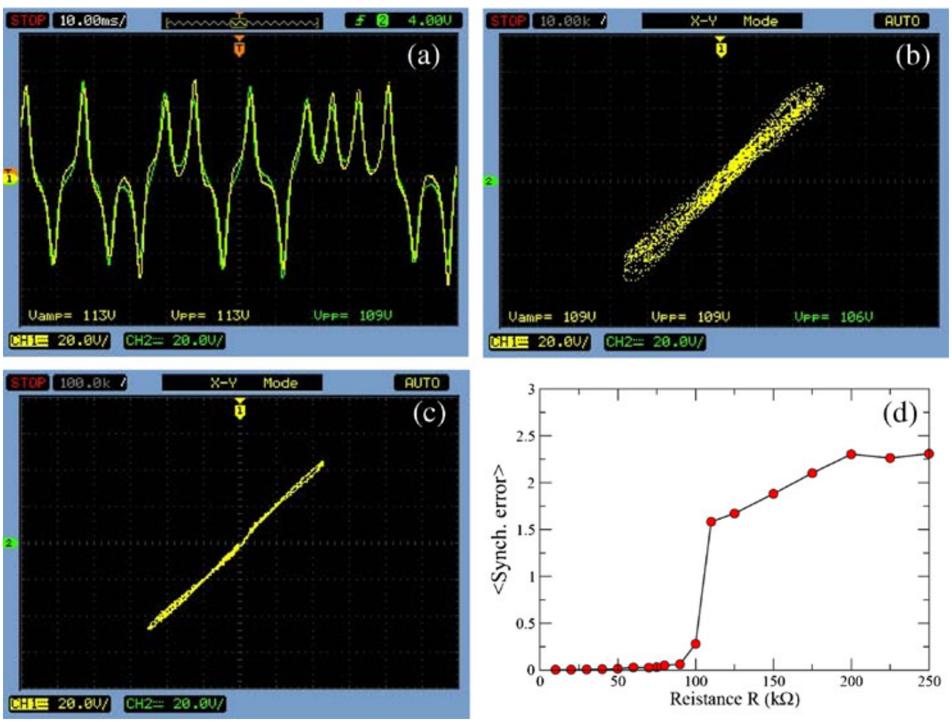

Figure 3. (**a**) Signal of the first variables x_{11} (yellow) and x_{21} (green) of two indirectly coupled chaotic Lorenz systems showing in-phase synchronization at $R_1 \approx R_{11} \approx 81$ kΩ, $R_{22} \approx R_{23} \approx 162$ kΩ. (**b**) Phase relationship between x_{11} and x_{21} at $R_1 \approx R_{11} \approx 81$ kΩ. (**c**) Phase relationship of outputs $R_1 \approx R_{11} \approx 45$ kΩ. (**d**) Average synchronization error of both the outputs x_{11}, x_{21} with resistance R ($\approx R_1 \approx R_{11} \approx R_{22}/2 \approx R_{23}/2$).

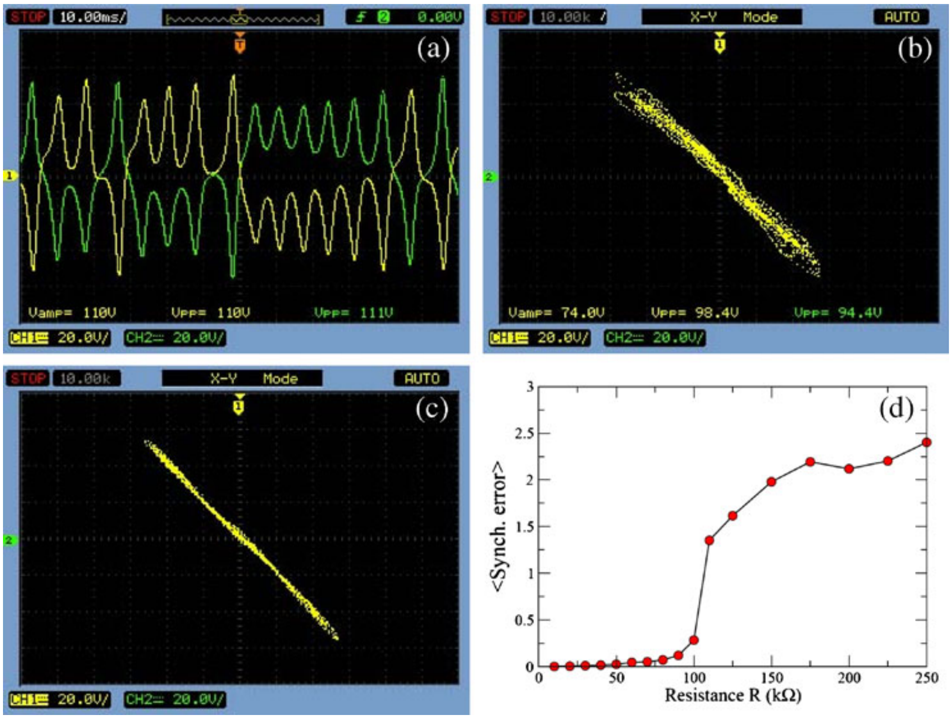

Figure 4. **(a)** Signal of the first variables x_{11} (yellow) and x_{21} (green) of two indirectly coupled chaotic Lorenz systems showing anti-phase synchronization at $R_1 \approx R_{11} \approx$ 92 kΩ, $R_{22} \approx R_{23} \approx$ 183 kΩ. **(b)** Phase relationship between x_{11} and x_{21} at $R_1 \approx R_{11} \approx$ 92 kΩ. **(c)** Phase relationship of outputs $R_1 \approx R_{11} \approx$ 46 kΩ. **(d)** Average synchronization error of both the outputs x_{11}, x_{21} with resistance R ($\approx R_1 \approx R_{11} \approx R_{22}/2 \approx R_{23}/2$).

(with decreasing R), the dynamics of both the oscillators follow each other and synchronize at a critical value of resistance R_c. At $R_1 \approx R_{11} \approx 81$ kΩ, and $R_{22} \approx R_{23} \approx 162$ kΩ the output voltage of OPAMP U_1 and U_5 shows the in-phase dynamics. The time series of output voltage (from U_1 and U_5) and phase relationship of x variables of both Lorenz oscillator circuits are shown in figures 3a and b. While at higher coupling strength, $R_1 \approx R_{11} \approx 45$ kΩ and $R_{22} \approx R_{23} \approx 90$ kΩ, the Lorenz oscillators show complete synchronization (see figure 3c). Synchronization error with resistance is shown in figure 3d.

Similarly, in the case of anti-synchronization (figure 2), at weak coupling strength ($R_1 \approx R_{11} \approx 92$ kΩ and $R_{22} \approx R_{23} \approx 183$ kΩ) the output of both the oscillators shows anti-phase synchronization. Time series and phase relationship of the output voltages of x_{11} and x_{21} are shown in figures 4a and b. At higher coupling strength ($R_1 \approx R_{11} \approx 46$ kΩ and $R_{22} \approx R_{23} \approx 92$ kΩ) both oscillators show complete anti-synchronization (figure 4c). Synchronization error with resistance is shown in figure 4d. We have also studied piece-wise Rössler electronic circuit and similar results are obtained as shown in figure 5.

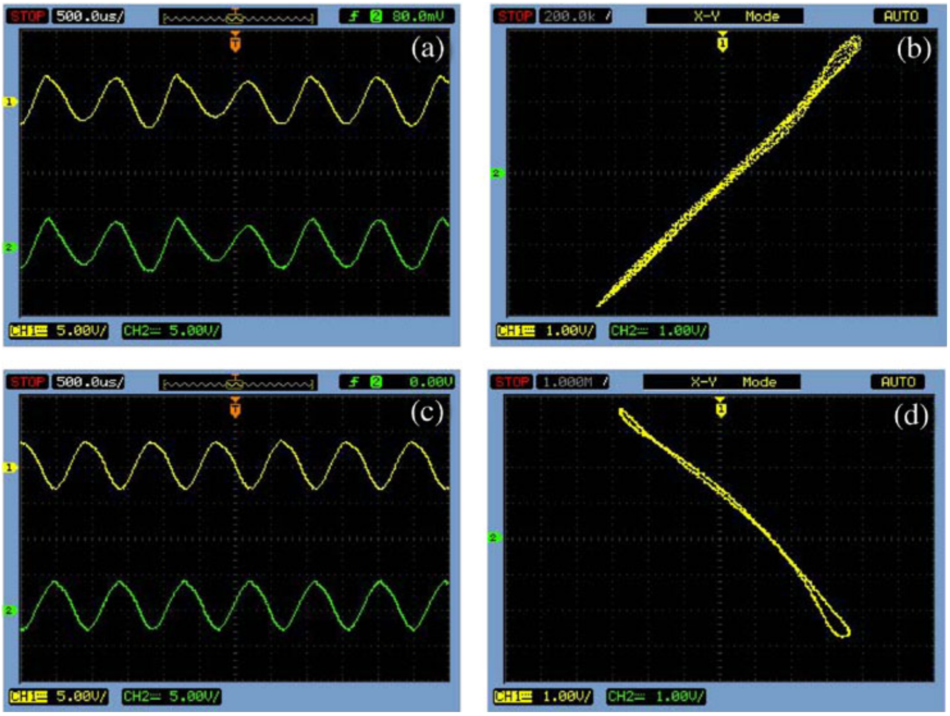

Figure 5. Time series and phase relationship of the two state variables of indirectly coupled piece-wise Rössler systems showing **(a, b)** in-phase and **(c, d)** anti-phase synchronizations.

5. Conclusion

It has been shown analytically and numerically for two chaotic systems coupled indirectly through dynamics environment that the two systems synchronize to in-phase or anti-phase states depending on the nature of coupling [20]. We report the experimental verification of the in-phase and anti-phase synchronizations of two identical Lorenz electronic circuits with small mismatch in parameters by attractive and repulsive indirect couplings through a common linear dynamical circuit. Our experimental results are in good agreement with numerical results. The different states of synchronization, i.e. in-phase and anti-phase, are achieved between the indirectly coupled chaotic circuit systems.

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References

[1] A S Pikovsky, M G Rosenblum and J Kurths, *Synchronization: A universal concept in nonlinear sciences (Cambridge Nonlinear Science Series)* (Cambridge University Press, Cambridge, 2001)
[2] C Huygen, *The pendulum clock* (Iowa State Univesity Press, Ames, 1986)
[3] H M Smith, *Science* **82**, 151 (1935)
[4] K Matsumoto and I Tsuda, *J. Stat. Phys.* **31**, 87 (1983)
[5] I Fisher, Y Liu and P Davis, *Phys. Rev.* **A62**, 011801(R) (2000)
[6] C Schafer, M G Rosenblum, J Kurths and H H Abel, *Nature* **392**, 239 (1998)
[7] B Blasius, A Huppert and L Stone, *Nature* **399**, 354 (1999)
B Blasius and L Stone, *Int. J. Bifurcat. Chaos* **10**, 2361 (2000)
[8] P Tass, M G Rosemblum, M G Weule, J Kurths, A Pikovsky, J Volkmann, A Schnitzler and H J Freund, *Phys. Rev. Lett.* **81**, 3291 (1998)
[9] A Neiman, X Pei, D Russell, W Wojtenek, L Wilkens, F Moss, H A Braun, M T Huber and K Voigt, *Phys. Rev. Lett.* **82**, 660 (1999)
[10] H Fujisaka and T Yamada, *Prog. Theor. Phys.* **69**, 32 (1983)
[11] M G Rosenblum, A S Pikovsky and J Kurths, *Phys. Rev. Lett.* **76**, 1804 (1996)
E R Rosa, E Ott and M H Hess, *Phys. Rev. Lett.* **80**, 1642 (1998)
[12] J Liu, C Ye, S Zhang and W Song, *Phys. Lett.* **A274**, 27 (2000)
[13] M G Rosenblum, A S Pikovsky and J Kurths, *Phys. Rev. Lett.* **78**, 4193 (1997)
[14] N F Rulkov, M M Sushchik, L S Tsimring and H D I Abarbanel, *Phys. Rev.* **E51**, 980 (1995)
L Kocarev and U Parlitz, *Phys. Rev. Lett.* **76**, 1816 (1996)
[15] S Sivaprakasam, I Pierce, P Rees, P S Spencer, K A Shore and A Valle, *Phys. Rev.* **A64**, 013805 (2001)
C M Kim, S Rim, W H Kye, J W Ryu and Y J Park, *Phys. Lett.* **A320**, 39 (2003)
H Zhu and B Cui, *Chaos* **17**, 043122 (2007)
[16] J Javaloyes, M Perrin and A Politi, *Phys. Rev.* **E78**, 011108 (2008)
[17] D Gonze, S Bernard, C Waltermann, A Kramer and H Herzel, *Biophys. J.* **89**, 120 (2005)
[18] A Kuznetsov, M Kaern and N Kopell, *SIAM J. Appl. Math.* **65**, 392 (2004)
R Wang and L Chen, *J. Biol. Rhythms* **20**, 257 (2005)
[19] R Toth, A F Taylor and M R Tinsley, *J. Phys. Chem.* **110**, 10170 (2006)
[20] V Resmi, G Ambika and R K Amritkar, *Phys. Rev.* **E81**, 046216 (2010)
[21] V Resmi, G Ambika, R E Amritkar and G Rangarajan, arXiv:1011.4143v1 (2010)
[22] L O Chua, *IEICE Trans. Fundamentals* **76**, 704 (1993)
K Murali, M Lakshmanan and L O Chua, *Int. J. Bifurcat. Chaos* **4**, 1511 (1994)
K Murali, M Lakshmanan and L O Chua, *IEEE Trans. Circuits Syst. I* **41**, 462 (1994)
G Sarafian and B Z Kaplan, *Int. J. Bifurcat. Chaos* **7**, 1665 (1997)
K Tamilmaran, M Lakshmanan and K Murali, *Int. J. Bifurcat. Chaos* **10**, 1175 (2000)
J C Sprott, *Phys. Lett.* **A266**, 19 (2000)
[23] L M Pecora and T L Carroll, *Phys. Rev. Lett.* **64**, 821 (1990)
T L Carroll and L M Pecora, *IEEE Trans. Circuits Syst.* **40**, 646 (1993)
[24] J F Heagy, T L Carroll and L M Pecora, *Phys. Rev.* **E50**, 1874 (1994)
[25] L O Chua, L Kocarev, K Eckert and M Itoh, *Int. J. Bifurcat. Chaos* **2**, 705 (1992)
[26] C K Volos, I M Kyprianidis and I N Stouboulos, *Int. J. Circuits, Systems, and Signal Processing* **1**, 274 (2007)
[27] S Oancea, *J. Optoelectron. Adv. Mater.* **7**, 2919 (2005)
[28] A Prasad, S K Dana, R Karnatak, J Kurths, B Blasius and R Ramaswamy, *Chaos* **18**, 023111 (2008)
[29] I Grosu, E Padmanaban, P K Roy and S K Dana, *Phys. Rev. Lett.* **100**, 234102 (2008)
[30] K M Cuomo, A V Oppenheim and S H Strogatz, *IEEE Trans. Circuits Syst. II* **40**, 626 (1993)