Multiple dynamical time-scales in networks with hierarchically nested modular organization

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Abstract. Many natural and engineered complex networks have intricate mesoscopic organization, e.g., the clustering of the constituent nodes into several communities or modules. Often, such modularity is manifested at several different hierarchical levels, where the clusters defined at one level appear as elementary entities at the next higher level. Using a simple model of a hierarchical modular network, we show that such a topological structure gives rise to characteristic time-scale separation between dynamics occurring at different levels of the hierarchy. This generalizes our earlier result for simple modular networks, where fast intramodular and slow intermodular processes were clearly distinguished. Investigating the process of synchronization of oscillators in a hierarchical modular network, we show the existence of as many distinct time-scales as there are hierarchical levels in the system. This suggests a possible functional role of such mesoscopic organization principle in natural systems, viz., in the dynamical separation of events occurring at different spatial scales.

1. Introduction

Complex networks are ubiquitous [1]. They are seen in phenomena studied by physicists from resistor networks to polymer contact structure to spin interactions in disordered systems [2], in engineered systems (e.g. the internet and the power grid) and in the biological world from the intracellular signalling system to neuronal networks to ecological food webs [3]. Analysis of such networks can lead not only to the fundamental understanding of how complex systems function, but may also have immediate practical relevance. For example, understanding the network of intracellular communication leading to identification of the key players can be used to develop drugs targeted specifically towards these molecules [4]. On a much larger scale, by revealing how the contact structure among susceptible and infected individuals govern the propagation of epidemics, the manipulation of networks may play a significant role in public health.

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Over the last decade, beginning in 1998–99, interest in complex networks has grown by leaps and bounds among physicists. However, the focus of research activity related to such systems has been on either the microscopic properties of individual nodes such as the distribution of degree (i.e., the number of links for a node) [5] or the macroscopic properties characterizing the entire network in terms of a global value, such as average path length or clustering coefficient [6]. Recent research, on the other hand, has revealed that networks which are indistinguishable at either of these two scales may nevertheless have radically different behaviour [7]. The origin of this difference lies in their mesoscopic organization which can be structurally manifested as patterns in the arrangement of links between subparts of the network [8].

One of the prominent examples of such organizing principles operating in networks is the existence of communities, also referred to as modularity [9]. Modules can be defined as subnetworks whose components are much more densely and/or strongly connected with each other compared to connections with components that belong to different sub-networks. In an earlier work we have shown that existence of modules can give rise to two distinct time-scales for the dynamics on such networks: a strongly modular character of the system, as indicated by a low value for the ratio of within-module to between-module connections, results in a sharp distinction between fast intramodular and slower intermodular processes [7]. Another mesoscopic organizing principle observed in many networks is hierarchy, i.e., an arrangement in which entities are ordered in several levels or layers [10]. We are most familiar with hierarchy as seen in social organization; however, they are also present in other systems, as for example in the neuronal communication network which has a clearly defined direction of information flow, beginning with the sensory organs and culminating with motor response. While both modular and hierarchical features are increasingly being reported in a wide range of systems, it is still not clear what role such patterns play in the dynamics and function of the corresponding systems. Thus, the second wave of research in complex networks that is just beginning, focusses on these mesoscopic aspects of networks.

There can be even more complicated intermediate-scale features in networks, in particular, hierarchically nested modularity, where modules and metamodules (composed of multiple strongly connected modules) may occur at different hierarchical levels. Such hierarchical modularity can be observed in many naturally occurring systems, including biological metabolic networks [11], ecological food webs [12], social groups [13,14], financial markets [15] and brain functional networks [16,17]. Figures 1a–b shows empirical evidence for the occurrence of hierarchical modular organization in the network of connections between cortical regions in the cat [18] and macaque [19] brains obtained from anatomical studies. Theoretical understanding of such systems is still at an early stage. A simple-minded deterministic construction approach towards such networks claimed that such structures have a specific signature in the power-law dependence of the clustering coefficient of nodes having degree k on k [20]. We demonstrated, using simple counter-examples, that this is true only for a very specific set of artificially generated networks and does not hold in general [8]. This illustrates the importance of acquiring a clearer understanding of the properties that the hierarchical nesting of modular structures impart to a system. We focus specifically on the role that such mesoscopic organization plays in synchronization dynamics on the network.

In this paper we show that in a hierarchical modular network of oscillators arranged into $l = h_{\text{lev}}$ levels, there exist h_{lev} distinct time-scales of the synchronization dynamics. The
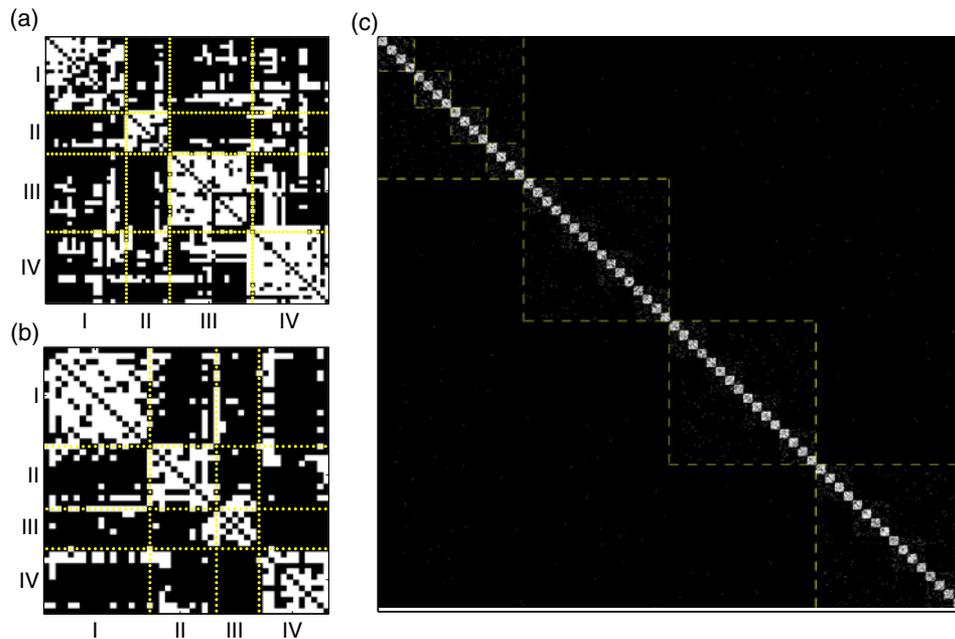
Figure 1. The adjacency matrix corresponding to the network of cortico-cortical connections in the **(a)** cat and **(b)** macaque brains. The largest-scale modules are indicated by dotted lines and labelled with Roman numerals. **(c)** Adjacency matrix for the hierarchical modular network of $N = 1024$ nodes with average degree $\langle k \rangle$. There are $h_{\text{lev}} = 3$ hierarchical levels, with a branching ratio of $q = 4$ at each level. The ratio of connection density between two successive hierarchical levels, $r = 0.05$. The modules at each hierarchical level are indicated by broken lines.

oscillators belonging to the elementary modules ($l = 0$) synchronize earliest, followed by those belonging to the same metamodule ($l = 1$) and so on, with global synchronization being achieved last [20a]. In the next section we discuss the simple network model with hierarchically nested modules which we have used. Our results on the synchronization dynamics on such a network are described in §3. We have carried out spectral analysis of the Laplacian matrix for the network to give a theoretical background for the explicit numerical simulations. We conclude with a short discussion on the possible functional relevance for the separation of time-scales through hierarchical modular organization in the biological world.

2. The model

We have used a general model for networks having modular organization at several hierarchically arranged levels that has been introduced in ref. [8] (figure 1c). At the lowest level ($l = 0$), the network consists of M modules, each containing n nodes. The connectivity, or, the probability of a link, between a pair of nodes is much higher for nodes belonging

to the same module compared to those belonging to different modules. Let us assume the connectivity within each of the modules to be $0 < \rho_1 \leq 1$. We now introduce the next hierarchical level ($l = 1$) by grouping together m modules into 'metamodules'. The connection density between nodes belonging to different modules in the same metamodule is $\rho_2 (\leq \rho_1)$. Thus, while the oscillators in different modules within each of the M/m metamodules have a lower probability of connection (ρ_2) between them compared to the probability of connections between oscillators in the same module (ρ_1), it is higher than the connectivity for oscillators belonging to different metamodules. To increase the number of hierarchical levels to three, we group q such metamodules into 'meta-metamodules' ($l = 2$). The connectivity between oscillators belonging to different metamodules but in the same meta-metamodule is $\rho_3 (\leq \rho_2)$. This procedure of grouping together q clusters of oscillators at higher and higher levels of organization can be continued up to a maximum number of layers, h_{lev}, implicitly given by the relation $q^{h_{\text{lev}}-1} m = M$. The parameter q is the branching at each level (except the lowest one), corresponding to the number of clusters at one level that are grouped together to form the cluster at the next higher level.

To reduce the number of model parameters, we further assume that the connectivities within and between clusters at different levels, $\rho_1, \rho_2, \ldots, \rho_{h_{\text{lev}}}$ are related as

$$\frac{\rho_2}{\rho_1} = \frac{\rho_3}{\rho_2} = \cdots = \frac{\rho_{h_{\text{lev}}}}{\rho_{h_{\text{lev}}-1}} = r, \tag{1}$$

where $0 \leq r \leq 1$ is the ratio of intercluster connectivities at two successive hierarchical levels. Varying r from 0 to 1, one obtains a whole range of hierarchical modular networks, with the special case of M isolated modules at one extreme ($r = 0$) and a homogeneous Erdos–Renyi random network at the other ($r = 1$). Figure 1c shows an example of a hierarchical network with three levels and $r = 0.05$ constructed following the above procedure.

The model allows the modular character of the network to be changed (by varying r) independent of the hierarchical complexity which is governed by the number of levels, h_{lev}. The stochastic construction process of the network implies that reasonable statistical averages can be done over an ensemble of random networks with the same hierarchical modularity.

3. Results

Most networks occurring in nature have dynamics associated with their nodes [21]. In particular, many systems comprise oscillating elements, such as, pancreatic beta cells, neurons or cardiac pacemaker cells [22,23]. Synchronization of the periodic activity exhibited by the nodes in such networks may play a vital functional role. To observe how the hierarchical modular organization of a network can affect its synchronization behaviour, we consider a population of N coupled oscillators. The time evolution of the phase of the ith oscillator having a frequency ω_i and which is connected to k_i other oscillators, can be described by [24]:

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{1}{k_i} \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i). \tag{2}$$

The coupling between the different oscillators is described by the matrix $\mathbf{K} = \{K_{ij}\} = \kappa \mathbf{A}$, where $\kappa (=1)$ is the strength of interaction between the oscillators and $\mathbf{A}$ is the network adjacency matrix, i.e., $A_{ij} = 1$ if nodes i, j are connected, and 0 otherwise. In our analysis, we shall mostly consider the simplified case of oscillators with identical frequency, i.e., $\omega_i = \omega$. However, we have explicitly verified that randomly varying the frequencies of different oscillators over a small range do not qualitatively affect our results.

The attractor for the network dynamics described by eq. (2) is the fully synchronized state $\theta_i = \theta$, $\forall i$. However, for a network with mesoscopic organization, the convergence to global synchronization occurs in a series of steps which is intrinsically related to the underlying connection topology (and has been seen earlier in simple modular networks [25]). We observe that, starting from a random distribution of initial phases for different oscillators, the nodes within each module synchronize first, followed by larger and larger structures at the higher hierarchical levels in sequence till the entire system oscillates in the same phase. The distinct time-scales of synchronization for the different levels reflect the structural organization of the network.

To analyse the time evolution of the synchronization process, we measure a local order parameter using the pair-correlation function between the oscillator phases:

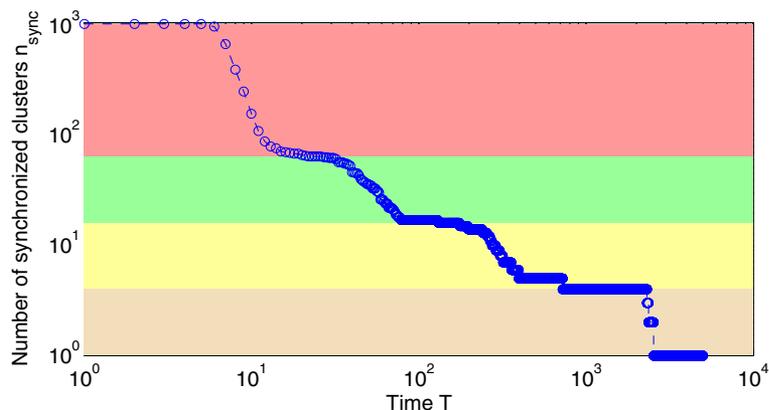$$\rho_{ij}(t) = \langle \cos[\theta_i(t) - \theta_j(t)] \rangle, \tag{3}$$

Figure 2. Time-evolution of the number of synchronized clusters of oscillators for hierarchical modular network with $N = 1024$ nodes of average degree $\langle k \rangle = 14$. The synchronization occurs over three distinct time-scales, reflecting the number of hierarchical levels $h_{lev} = 3$ of the network. First, local synchronization occurs by $T \simeq 20$ among the oscillators within each of the 64 modules (having 16 oscillators each) at the lowest level, $l = 0$. By $T \simeq 80$, groups of four modules (corresponding to a total of 64 oscillators each), which form the metamodule in the next hierarchical level ($l = 1$) get synchronized. By $T \simeq 720$, synchronization is observed at the level $l = 2$, where each of the synchronized groups comprise 256 oscillators. Finally global synchronization among all 1024 oscillators occurs at $T \sim 2500$. The branching at each level is $q = 4$ and the ratio of intermodular connections between two successive hierarchical levels, $r = 0.03$.

where $\langle \cdots \rangle$ is an average over random initial phases. Introducing a threshold T, the correlation matrix is converted into a dynamic connectivity matrix $\mathcal{D}_t(T)$, where $\mathcal{D}_{ij} = 1$ if $\rho_{ij} > T$ and $= 0$ otherwise. The number of distinct synchronized communities at any given time is given by the number of disconnected clusters in $\mathcal{D}$. Initially, when the phases of the oscillators are chosen from a random distribution, none of the elements are synchronized and the number of clusters is trivially equal to the number of oscillators, N. However, with time, more and more oscillators get successively synchronized and when global synchronization is achieved, there is only a single cluster of synchronized elements. In contrast to homogeneous random networks, where there is a smooth, continuous transition to global synchronization with time, for a hierarchical modular network we observe a series of step-like transitions as larger and larger clusters get synchronized at different time-scales (figure 2).

For a network having h_{lev} hierarchical levels, the evolution to global synchronization exhibits h_{lev} intermediate time-scales as reflected by the occurrence of h_{lev} plateaus between N and 1 for the number of synchronized clusters (figure 2). First, at the relatively short time-scale of τ_m, disconnected clusters are observed to form in $\mathcal{D}$ which correspond to the modules at the lowest hierarchical level of the network. Thus, local synchronization within each module at level $l = 0$ is achieved relatively rapidly. Next, at a slightly
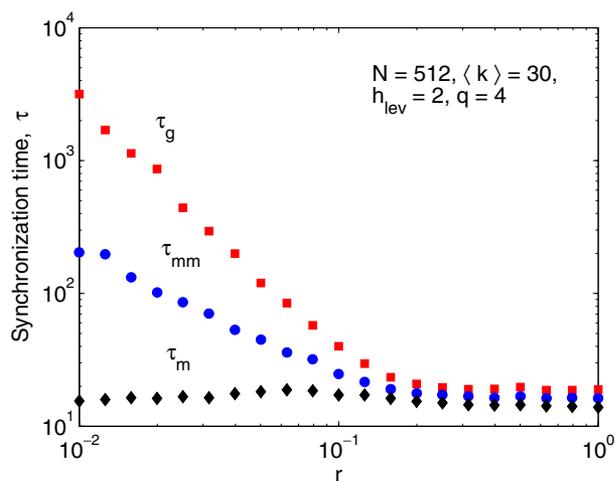

Figure 3. The variation of the synchronization time-scales at different hierarchical levels as a function of r for a network having $N = 512$ oscillators arranged into modules in $h_{\text{lev}} = 2$ levels. The time τ_m is that required for each module to synchronize the 32 oscillators that each of them have, while τ_{mm} is the time required for synchronization within a metamodule comprising four of the aforementioned modules and τ_g is the global synchronization time (i.e., the entire system comprising four metamodules). The three time-scales diverge sharply as the modularity at each level becomes more prominent with decreasing r. Conversely, as the network becomes more homogeneous when $r \to 1$ the three time-scales converge. The average number of connections among the oscillators $\langle k \rangle = 30$ and the branching at each level is $q = 4$.

longer time-scale of τ_{mm}, several of the clusters mentioned earlier merge into bigger clusters corresponding to the metamodules at the next hierarchical level $l = 1$ of the network. Therefore, the intermediate-scale synchronization over metamodules is a slower process than the local synchronization seen within each module, the initially synchronized clusters remaining fairly stable in the intervening time-period between τ_m and τ_{mm}. Successive hierarchical levels take longer and longer times to synchronize, and we can associate distinct synchronization time-scales τ with each such level. Finally, global synchronization is observed at the longest time-scale of τ_g.

As the existence of the distinct time-scales to synchronization is a result of the mesoscopic organization of the network, we expect that as the system becomes more homogeneous on increasing r, the different time-scales should converge towards the same value. This is indeed observed in figure 3, where the three time-scales for $h_{\text{lev}} = 2$, viz., τ_m, τ_{mm} and τ_g approach each other as $r \to 1$.

The existence of the distinct time-scales in a hierarchical modular network can be understood by analysing the linearized dynamics around the synchronized state of the system:

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \frac{1}{k_i} \sum_j L_{ij}\theta_j, \tag{4}$$
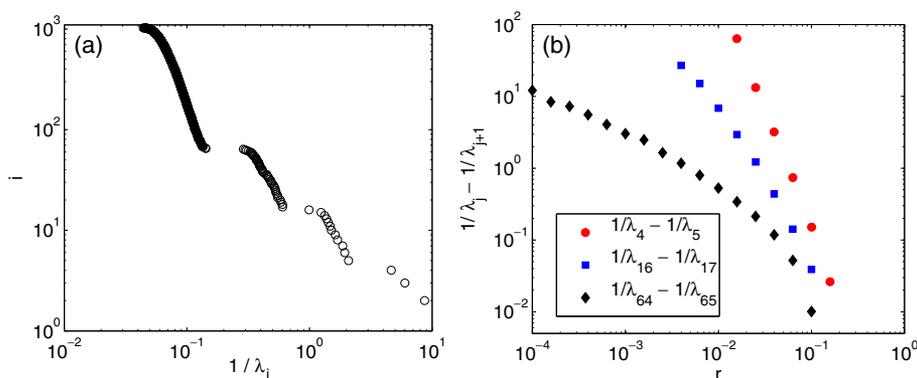

Figure 4. (a) The rank-ordered reciprocal eigenvalues for the Laplacian matrix of a hierarchical modular random network ($h_{\text{lev}} = 3$) with $N = 1024$ oscillators having $\langle k \rangle = 14$ connections (on average) for $r = 0.05$. The existence of three distinct spectral gaps reflects the three hierarchical levels in which the modular organization is arranged. (b) The width of the three Laplacian spectral gaps shown as a function of r for a hierarchical modular network with $N = 1024$ oscillators having $\langle k \rangle = 60$ connections (on average) arranged into $h_{\text{lev}} = 3$ levels. The spectral gap for each level increases with decreasing r, implying that as the hierarchical modular character of the network becomes more prominent, the synchronization times for different levels deviate significantly. The divergence of the gaps for sufficiently low values of r indicate that the system may not synchronize globally for very weak coupling among the clusters of modules. For all cases, the branching at each level is $q = 4$.

where $\mathbf{L}$ is the Laplacian of the network, with $L_{ii} = k_i$ and $L_{ij} = -A_{ij}$ for $i \neq j$. Solving eq. (4) in terms of the normal modes $\phi_i(t)$, we obtain

$$\phi_i(t) = \phi_i(0) \, e^{-\lambda_i t}, \quad i = 1, \ldots, N, \tag{5}$$

where λ_i are the eigenvalues of $\mathbf{L}' = \mathbf{D}^{-1}\mathbf{L}$ with $\mathbf{D}$ being a diagonal matrix having $D_{ii} = k_i$. All the eigenvalues of $\mathbf{L}'$ are real as the matrix is related to the symmetric normalized Laplacian $\mathbf{D}^{1/2} \, \mathbf{L}' \, \mathbf{D}^{-1/2}$ through a similarity transformation. Any difference in time-scales of different modes is manifested as gaps in the spectrum of this Laplacian, and reveals different levels of mesoscopic organization in the network. The mode corresponding to the smallest eigenvalue is associated with global synchronization. The other modes provide information about collective behaviour within different groups of oscillators. Typically, the size of the clusters considered decrease with increase in the corresponding eigenvalues.

In figure 4a we observe multiple gaps in the Laplacian spectrum for a hierarchical network, the number of gaps corresponding exactly to the number of hierarchical levels ($h_{\mathrm{lev}} = 3$) indicating that the existence of several distinct time-scales for the dynamical system has its origin in the hierarchical modular nature. As seen in figure 4b, the widths of the gaps increase with decreasing value of r, suggesting that the time-scales become more differentiated with increasing modularity, i.e., decreasing connectivity between the modules, between the metamodules, and so on.

4. Discussion

In this paper we have generalized our earlier result for time-scale separation of dynamics on simple modular networks by showing that for a system with hierarchically nested modules occurring at multiple levels, there exists a number of distinct time-scales (equal to the number of hierarchical levels). Any collective dynamical process taking place on such a system (e.g. coordination of nodal activity) will occur at many different temporal rates, the fastest events taking place at the lowest modular level and the slowest occurring at the global level. An important point to note is that the non-uniform synchronization observed in our system is rooted in its non-trivial mesoscopic organization. This distinguishes the behaviour from the 'hierarchical' synchronization observed in homogeneous scale-free networks, where oscillators having more connections (i.e., higher degree) show stronger phase-locking between them [26]. Thus, the latter phenomenon is related to the dispersion of microscopic properties (viz., degree) of the nodes, which by design is identical for all oscillators in our system. An interesting topic for future study is the effect of different varieties of degree distribution on synchronization dynamics in hierarchical modular networks.

Our results also have ramifications for the evolution of such mesoscopic organization in natural systems such as the brain. While synchronization of activity among elements belonging to the same cluster may have functional relevance, global synchronization of the system is often pathological and detrimental to the system [27,28]. Thus, the occurrence of a hierarchically nested modular organization may be nature's way to allow rapid local synchronization at small scales while making it difficult for activity in the system as a whole to get coordinated. Note that this possible functional role of hierarchical modularity

in the brain is distinct from the recent suggestion that it is related to the task of simultaneously segregating sensory signals and integrating information from multiple channels in the brain, as the latter requires the existence of hubs (i.e., nodes having a much higher number of connections than the average, which by design is absent in our model) in addition to the modular and hierarchical nature of cortico-cortical connections [29]. We have earlier demonstrated that the occurrence of modularity in the *C. elegans* nervous system may be directly related to the functional requirement of information processing [30]. A recent study of Hopfield-like associative memory models has also shown that modular network structure alters the attractor landscape of convergence dynamics in such systems, making the recall of stored patterns more efficient [31]. Generalizing these results to modules at more than one level should bring us closer to answering the question as to why hierarchical modularity is ubiquitous in the biological world.

Acknowledgements

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