

Avalanche transmission and critical behaviour in load-bearing hierarchical networks

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Abstract. The strength and stability properties of hierarchical load-bearing networks and their strengthened variants have been discussed in a recent work. Here, we study the avalanche time distributions on these load-bearing networks. The avalanche time distributions of the V-lattice, a unique realization of the networks, show power-law behaviour when tested with certain fractions of its trunk weights. All other avalanche distributions show Gaussian peaked behaviour. Thus the V-lattice is the critical case of the network. We discuss the implications of this result.

Keywords. Hierarchical lattices; avalanche time distributions; critical behaviour.

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1. Introduction

It is well known that many important systems such as the World Wide Web and the Internet, power grids and cellular networks can be modelled as complex networks [1,2]. Branching hierarchical networks constitute an important class of networks, and have been used as models of granular media [3], river networks [4], as models of the lung inflation process [5,6], directed percolation processes [7] and voter models [8]. Branching hierarchical networks have also been used as models of load-bearing networks, and strategies to improve their strength [9] as well as their tolerance to failure [10] were studied in detail. In this paper, we study avalanche propagation on the load-bearing branching hierarchical network, and show that the distribution on a specific realization of the lattice, known as the V-lattice, shows power-law behaviour. Thus the V-lattice constitutes the critical case of the load-bearing networks. We also discuss the relevance of this result in application contexts.

2. Hierarchical models

The distributions of avalanche time are studied for a 2D hierarchical lattice, henceforth to be called the original lattice, two of its variants enhanced by capacity enhancing strategies [9], and also for a specific realization of the original lattice, called the V-lattice.

The original lattice is, in fact, the special $q(0, 1)$ case of the Coppersmith model [3] of granular media and Scheidegger's river model [4], in which, a site in a layer is connected randomly to one of its two neighbours in the layer below, yielding a hierarchical river-like branched structure. We discuss here the version of the lattice set-up in ref. [9]. Each site in the lattice has the capacity to support unit weight if it is not connected to any site in the layer above, and has the capacity $w + 1$ if it is connected to sites whose capacities add up to w , in the layer above. Thus, the capacity $w(i^M)$ of the i th site at any layer M is given by

$$w(i^M) = l(i_l^{M-1}, i^M)w(i_l^{M-1}) + l(i_r^{M-1}, i^M)w(i_r^{M-1}) + 1, \quad (1)$$

where i_l^{M-1} and i_r^{M-1} are the left and right neighbours of the site i^M , in the $M - 1$ th layer. The quantity $l(i_l^{M-1}, i^M)$ takes the value 1 if a connection exists between i_l^{M-1} and i^M and 0 if a connection does not exist. Figure 1 shows a realization of the original network with lattice side $M = 8$. It is clear that the network consists of many clusters, where a cluster consists of a set of sites connected with each other. The largest such cluster of sites in the network is called the maximal cluster of the network. Figure 1 shows a representation of the network where the connections between sites are shown by solid lines, and the numbers in the brackets indicate the capacity of each site. C_1, C_2, C_3 and C_4 are the clusters seen in this realization of network, and C_2 is the maximal cluster. The beaded line denotes the trunk of the maximum cluster, where the trunk is the set of connected sites with highest weight bearing capacity in the maximal cluster.

2.1 The V-lattice network

The original lattice has a very special realization which bears the maximum trunk capacity compared to all possible realizations. This lattice bears a unique V-shaped cluster that includes all the sites in the first layer and $(M - I + 1)$ sites in the I th layer, where M

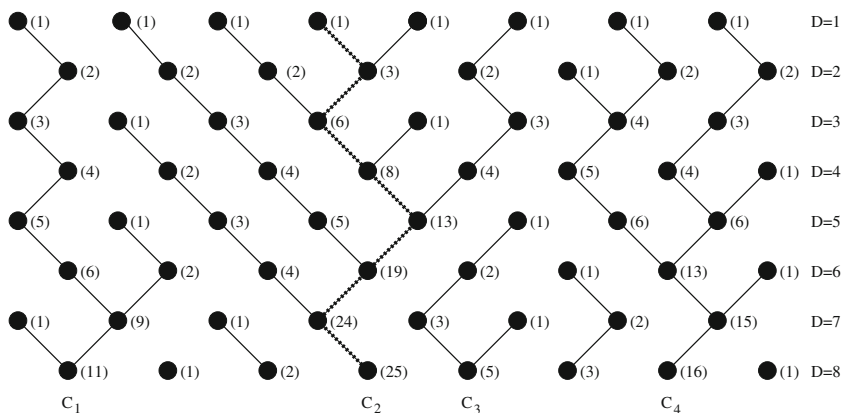


Figure 1. A network of $M = 8$ layers with eight sites per layer with connection probability $p = 1/2$. The beaded line is the trunk of the maximal cluster. The weight bearing capacity of the trunk $W_T = 99$.

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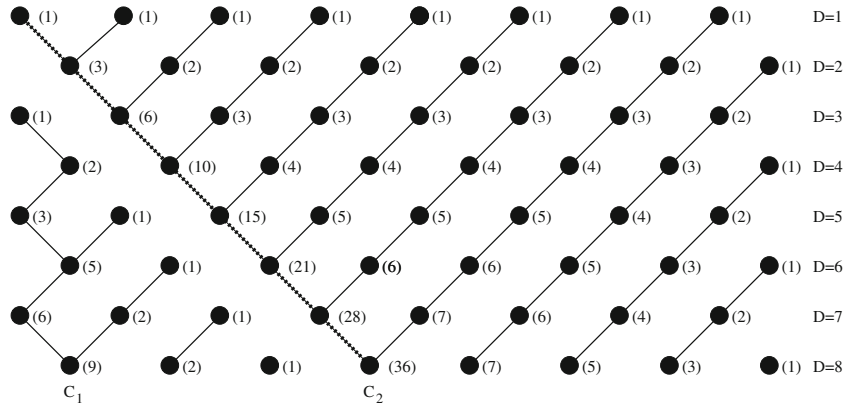


Figure 2. The V-lattice network of $D = 8$ layers with eight sites per layer which is the critical case of our original network. The beaded line is the trunk of the maximal cluster with weight bearing capacity $W_T = 120$.

is the total number of layers. One of the arms of the V constitutes the trunk, and all other connections run parallel to the arm of the V that is opposite to the trunk. Thus, this cluster includes the largest number of sites, and is thus the largest possible cluster the original lattice could have. We call this lattice the V-lattice, and the cluster the V-cluster. The V-lattice realization with lattice side $M = 8$ is shown in figure 2. The largest cluster here is the cluster C_2 . Every site in the cluster at the layer I has capacity I , except for the trunk site which has capacity $w_T(I) = w_T(I - 1) + I$ in the I th layer. We shall see that this network corresponds to the critical case of the original lattice.

3. Probability distributions of avalanche times

It is interesting to study the avalanche phenomena on the network. Our study is of relevance to any situation where threshold phenomena propagate on a network. Examples of this range from power propagation on grids, electrical impulses on neural networks, ventilation in respiratory networks to directed percolation and granular media.

We define avalanches on the network in terms of weight transmission. The weight transmission in the network takes place along the connections between sites. When a site in the first layer of the network receives a weight \mathcal{W} , it retains an amount equal to its capacity W_c and transmits the rest, i.e. $\mathcal{W} - W_c$, to the site it is connected to, in the layer below. Thus, the weight transmission is in the downward direction and the sites involved in this process with their connections constitute the path of transmission. Let P be one such path and P_D be the site on P in the D th layer. Then, the excess weight at a site P_D in the D th layer is given by

$$W^{\text{ex}}(P_D) = \mathcal{W} - \sum_{K=1}^D W_c(P_K).$$

If $W^{\text{ex}}(P_D) \leq 0$, then the transmission ends at the D th layer of the path P and is considered to be successful. On the other hand, if $W^{\text{ex}}(P_D) > 0$, the weight is transferred to P_{D+1} . Finally, if there is still excess weight left at the M th layer, it is then transmitted to the corresponding site in the first layer and the second cycle of the downward transmission begins as described above. This process continues till either there is no excess weight left, i.e. the transmission is successful, or the receiving site is not able to transmit the excess to the site in the layer below, and the transmission fails. Such a failure occurs when the transmitting site is connected to a site that has already received its share of the weight (i.e. saturated its capacity) in the first cycle of transmission, thus making further transmissions impossible. This process of weight transmission is defined as an avalanche. The time taken for an avalanche is defined as the number of layers traversed by the weight in the network.

In this paper we compare the probability distribution of avalanche time t between the original lattice, and its strongest realization, the V-lattice. This probability distribution as the name suggests is, in fact, the distribution of the number of layers traversed during all cycles of successful avalanche transmission by a test weight placed at a random site in the first layer for any lattice. This probability distribution in the case of the original lattice tested with the trunk weights, has been studied by Janaki and Neelima Gupte [9].

The probability distribution of avalanche time t for the original lattice is shown in figure 3 for an ensemble of 1000 successful weight transmissions for a weight equal to the trunk capacity W_T . In the original lattice, no avalanche is seen when $t/M < 1$ as there are no paths with capacity greater than trunk capacity W_T , i.e. there are no successful transmissions for $t/M < 1$. Also the avalanches of weight transmission in the original lattice can cycle as much as thrice through the network. When these distributions for the original networks of different sizes are scaled by their respective total number of layers M , they collapse on one another as shown in figure 3. Similar behaviour is seen in ref. [9].

The distribution of avalanche time t for the original lattice for the test weights which are fractions of trunk capacity W_T , is different from that for weights equal to trunk capacity as shown in figure 4. Figure 4 shows the distribution $P(t)$ tested for weights ranging from $0.1W_T$ to $0.9W_T$. In the distributions corresponding to $0.1W_T$ and $0.2W_T$ as shown in

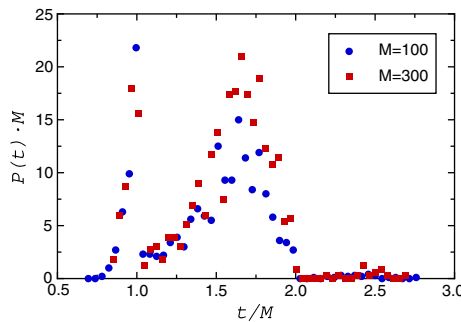


Figure 3. Scaled probability distribution of avalanche time t for the original lattice corresponding to 1000 realizations of networks of side $M = 100$ and $M = 300$. Both, the distributions for $M = 100$ and $M = 300$, scale by the total number of layers M in the network.

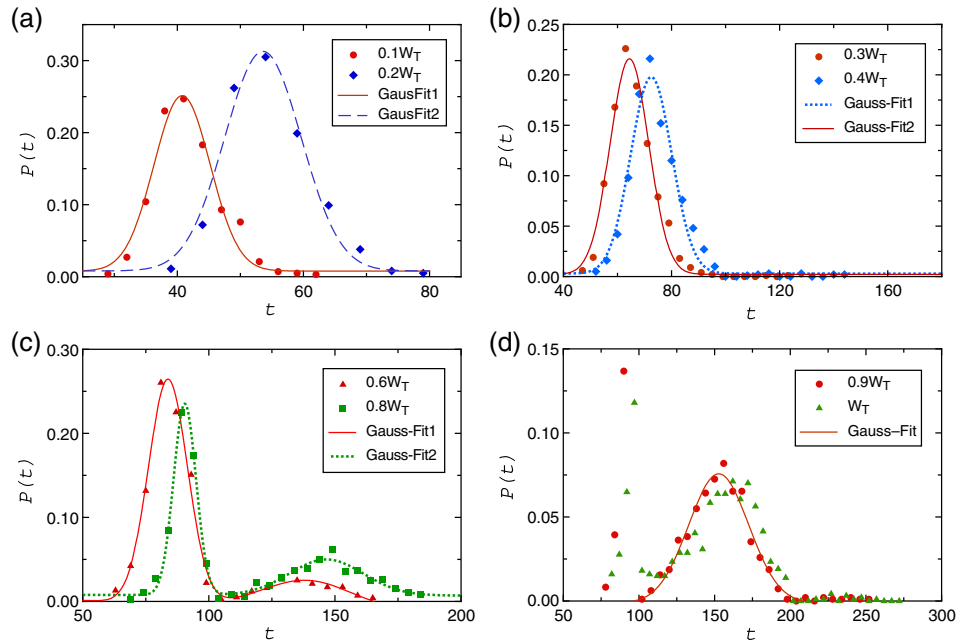


Figure 4. Probability distributions of avalanche time t for the original lattice corresponding to 1000 realizations of networks of side $M = 100$, fit to Gaussian given by eq. (2) when tested for weights equal to (a) $0.1W_T$ ($\sigma = 4.46$, $\chi^2 = 0.0028$) and $0.2W_T$ ($\sigma = 6.08$, $\chi^2 = 0.0024$), (b) $0.3W_T$ ($\sigma = 6.92$, $\chi^2 = 0.0016$) and $0.4W_T$ ($\sigma = 7.34$, $\chi^2 = 0.0024$), (c) $0.6W_T$ ($\sigma_1 = 7.75$, $\sigma_2 = 17.55$, $\chi^2 = 0.0013$) and $0.8W_T$ ($\sigma_1 = 4.46$, $\sigma_2 = 14.68$, $\chi^2 = 0.0062$), and (d) $0.9W_T$ ($\sigma = 19.82$, $\chi^2 = 0.0035$) and W_T .

figure 4a, there is only one cycle in the form of a Gaussian peak, which can be expressed by the equation

$$P(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right), \quad (2)$$

where a is a constant and σ is the standard deviation. The values of σ and χ^2 tested for the accuracy of the fits corresponding to $0.1W_T$ and $0.2W_T$ are shown in the caption of figure 4. In the distributions corresponding to $0.3W_T$ and $0.4W_T$, the emergence of second cycle is seen as shown in figure 4b. This second cycle persists till $0.9W_T$, then, it gradually attains the form of a new Gaussian peak as can be seen in figures 4c and 4d. In the distribution corresponding to $0.9W_T$, the first cycle completely loses its Gaussian form, and a new third cycle emerges out after the Gaussian peak corresponding to the second cycle, which is apparent from figure 4d. Thus, as the test weights start approaching the trunk capacity, we start seeing more and more transmissions which cycle more than once through the lattice.

The distribution of avalanche times for the V-lattice is found to be quite different from that of the original lattice. It does not show any systematic behaviour for weights equal to trunk capacity W_T , i.e. distinct lattice realizations show distinct behaviour. However, the

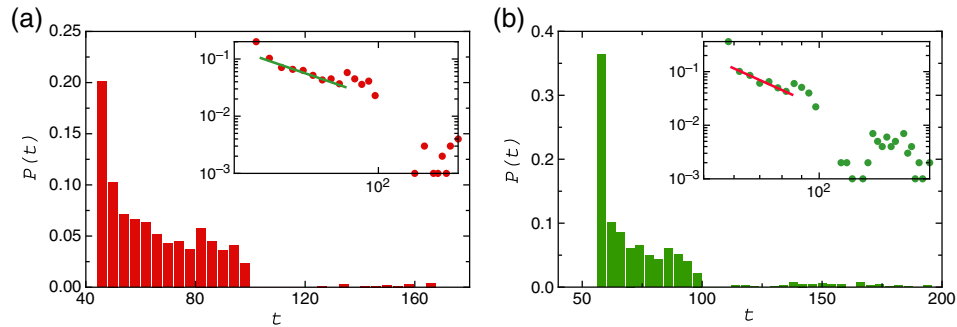


Figure 5. Probability distributions of avalanche time t corresponding to 1000 realizations for the V-lattice network of side $M = 100$ when tested for weights equal to (a) $0.1W_T$ and (b) $0.2W_T$. Small regimes for $0.1W_T$ (as shown in the inset of (a)) and $0.2W_T$ (as shown in the inset of (b)) display power-law behaviour with exponent $\alpha = 2.45$ and $\chi^2 = 21.136$, and $\alpha = 2.96$ and $\chi^2 = 52.197$ respectively.

V-lattice does show interesting behaviour for avalanche time distributions when it is tested for weights less than its trunk capacity. Figure 5 displays the probability distribution of avalanche times for the V-lattice (1000 realizations). It is clear from the figure that the V-lattice displays a power-law behaviour of the form $P(t) \sim t^{-\alpha}$ with exponent $\alpha = 2.45$ and 2.96 when weights equal to $0.1W_T$ (figure 5a) and $0.2W_T$ (figure 5b) are placed in the first layer, respectively. The power-law regime in the distribution of avalanche times gradually starts disappearing when the distribution is tested with values of weights placed approaching trunk capacity, and at the trunk capacity no stable behaviour in the distribution for different realizations is seen. The difference in the behaviour of the distribution of the V-lattice from the original lattice is due to the presence of a unique and asymmetric cluster, called the V-cluster, in the V-lattice network. The V-cluster includes sites of all levels of capacity. Hence transmissions on the lattice can achieve success at any one of the layers. This accounts for the power-law distribution and thus brings criticality to the distribution of avalanche time t . When the distribution for the V-lattice is tested for weights more than $0.2W_T$ the power-law regime of the avalanche time distribution starts dying down slowly

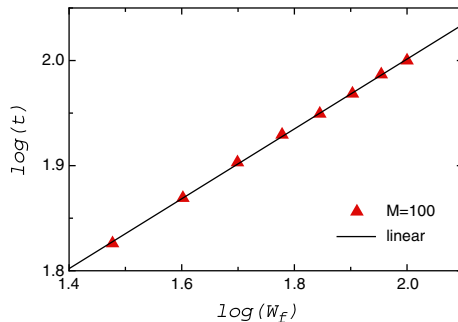


Figure 6. Avalanche time t displays power-law behaviour $t \sim W_f^\beta$ with $\beta = 0.33$ and $\chi^2 = 0.00001$ against weights placed as percentage increase in trunk capacity for the V-lattice for $M = 100$.

much before the instability in the distribution is seen at trunk capacity W_T . This behaviour of the existence and subsequent disappearance of the power-law regime in the distribution is one of the indications that the V-lattice is indeed a critical case of the original lattice. Such behaviour has not been seen in any of realization of the original lattice except the V-lattice. Even the two connectivity strategies studied by Janaki and Neelima Gupte [9] do not show the existence of such behaviour.

The study of avalanche time t of successful weight transmissions along the trunk path in the V-lattice is shown in figure 6. From this figure it is clear that the avalanche time t along the trunk is also governed by a power law $t \sim W_f^\beta$ with $\beta = 0.33$, against W_f which are test weights ranging from $0.1W_T$ to W_T .

4. Conclusions

Threshold phenomena propagating on branching hierarchical lattices are of interest in a diverse variety of application contexts. Models of this kind have been proposed in the context of respiration networks [5,6], voter models [8], granular media [3], power networks [11], river networks [4], as well as directed percolation [7]. Avalanche distributions in these contexts have significance for ventilation strategies for respiratory networks, opinions or preferences cascading on voter networks as well as numerous percolation contexts. It is interesting to note the significance of the V-lattice configuration as the critical configuration in these contexts. It is clear from our results that a V-shaped structure which spans the breadth of the lattice at one extreme, with the trunk, i.e. the collection of the strongest sites along one arm, is best able to support cascades of all possible scales. The utility of this configuration for optimizing the connectivity structure of the networks deserves to be explored further. We hope to explore some of these directions in future work.

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