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Synchronization and emergence in complex systems

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Abstract. We show how novel behaviour can emerge in complex systems at the global level through synchronization of the activities of their constituent units. Two mechanisms are suggested for the emergence, namely non-diffusive coupling and time delays. In this way, simple units can synchronize to display complex dynamics, or conversely, simple dynamics may arise from complex constituents.

Keywords. Synchronization; emergence; delay; chimera.

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1. Introduction

Many natural and man-made systems are intrinsically complex entities consisting of a large number of interacting components. A main concept in the study of complex systems is emergence, i.e., how novel behaviour appears at the system level from the many low-level interactions of the constituent components [1]. Although it may be difficult to fully understand the whole scale of complexity in real systems, one can nevertheless study certain aspects through abstract models. This paper deals with emergence in the context of a particular but important behaviour of networked systems, namely synchronization. Here, (complete) synchronization refers to the state where all units in the network display identical behaviour. An important aspect of synchronization is the amplification of the output signal from the system. Indeed, our experience of complex systems is often through some aggregate system activity at a higher level rather than at the level of constituent units. As prominent examples one can mention the EEG/MEG measurements of the brain or the dynamics of financial markets, among many others. Synchronized activity of the components is a way the system can yield amplified, and hence conspicuous, signals to the outside world.

In most analytical studies of synchronization, there is a particular model of interaction that works to steer the system towards synchronous activity, which we shall refer to as 'diffusive coupling' in this paper. Essentially, such a coupling acts as a forcing term on unit i that depends on the differences of the states, $x_j(t) - x_i(t)$, with other units j that are in interaction with i. The idea is much like that of 'feedback control', where the discrepancy between the desired and the actual states is used as a forcing term to steer the system to

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some target state. Hence, as long as $x_j(t) - x_i(t)$ are nonzero, the units will experience a forcing until the differences are reduced to zero, i.e., the system is synchronized. This procedure has two important features. The first one is that each unit typically interacts with a few others, that is, the interaction is 'local', whereas the outcome, if it is realized, is a 'global' state where the whole system exhibits coordinated behaviour. In other words, the units self-organize themselves to collective behaviour using only local interactions without a central commander. This first aspect has been well studied from many perspectives in numerous papers in the literature and the conditions for synchronization in many cases are well-understood. The second feature is more interesting as far as this paper is concerned, and stems from the fact that in the synchronized state, the differences $x_j(t) - x_i(t)$ are zero: If diffusive coupling is the sole interaction mode within the system, then in the synchronized state all interactions will vanish and the units will effectively be governed by their own internal dynamics (and also possibly by factors exterior to the system, depending on the model). In other words, the synchronized system cannot display any novel behaviour beyond what the units are already capable of. Hence, the diffusive coupling mechanism, while facilitating synchronization and self-emergence of coordination in the system, is also responsible for the absence of any emergent behaviour. How can one then reconcile synchronization and emergence in this context?

This paper will present two mechanisms as answers to this question. The first mechanism is non-diffusive coupling and the second one is the presence of time delays in the system. In the first case the interaction between the units is not designed for cooperation, and in the second case the units are only aware of a past state of the network. Hence, it is not obvious whether such systems can synchronize their actions at all. Interestingly, it turns out that in both cases the system can indeed exhibit synchronization under appropriate conditions. What is more important, the synchronized network can display a rich range of dynamics much different from that of individual units. Thus, it is possible that dramatically different dynamics can emerge at the global level through synchronization of the system.

The paper is organized as follows. In §2 we make the above notions precise and distinguish between diffusive and non-diffusive types of coupling in the context of coupled map networks. Section 3 studies synchronization and the emergence of novel dynamics under a specific type of non-diffusive coupling, while §4 does the same when signal transmission delays are taken into account. Finally, §5 considers distance-dependent delays in networks of phase oscillators and exhibits clustered chimera states, where both synchronized and incoherent behaviour co-exist in the same system at different spatial locations.

2. Synchronization of coupled map networks

We specify a concrete class of systems to make the discussion precise. Consider a dynamical system whose state $x \in \mathbb{R}^n$ evolves according to some rule f,

$$x(t+1) = f(x(t)). \tag{1}$$

(Here we take time to be discrete, $t \in \mathbb{Z}$, but a similar development is possible also in continuous time.) If several such identical systems, indexed by $i = 1, \ldots, N$, come into interaction with each other, the collective dynamics might be modelled by

$$x_i(t+1) = f(x_i(t)) + \varepsilon \sum_{j=1}^{N} a_{ji} g(x_j(t), x_i(t)). \tag{2}$$

The scalar ε denotes the coupling strength and the function g describes the pairwise interaction of the individual units, which is assumed to be identical for each pair, except possibly with different scalar weights a_{ji} modelling the strength of the influence of unit j on i. The indices i, j for which $a_{ij} \neq 0$ define a 'neighborhood relation' in the network emphasizing the local nature of interactions. Note that we allow a_{ij} to have both positive and negative signs (excitatory and inhibitory connections); furthermore, $a_{ij} \neq a_{ji}$ in general (directed links).

We shall be interested in a particular type of solutions of (2), namely 'synchronized solutions' which have the form

$$x_i(t) = s(t) \qquad \forall i, t. \tag{3}$$

The existence of synchronized solutions requires additional assumptions. Indeed, substituting (7) into (2) shows that $s(t)$ should satisfy

$$s(t+1) = f(s(t)) + \varepsilon g(s(t), s(t)) \sum_{j=1}^{N} a_{ji}, \qquad \forall i. \tag{4}$$

Since only the summation term involves the index i, for consistency we need either $g(s(t), s(t)) = 0$ for all t, or that there is some regularity in the connection structure, namely $\sum_{j=1}^{N} a_{ji} = k \ \forall i$ for some constant k. The former case is typical for interactions that resemble a diffusion process. Thus, we say that g satisfies a generalized diffusive condition if

$$g(x, x) = 0 \quad \forall x. \tag{5}$$

Some common cases of interaction function that satisfy (5) are, for instance, $g(x, y) = x - y$, which corresponds to simple linear diffusion, or

$$g(x, y) = f(x) - f(y), \tag{6}$$

corresponding to the coupled map lattice model introduced by Kaneko [2]. A particular feature of the diffusive condition (5) is that a synchronized solution always exists, and by substitution into (2), satisfies

$$s(t+1) = f(s(t)). \tag{7}$$

Note that in this case the synchronized dynamics is the same as the dynamics (1) of the individual unit.

If the interaction function g does not satisfy the diffusive condition (5), then for the existence of synchronized solutions we stipulate that $\sum_{j=1}^{N} a_{ji} = k$ for all i, that is, all units have the same sum k of incoming connection weights. In this case, after subsuming the constant k into the coupling strength ε, the synchronized dynamics takes the form

$$s(t+1) = f(s(t)) + \varepsilon g(s(t), s(t)). \tag{8}$$

It may be expected that the dynamics of (8) can now be very different from that of the isolated units (1).

Whether a synchronized solution can actually be observed in experiments or simulations depends on its robustness against small perturbations. The system is said to (locally, asymptotically) synchronize if $x_i(t) \to s(t)$ as $t \to \infty$ for all i, starting from a suitably large set of initial conditions near $s(t)$. Intuitively, diffusive-type interaction functions can help drive the system toward synchronization, and the well-known synchronization conditions exhibit this property (e.g. [3]). The question whether the system can synchronize under non-diffusive coupling is more involved, but has a positive answer under appropriate conditions; we refer the reader to [4] for the synchronization conditions for a general interaction function g. For the purposes of this paper the important observation is the difference between (7) and (8): Under diffusive coupling the dynamics of the synchronized solution (7) is no different from that of an isolated unit (1); so nothing new emerges through synchronization, but the situation is very different when the coupling is not diffusive. We proceed to show by examples that the system (2) can both synchronize and display novel dynamics under suitable conditions.

3. Non-diffusive coupling

In this section we take the interaction function g to depend only on one argument,

$$g(x, y) = \hat{g}(x)$$

so that the system has the form

$$x_i(t+1) = f(x_i(t)) + \varepsilon \sum_{j=1}^{N} a_{ji}\hat{g}(x_j(t)). \tag{9}$$

This form of interaction, which does not depend on the state of the node itself but only on that of its neighbours, is sometimes called direct coupling, and has been used to model e.g. neuronal networks. As an example, we consider a coupled system of simple scalar units by taking

$$f(x) = \gamma x + \Theta, \tag{10}$$

where $x \in \mathbb{R}$, $0 < \gamma < 1$, and Θ is a constant. The dynamics (1) is then the simplest non-trivial behaviour one can get; namely, there is a single fixed point $x = \Theta/(1-\gamma)$ which globally attracts every trajectory. One can regard such a system as an abstract model of e.g. a leaky neuron which relaxes at a rate γ under the influence of an external input Θ. We take interaction function to have a sigmoidal form

$$\hat{g}(x) = \frac{1}{1+\exp(-\kappa x)} - \frac{1}{2}, \qquad \kappa > 0. \tag{11}$$

With the choices (10) and (11), we simulate system (9) over a directed random network with predominantly positive weights (with probability 0.25) but also negatively-weighted connections (with probability 0.01). Figure 1 shows that the network synchronizes to a chaotic trajectory starting from random initial conditions, although the isolated units are simple linear maps.

The chaotic nature of the trajectory in figure 1 is indicated, as usual, by a positive Lyapunov exponent of the synchronized dynamics

$$s(t+1) = \gamma s(t) + \Theta + \varepsilon\hat{g}(s(t)) \tag{12}$$

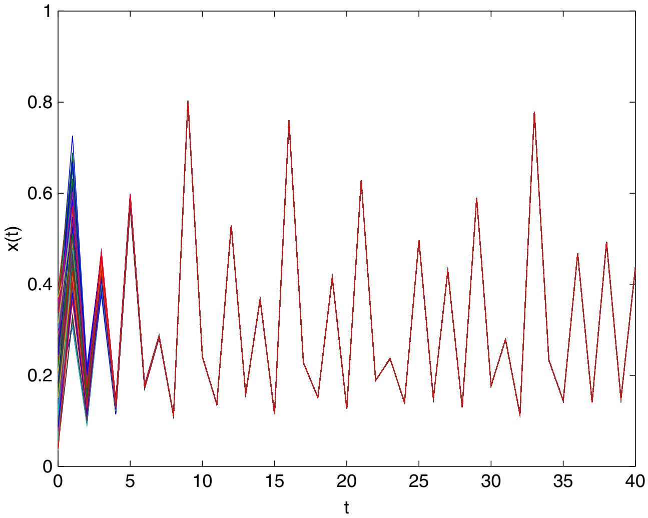

Figure 1. Synchronization of simple linear maps (10) to a chaotic trajectory under direct coupling. A random directed network of 500 nodes with both positive and negative weights is simulated starting from random initial conditions from the interval (0.04, 0.4). The parameter values are $\gamma = 0.3$, $\Theta = 4$, $\varepsilon = -8$, $\kappa = 20$.

for the chosen parameter values. It is interesting to note that the value of the exponent depends rather sensitively on the coupling strength, as depicted in figure 2. This shows that the network with non-diffusive coupling is capable of exhibiting a wide range of dynamics, both chaotic and non-chaotic, when it is synchronized. Furthermore, these different states can be attained solely by tuning the coupling strength, often by a small amount. By contrast, the synchronized dynamics under diffusive coupling is independent of the coupling

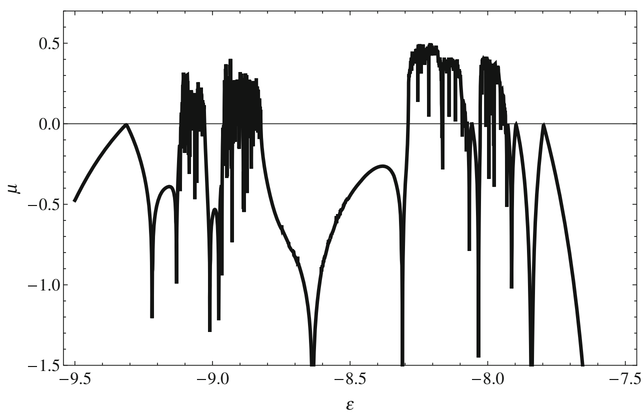

Figure 2. The Lyapunov exponent μ of the synchronized solution (12) as a function of the coupling strength ε. Other parameter values are as in figure 1.

strength, as it is limited to the dynamics of the isolated unit. (Note, however, that the synchronizability of the system depends on the coupling strength in general, even when the resulting synchronized dynamics does not.) For further discussion and examples on the range of dynamics that can be obtained by synchronization, the reader is referred to [5].

4. Delay coupling

We have noted in §2 that when the diffusively-coupled network synchronizes, the synchronized dynamics is identical to that of the isolated map. This observation is no longer true if we consider time delays arising from the finite speed of information transmission in the system. Suppose now that the signal transmission along a link requires a time span of τ, here for simplicity assumed to be an integer whose value is identical for all the links. Then in (2), the interaction terms under the summation sign take the form $g(x_j(t-\tau), x_i(t))$, and the synchronized dynamics, previously given by (8), is now governed by

$$s(t+1) = f(s(t)) + \varepsilon g(s(t-\tau), s(t)).$$

Clearly, even for the diffusive-type coupling (5), the last term does not vanish in general for non-constant s. For example, when the interaction function g is given by (6), the diffusive condition (5) is satisfied but the synchronized solution obeys

$$s(t+1) = (1-\varepsilon) f(s(t)) + \varepsilon f(s(t-\tau)) \tag{13}$$

rather than (7). Hence, it can be expected that novel behaviour may arise rather naturally in the presence of time delays in the model. The question remains, as before, whether the coupled system can sustain synchrony under perturbations, since the diffusive coupling now compares the states of the neighbouring units at different times and there is no *a priori* reason that such differences should act to drive the system towards synchrony. Interestingly, it turns out that the system with time delays can still synchronize under appropriate conditions, and in fact sometimes better than the undelayed network [6]. Here, we focus on the range of dynamics that can be obtained from a synchronized system.

We take a network of chaotic logistic maps with $f(x) = 4x(1-x)$, coupled through the scheme (6) and time delay $\tau = 1$. Under appropriate coupling topologies, the network can synchronize for a range of coupling strengths $\varepsilon \in (0.6, 1)$ [6]. The synchronized solutions (13) are plotted as a function of the coupling strength over this range in the bifurcation diagram of figure 3. Various regimes of chaotic and periodic behaviour are evident in the figure, which can be attained by slight changes in the coupling strength. Note that in the undelayed case the dynamics of (13) reduces to that of the isolated unit (1), and is thus independent of the coupling strength ε. Hence, in the presence of time delays, the system can not only synchronize, but also exhibit a wide range of emergent behaviour as a result of synchronization.

As seen in §3, even the simplest maps can synchronize to yield chaotic behaviour. Sometimes the converse is also interesting, that is, systems with complex dynamics synchronizing to yield simple behaviour. Figure 3 already illustrates that periodic activity can arise from synchronization of chaotic maps with delays. Even simpler behaviour, namely spatially uniform equilibrium solutions, are possible and are relevant in areas such as chaos control. Precise conditions for the stability of equilibria in diffusively coupled networks

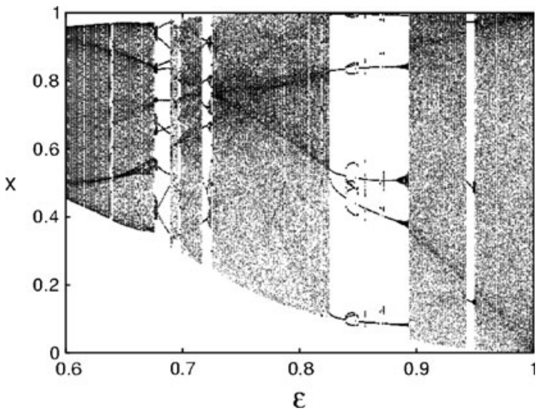

Figure 3. The bifurcation diagram of synchronized solutions (13) of delay-coupled chaotic logistic maps as a function of the coupling strength ε.

with delays are given in [7] and [8] for scalar and higher-dimensional maps, respectively, and a related analysis is provided for differential equations [9]. In particular, it is known that the largest eigenvalue of the graph Laplacian completely determines the effect of connection structure. However, it is ultimately the time delays that play a crucial role in driving the synchronous solution to a fixed point, as stabilization through diffusive coupling is not possible in the undelayed case [7,8]. For a general discussion on time delays in complex systems and networks, the interested reader is referred to [10].

5. Chimera states

So far, the emerging novelty through synchronization was a temporal one, that is, it manifested itself in the time evolution of the nodes, while spatially the system remained simple through the uniformity brought in by synchronization. We now look at the opposite situation where the nodes have a simple time evolution but the system exhibits complex spatial arrangement of dynamical states. Such cases are of course not completely rare, as unsynchronized networks would typically exhibit complicated spatial structure. However, keeping to the topic of the present paper, we make a connection with synchronization: Our network exhibits local clusters of nodes that are synchronized among themselves and are separated from other clusters by boundary layer nodes exhibiting incoherent motion. These so-called ‘chimera states’ can be observed in the simplest phase oscillators [11,12]. Here, we include distance-dependent delays into the model and show the formation of clustered chimera states.

Phase oscillators are simple models of periodic motion described by a phase variable ϕ evolving on the circle S^1 with constant velocity ω [13,14]. A network of interacting identical oscillators can be described by the equation

$$\dot{\phi}_i(t) = \omega + \varepsilon \sum_{j=1}^{N} a_{ji} \sin(\phi_j(t - \tau_{ji}) - \phi_i(t)), \tag{14}$$

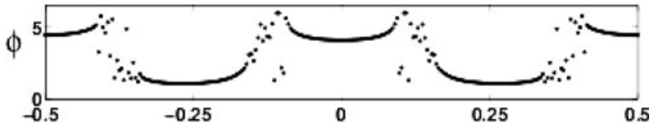

Figure 4. Clustered chimera state in a ring of phase oscillators with distance-dependent delays. The horizontal axis is the position along the ring and so the left and right edges of the figure correspond to the same oscillator. The vertical axis is the phase of each oscillator, which takes values in the interval $[0, 2\pi)$.

where τ_{ji} is the time delay over the link between nodes j and i. In case of uniform delays, $\tau_{ij} = \tau\ \forall i, j$, synchronization conditions are given in [15]. By contrast, we consider a network of oscillators in physical space, and take the delay to be proportional to physical distance, say $\tau_{ij} = d(i, j)/v$, where $d(i, j)$ is the distance between oscillators i and j and v is the speed of signal propagation. Furthermore, the connectivity weights a_{ij} are taken also as distance-dependent, $a_{ij} = \exp(-kd(i, j))$ for some decay constant k. Larger values of k thus indicate highly local interaction, whereas $k = 0$ corresponds to an all-to-all coupled network. For this distance-dependent case, the conditions for complete synchronization are given in [16]. In addition to synchronization, for some parameter ranges there also exist clustered chimera states that can be reached from certain initial conditions [17]. Figure 4 shows such a state for a network of oscillators arranged in a ring. Here, there are four clusters of oscillators which are synchronized (phase-locked) among themselves. The neighbouring clusters are in anti-phase relation to each other, and are separated by narrower bands of oscillators whose phases drift incoherently. Thus, the diffusive-type coupling in (14) together with distance-dependent delays result in the emergence of a non-trivial structure that involves both synchronized and incoherent dynamics in the same system at different spatial locations.

6. Conclusion

We have seen that, through synchronization, a network of dynamical systems can exhibit a wide range of novel dynamics at the global level well beyond what its constituent units are capable of. The emergent behaviour can manifest itself both temporally or spatially. While synchronization, in its various forms, is in general neither necessary nor sufficient for emergence, it is an important mechanism since it shows what can be achieved through self-organized cooperative behaviour, and yields amplified and conspicuous output signals in real systems that are observable to the outside world.

References

[1] J Jost, N Bertschinger and E Olbrich, *New Ideas in Psychology* **28(3)**, 265 (2010)
[2] K Kaneko (ed.), *Theory and applications of coupled map lattices* (Wiley, New York, 1993)
[3] J Jost and M P Joy, *Phys. Rev.* **E65**, 016201 (2002)
[4] F Bauer, F M Atay and J Jost, *Nonlinearity* **22(1)**, 2333 (2009)
[5] F Bauer, F M Atay and J Jost, *EPL Europhys. Lett.* **89(2)**, 20002 (2010), http://stacks.iop.org/0295-5075/89/20002

[6] F M Atay, J Jost and A Wende, *Phys. Rev. Lett.* **92(14)**, 144101 (2004)
[7] F M Atay and Ö Karabacak, *SIAM J. Appl. Dynam. Syst.* **5(3)**, 508 (2006)
[8] F M Atay, *Afrika Matematika* (2011), DOI: 10.1007/s13370-011-0024-z, URL: http://dx.doi.org/10.1007/s13370-011-0024-z
[9] F M Atay, *J. Differ. Eq.* **221(1)**, 190 (2006)
[10] F M Atay (ed.), *Complex time-delay systems* (Springer, Berlin Heidelberg, 2010)
[11] Y Kuramoto and D Battogtokh, *Nonlinear Phenomena in Complex Systems* **5(4)**, 380 (2002)
[12] D M Abrams and S H Strogatz, *Phys. Rev. Lett.* **93(17)**, 174102 (2004)
[13] A T Winfree, *The geometry of biological time* (Springer, New York, 1980)
[14] Y Kuramoto, *Chemical oscillations, waves, and turbulence* (Springer, Berlin, 1984)
[15] M G Earl and S H Strogatz, *Phys. Rev.* **E67**, 036204 (2003)
[16] G C Sethia, A Sen and F M Atay, *Phys. Rev.* **E81**, 056213 (2010)
[17] G C Sethia, A Sen and F M Atay, *Phys. Rev. Lett.* **100**, 144102 (2008), http://link.aps.org/abstract/PRL/v100/e144102