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Invariance analysis and conservation laws of the wave equation on Vaidya manifolds

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Abstract. In this paper we discuss symmetries of classes of wave equations that arise as a consequence of some Vaidya metrics. We show how the wave equation is altered by the underlying geometry. In particular, a range of consequences on the form of the wave equation, the symmetries and number of conservation laws, *inter alia*, are altered by the manifold on which the model wave rests. We find Lie and Noether point symmetries of the corresponding wave equations and give some reductions. Some interesting physical conclusions relating to conservation laws such as energy, linear and angular momenta are also determined. We also present some interesting comparisons with the standard wave equations on a flat geometry. Finally, we pursue the existence of higher-order variational symmetries of equations on nonflat manifolds.

Keywords. Wave equations on nonflat manifolds; symmetry analysis; conservation laws.

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1. Introduction

The well-known Vaidya metric representing a model for the spherically symmetric solution of the Einstein equations with geometrical optics stress energy tensor of radiation is widely discussed in the literature [1–3]. A special case of the metric is the well-known Papapetrou model [4] and a study involving the Carter constant and Petrov classification is conducted in [5]. The references in the latter includes studies on the nature of the Killing tensors and well-known notion of the isometries of the metric which are the diffeomorphisms of the manifold onto itself which preserve the metric tensor [6].

The Lie and Noether symmetries of the geodesic equations have been discussed in detail in [7], the more interesting case being the latter since these lead to conservation laws via Noether's theorem [8]. We showed that we totally recover the information regarding the isometries of the metric from a study of the Noether symmetries associated with the corresponding natural Lagrangian, L. The Noether symmetries of a system are a one-parameter

Lie groups of transformations that preserve the action $\mathcal{L} = \int L$ and are determined by a Killing-type equation given later. That is, a larger algebra of generators of symmetry are obtained and, hence, more conservation laws are classified. The Lie symmetries, on the other hand, leave the system invariant under the transformation and contain the Noether symmetries [9].

The standard wave equation in (1+3) dimensions has been extensively studied in the literature from the point of view of its Lie point symmetries. A detailed symmetry analysis of this equation is discussed by Ibragimov [10]. It is well known that in three-dimensional Euclidean space, the linear wave equation admits a maximal 16-dimensional Lie algebra of point symmetries excluding the infinite symmetry. In this work, we use a purely geometric consideration to construct the wave equation in a curved background geometry in such a way that the wave equation inherits nonlinearities of the respective geometry in a natural way. Keeping in mind that the wave equation in four-dimensional spacetime may be of more physical significance, we use the Vaidya manifold for our purposes.

We present some salient features of an Euler–Lagrange system of differential equations [8,9]. Consider an rth-order system of partial differential equations of n independent and m dependent variables,

$$E^{\beta}(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \beta = 1, \ldots, m. \tag{1}$$

A conservation law of (1) is a closed form of the equation given by

$$D_i T^i = 0, \tag{2}$$

on the solutions of (1), where D_i is the total derivative operator given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \cdots, \quad i = 1, \ldots, n,$$

where u_i^{α} represents the first derivative of u^{α} with respect to the ith independent variable x^i and u_{ij}^{α} represents all second derivatives of u^{α} and so on and $T = (T^1, \ldots, T^n)$ is a conserved vector of (1). Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie–Bäcklund operator (vector field) is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \cdots, \tag{3}$$

where $\xi^i, \ \eta^{\alpha} \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned} \zeta_i^{\alpha} &= D_i(W^{\alpha}) + \xi^j u_{ij}^{\alpha}, \\ \zeta_{i_1 i_2}^{\alpha} &= D_{i_1} D_{i_2}(W^{\alpha}) + \xi^j u_{j i_1 i_2}^{\alpha}, \end{aligned} \tag{4}$$

and so forth, where W^{α} is the Lie characteristic function defined by

$$W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}. \tag{5}$$

In this paper, we shall assume that X is a Lie point symmetry operator, that is, ξ and η are functions of x and u and are independent of derivatives of u. The Euler–Lagrange equations, if they exist, associated with (1) is the system,

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{6}$$

where $\delta/\delta u^\alpha$ is the Euler–Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{7}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional,

$$\mathcal{L} = \int_\Omega L(x, u, u_{(1)}, \ldots, u_{(r)})\mathrm{d}x, \tag{8}$$

defined over Ω. If we include point-dependent gauge terms $g_1, \ldots, g_n$, then the Noether symmetries X are given by a Killing-type equation,

$$XL + LD_i\xi_i = D_i g_i. \tag{9}$$

Corresponding to each X, a conserved vector $T = (T^1, \ldots, T^n)$ is obtained via Noether's theorem.

The paper is organized in the following form. In §2 we discuss symmetries of classes of wave equations that arise as a consequence of some Vaidya metrics. The objective is to show how the respective geometry affects the Lie point symmetry algebras of the corresponding wave equation. Then, in §3, the Noether point symmetries of the corresponding wave equations are determined via a Lagrangian. It will be shown that the algebras of Noether point symmetries are also reduced when compared with the algebra of the standard wave equation (radically, in some cases). Via Noether's theorem, some conserved flows are constructed. Finally, in §4, we pursue the existence of higher-order variational symmetries of wave equations on the respective manifolds.

Firstly, we note that the Vaidya metric [3] is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(t)}{r}\right)\mathrm{d}t^2 - 2\,\mathrm{d}t\,\mathrm{d}r + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{10}$$

for which a special case, $m(t) = t$, is known as the Papapetrou model [4].

A wave equation on a Lorentzian manifold endowed with a metric g_{ij} is given by the expression

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)u\,(\bar{x}, t) = \left\{g^{00}\partial^2_{00} + \frac{1}{2}g^{ij}\left[g^{00}(\partial_i g_{00})\partial_j + \partial_{ij}\Gamma^k_{ij}\partial_k\right]\right\}u\,(\bar{x}, t) = 0,$$

where $u(\bar{x}, t)$ is some given wave function,

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}\left(\partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij}\right),$$

Table 1. The Lie point symmetries of the wave equation for various cases of $m(t)$.

$m = 0$	$m = k$	$m = t$	$m = e^t$	$m = m(t)$
∂_t	∂_t	$t\partial_t + r\partial_r$	$u\partial_u$	$u\partial_u$
$t\partial_t + r\partial_r$	$u\partial_u$	$u\partial_u$	∂_ϕ	∂_ϕ
$t^2\partial_t + 2r(r+t)\partial_r - 2u(r+t)\partial_u$	∂_ϕ	∂_ϕ		
$u\partial_u$				
∂_ϕ				

represents the Christoffel symbols, g^{ij} is the inverse of the metric g_{ij} with spherical variables r, θ, ϕ, and

$$g_{ij} = \begin{pmatrix} -(1-\frac{2m(t)}{r}) & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

Consequently, the wave equation on a curved Vaidya background takes the form

$$- r^2 \sin\theta u_{rt} - 2r\sin\theta u_t + 2r\sin\theta\left(1 - \frac{2m(t)}{r}\right)u_r + 2\sin\theta m(t) u_r$$
$$+ r^2\sin\theta\left(1 - \frac{2m(t)}{r}\right)u_{rr} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} + u_{\phi\phi} = 0. \quad (11)$$

We shall classify the Lie and Noether point symmetries of this equation and show the effect of the curved background on the respective symmetry algebras.

2. Lie symmetries

We determine the Lie point symmetry generators of the wave equation (11) and split these into various cases (table 1). The principle Lie algebra is stated below. The most significant result we note is the reduction in the dimension of the algebra of Lie point symmetries when compared with the algebra of the wave equation on a flat manifold. This, as will be seen later, also has consequences on the number of standard conservation laws (usually first-order) of (11). Furthermore, the number of exact or invariant solutions are reduced drastically. For illustration, we perform a reduction corresponding to some two-dimensional subalgebras.

It is well known that the case $m = 0$ is isomorphic to a flat manifold and one would expect a maximal 16-dimensional algebra. It is clear that the wave equation is somewhat distorted even in this case and the number conservation laws will be reduced (see §2.1 for a confirmation of this).

2.1 *Reduction of order – some illustrations*

We demonstrate two reductions of the wave equation using two-dimensional subalgebras for the case $m = t$. In both cases, this leads to a partial differential equation in just two independent variables which can be further analysed using another Lie symmetry reduction or an appropriate, alternative method.

(i) If $\bar{X}_1 = \partial_\phi + u\partial_u$, $[\bar{X}_1, X_2] = 0$ so that reducing may begin with either $\bar{X}_1$ or X_2. The respective wave equation,

$$-2r^2 \sin\theta u_{rt} - 2r \sin\theta u_t + 2r \sin\theta \left(1 - \frac{2t}{r}\right) u_r + 2t \sin\theta u_r + r^2 \sin\theta \left(1 - \frac{2t}{r}\right) u_{rr} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} + u_{\phi\phi} = 0, \quad (12)$$

becomes

$$-2r^2 \sin\theta w_{rt} - 2r \sin\theta w_t + 2r \sin\theta \left(1 - \frac{2t}{r}\right) w_r + 2t \sin\theta w_r + r^2 \sin\theta \left(1 - \frac{2t}{r}\right) w_{rr} + \cos\theta w_\theta + \sin\theta w_{\theta\theta} + w = 0 \quad (13)$$

via the generator $\bar{X}_1$ since

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}\theta}{0} = \frac{\mathrm{d}\phi}{1} = \frac{\mathrm{d}u}{u}$$

leads to the new dependent variable w defined by $u = w(t, r, \theta)\mathrm{e}^\phi$. From $X_2 = t\partial_t + r\partial_r$ we get

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}r}{r} = \frac{\mathrm{d}\theta}{0} = \frac{\mathrm{d}w}{0},$$

so that $w = W(\alpha, \theta)$ leads to the partial differential equation,

$$(2\alpha^3 + \alpha^2 - \alpha) \sin\theta W_{\alpha\alpha} + (4\alpha^2 + 2\alpha - 2) \sin\theta W_\alpha + \cos\theta W_\theta + \sin\theta W_{\theta\theta} + W = 0, \quad (14)$$

where $\alpha = r/t$.

(ii) If we first reduce using $X_1 = \partial_\phi$ we get

$$-2r^2 \sin\theta u_{rt} - 2r \sin\theta u_t + 2r \sin\theta \left(1 - \frac{2t}{r}\right) u_r + 2t \sin\theta u_r + r^2 \sin\theta \left(1 - \frac{2t}{r}\right) u_{rr} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} = 0, \quad (15)$$

and then $\bar{X}_2 = u\partial_u + t\partial_t + r\partial_r$ leads to

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}r}{r} = \frac{\mathrm{d}\theta}{0} = \frac{\mathrm{d}u}{u},$$

so that with invariants $\alpha = r/t$ and $w(\theta, \alpha) = tu$ we get the reduced partial differential equation

$$(2\alpha^3 + \alpha^2 - 2\alpha)\sin\theta w_{\alpha\alpha} + (2\alpha^2 + 2\alpha - 2)\sin\theta w_\alpha \\ - 2\alpha\sin\theta w + \cos\theta w_\theta + \sin\theta w_{\theta\theta} = 0. \quad (16)$$

3. Noether symmetries and conservation laws

Since (11) is variational, we determine the Noether symmetries using (9) and the corresponding conserved flows $(T^t, T^r, T^\theta, T^\phi)$ via Noether's theorem. First, it can be shown that a Lagrangian of (11) is given by

$$L = -r^2\sin\theta u_t u_r + \frac{1}{2}r^2\sin\theta\left(1 - \frac{2m\,(t)}{r}\right)u_r^2 + \frac{1}{2}\sin\theta u_\theta^2 + \frac{1}{2}u_\phi^2. \quad (17)$$

(i) The case $m = 0$.

(a) $X_1^1 = \partial_t$

$$T^t = \frac{1}{2}(r^2\sin\theta u_r u_t - u(u_{\phi\phi} + \cos\theta u_\theta \\ + \sin\theta(u_{\theta\theta} + r(2u_r + ru_{rr} - 2u_t - ru_{tr})))),$$

$$T^r = -\frac{1}{2}r^2\sin\theta(u_r u_t - u_t^2 + u(-u_{tr} + u_{tt})),$$

$$T^\theta = -\frac{1}{2}\sin\theta(u_\theta u_t - uu_{t\theta}),$$

$$T^\phi = \frac{1}{2}(-u_\phi u_t + uu_{t\phi}).$$

(b) $X_2^1 = \partial_\phi$

$$T^t = \frac{1}{2}r^2\sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = -\frac{1}{2}r^2\sin\theta(u_\phi(u_r - u_t) + u(-u_{r\phi} + u_{t\phi})),$$

$$T^\theta = -\frac{1}{2}\sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \frac{1}{2}(-{u_\phi}^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} \\ + r(2u_r + ru_{rr} - 2(u_t + ru_{tr}))))).$$

(c) $X_3^1 = u\partial_u + t\partial_t + r\partial_r$

$$T^t = \frac{1}{2}(r^2 \sin\theta u_r(ru_r + tu_t) - u(tu_{\phi\phi} + t\cos\theta u_\theta + \sin\theta(tu_{\theta\theta} + r((r+2t)u_r + r(r+t)u_{rr} - t(2u_t + ru_{tr}))))),$$

$$T^r = -\frac{1}{2}r(r\sin\theta(u_r - u_t)(ru_r + tu_t) + u(u_{\phi\phi} + \cos\theta u_\theta + \sin\theta(u_{\theta\theta} - r(-u_r + u_t + ru_{tr} + tu_{tr} - tu_{tt})))),$$

$$T^\theta = -\frac{1}{2}\sin\theta(u_\theta(ru_r + tu_t) - u(ru_{r\theta} + tu_{t\theta})),$$

$$T^\phi = \frac{1}{2}(-u_\phi(ru_r + tu_t) + u(ru_{r\phi} + tu_{t\phi})).$$

(d) $X_4^1 = t^2\partial_t + 2r(r+t)\partial_r - 2u(r+t)\partial_u$

$$T^t = \frac{1}{2}(2r^2\sin\theta u^2 - r^2\sin\theta u_r(2r(r+t)u_r + t^2u_t) + u(t^2u_{\phi\phi} + t^2\cos\theta u_\theta + \sin\theta(t^2u_{\theta\theta} + r(2(2r^2 + rt + t^2)u_r + r(2r^2 + 2rt + t^2)u_{rr} - t^2(2u_t + ru_{tr}))))),$$

$$T^r = \frac{1}{2}r(r\sin\theta(u_r - u_t)(2r(r+t)u_r + t^2u_t) + u(2(r+t)u_{\phi\phi} + 2(r+t)\cos\theta u_\theta + \sin\theta(2(r+t)u_{\theta\theta} - r(-2(r+t)u_r + 2(2r+t)u_t + 2r^2u_{tr} + 2rtu_{tr} + t^2u_{tr} - t^2u_{tt})))),$$

$$T^\theta = \frac{1}{2}\sin\theta(u_\theta(2r(r+t)u_r + t^2u_t) - u(2r(r+t)u_{r\theta} + t^2u_{t\theta})),$$

$$T^\phi = \frac{1}{2}(u_\phi(2r(r+t)u_r + t^2u_t) - u(2r(r+t)u_{r\phi} + t^2u_{t\phi})).$$

(e) $X_5^1 = g(t,r,\theta,\phi)\partial_u$ where the function $g(t,r,\theta,\phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} + r(2g_r + rg_{rr} - 2(g_t + rg_{tr})))$.

$$T^t = r^2\sin\theta(ug_r - gu_r),$$

$$T^r = r^2\sin\theta(u(-g_r + g_t) + g(u_r - u_t)),$$

$$T^\theta = \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = -ug_\phi + gu_\phi.$$

(ii) The case $m = k$ where k is an arbitrary constant.

(a) $X_1^2 = \partial_t$

$$T^t = \frac{1}{2}(r^2 \sin\theta u_r u_t - u(u_{\phi\phi} + \cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(k-r)u_r + r((-2k+r)u_{rr} - 2u_t - ru_{tr})))),$$

$$T^r = -\frac{1}{2} r \sin\theta((-2k+r)u_r u_t - ru_t^2 + u((2k-r)u_{tr} + ru_{tt})),$$

$$T^\theta = -\frac{1}{2}\sin\theta(u_\theta u_t - uu_{t\theta}),$$

$$T^\phi = \frac{1}{2}(-u_\phi u_t + uu_{t\phi}).$$

(b) $X_2^2 = \partial_\phi$

$$T^t = \frac{1}{2} r^2 \sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = -\frac{1}{2} r \sin\theta(u_\phi((-2k+r)u_r - ru_t) + u((2k-r)u_{r\phi} + ru_{t\phi})),$$

$$T^\theta = -\frac{1}{2}\sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \frac{1}{2}(-u_\phi^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(k-r)u_r + r((-2k+r)u_{rr} - 2(u_t + ru_{tr}))))).$$

(c) $X_3^2 = g(t, r, \theta, \phi)\partial_u$ where the function $g(t, r, \theta, \phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} - 2(k-r)g_r + r((-2k+r)g_{rr} - 2(g_t + rg_{tr})))$

$$T^t = r^2 \sin\theta(ug_r - gu_r),$$

$$T^r = r\sin\theta(u((2k-r)g_r + rg_t) + g((-2k+r)u_r - ru_t)),$$

$$T^\theta = \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = -ug_\phi + gu_\phi.$$

(iii) The case $m = t$.

(a) $X_1^3 = \partial_\phi$

$$T^t = \frac{1}{2} r^2 \sin\theta (u_\phi u_r - u u_{r\phi}),$$

$$T^r = -\frac{1}{2} r \sin\theta (u_\phi ((r-2t)u_r - r u_t) + u(-(r-2t)u_{r\phi} + r u_{t\phi})),$$

$$T^\theta = -\frac{1}{2} \sin\theta (u_\phi u_\theta - u u_{\theta\phi}),$$

$$T^\phi = \frac{1}{2}(-u_\phi^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} + 2(r-t)u_r + r((r-2t)u_{rr} - 2(u_t + r u_{tr}))))).$$

(b) $X_2^3 = u\partial_u + t\partial_t + r\partial_r$

$$T^t = \frac{1}{2}(r^2 \sin\theta u_r (r u_r + t u_t) + u(-t u_{\phi\phi} - t\cos\theta u_\theta + \sin\theta(-t u_{\theta\theta} - (r^2 + 2rt - 2t^2)u_r + (2rt^2 - r^3 - r^2 t)u_{rr} + t(2u_t + r u_{tr}))))),$$

$$T^r = \frac{1}{2} r(\sin\theta(r(2t-r)u_r^2 + (r^2 - rt + 2t^2)u_r u_t + r t u_t^2) - u(u_{\phi\phi} + \cos\theta u_\theta + \sin\theta(u_{\theta\theta} + r(u_r - u_t) - r^2 u_{tr} - r t u_{tr} + 2t^2 u_{tr} + r t u_{tt}))),$$

$$T^\theta = -\frac{1}{2} \sin\theta (u_\theta (r u_r + t u_t) - u(r u_{r\theta} + t u_{t\theta})),$$

$$T^\phi = \frac{1}{2}(-u_\phi (r u_r + t u_t) + u(r u_{r\phi} + t u_{t\phi})).$$

(c) $X_3^3 = g(t, r, \theta, \phi)\partial_u$ where the function $g(t, r, \theta, \phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} + 2(r-t)g_r + r((r-2t)g_{rr} - 2(g_t + r g_{tr})))$

$$T^t = r^2 \sin\theta (u g_r - g u_r),$$

$$T^r = r\sin\theta(u(-(r-2t)g_r + r g_t) + g((r-2t)u_r - r u_t)),$$

$$T^\theta = \sin\theta(-u g_\theta + g u_\theta),$$

$$T^\phi = -u g_\phi + g u_\phi.$$

(iv) The case $m = e^t$.

(a) $X_1^4 = \partial_\phi$

$$T^t = \frac{1}{2}r^2 \sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = -\frac{1}{2}r \sin\theta(u_\phi((-2e^t + r)u_r - ru_t) + u((2e^t - r)u_{r\phi} + ru_{t\phi})),$$

$$T^\theta = -\frac{1}{2} \sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \frac{1}{2}(-u_\phi^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(e^t - r)u_r + r((-2e^t + r)u_{rr} - 2(u_t + ru_{tr}))))).$$

(b) $X_2^4 = g(t, r, \theta, \phi)\partial_u$ where the function $g(t, r, \theta, \phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} - 2(\mathrm{e}^t - r)g_r + r((-2\mathrm{e}^t + r)g_{rr} - 2(g_t + rg_{tr})))$

$$T^t = r^2 \sin\theta(ug_r - gu_r),$$

$$T^r = r \sin\theta(u((2e^t - r)g_r + rg_t) - g((2e^t - r)u_r + ru_t)),$$

$$T^\theta = \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = -ug_\phi + gu_\phi.$$

(v) The case $m = m(t)$ where $m(t)$ is an arbitrary function of t.

(a) $X_1^5 = \partial_\phi$

$$T^t = \frac{1}{2}r^2 \sin\theta(u_\phi u_r - uu_{r\phi}),$$

$$T^r = -\frac{1}{2}r \sin\theta(u_\phi((r - 2m(t))u_r - ru_t) + u((2m(t) - r)u_{r\phi} + ru_{t\phi})),$$

$$T^\theta = -\frac{1}{2} \sin\theta(u_\phi u_\theta - uu_{\theta\phi}),$$

$$T^\phi = \frac{1}{2}(-u_\phi^2 - u(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} + 2(r - m(t))u_r + r((r - 2m(t))u_{rr} - 2(u_t + ru_{tr}))))).$$

(b) $X_2^5 = g(t, r, \theta, \phi)\partial_u$ where the function $g(t, r, \theta, \phi)$ satisfies the equation $g_{\phi\phi} + \cos\theta g_\theta + \sin\theta(g_{\theta\theta} + 2(r - m(t))g_r + r((r - 2m(t))g_{rr} - 2(g_t + rg_{tr})))$

$$T^t = r^2 \sin\theta(ug_r - gu_r),$$

$$T^r = r \sin\theta(u(-(r - 2m(t))g_r + rg_t) + g((r - 2m(t))u_r - ru_t)),$$

$$T^\theta = \sin\theta(-ug_\theta + gu_\theta),$$

$$T^\phi = -ug_\phi + gu_\phi.$$

4. Higher-order symmetries and conserved densities

In the standard (1+3) wave equation there exist higher-order variational symmetries which lead to nontrivial conserved flows. Thus, even though there is a radical reduction in the number of variational point symmetries and conservation laws, one could analyse the curved wave equation on a knowledge of the higher-order symmetries and conservation laws. In this section we list some of these variational symmetries $\mathcal{X} = \eta(x, u, u_{(1)}, u_{(2)}, u_{(3)})\partial_u$ and nontrivial conserved flows $(\mathcal{T}^t, \mathcal{T}^r, \mathcal{T}^\theta, \mathcal{T}^\phi)$ wherein the conserved density is given by $\mathcal{T}^t$ (see [9] for a discussion on recursion operators and generalized symmetries).

(i) The case $m = 0$.

$$\mathcal{X}_1^1 = (2u_{tt} + tu_{ttt} + ru_{rtt})\partial_u,$$

$$\begin{aligned}
\mathcal{T}^t = \frac{1}{6}(&2\sin\theta u_{\theta\theta}u_t + 2r\sin\theta u_r u_t - ru_{r\phi\phi}u_t - r\cos\theta u_{r\theta}u_t \\
&- r\sin\theta u_{r\theta\theta}u_t - 2r^2\sin\theta u_{rr}u_t - r^3\sin\theta u_{rrr}u_t \\
&- 2r\sin\theta u_t^2 + 2uu_{t\phi\phi} - ru_r u_{t\phi\phi} - 3tu_t u_{t\phi\phi} \\
&+ 2\cos\theta uu_{t\theta} - r\cos\theta u_r u_{t\theta} - 3t\cos\theta u_t u_{t\theta} \\
&+ 2\sin\theta uu_{t\theta\theta} - r\sin\theta u_r u_{t\theta\theta} - 3t\sin\theta u_t u_{t\theta\theta} \\
&+ 8r\sin\theta uu_{tr} + 2r\sin\theta u_{\theta\theta}u_{tr} + 2r^2\sin\theta u_r u_{tr} \\
&+ 2r^3\sin\theta u_{rr}u_{tr} - 2r^2\sin\theta u_t u_{tr} - 6rt\sin\theta u_t u_{tr} \\
&- 4r^3\sin\theta u_{tr}^2 + 2ruu_{tr\phi\phi} + 2r\cos\theta uu_{tr\theta} \\
&+ 2r\sin\theta uu_{tr\theta\theta} + 10r^2\sin\theta uu_{trr} - r^3\sin\theta u_r u_{trr} \\
&+ 2r^3\sin\theta u_t u_{trr} - 3r^2t\sin\theta u_t u_{trr} + 2r^3\sin\theta uu_{trrr} \\
&- 8r\sin\theta uu_{tt} + 3t\sin\theta u_{\theta\theta}u_{tt} - 4r^2\sin\theta u_r u_{tt} \\
&+ 6rt\sin\theta u_r u_{tt} + 3r^2t\sin\theta u_{rr}u_{tt} - 6r^2t\sin\theta u_{tr}u_{tt} \\
&+ u_{\phi\phi}(2u_t + 2ru_{tr} + 3tu_{tt}) + \cos\theta u_\theta(2u_t + 2ru_{tr} + 3tu_{tt}) \\
&+ 3tuu_{tt\phi\phi} + 3t\cos\theta uu_{tt\theta} + 3t\sin\theta uu_{tt\theta\theta} - 7r^2\sin\theta uu_{ttr} \\
&+ 6rt\sin\theta uu_{ttr} - r^3\sin\theta u_r u_{ttr} + 6r^2t\sin\theta u_t u_{ttr} \\
&- r^3\sin\theta uu_{ttrr} + 3r^2t\sin\theta uu_{ttrr} - 6rt\sin\theta uu_{ttt} \\
&- 3r^2t\sin\theta u_r u_{ttt} - 3r^2t\sin\theta uu_{tttr}),
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^r = \frac{1}{6} r(&u_{\phi\phi}u_{tt} + \cos\theta u_\theta u_{tt} + \sin\theta u_{\theta\theta}u_{tt} + 8r\sin\theta u_r u_{tt}\\
&+ r^2\sin\theta u_{rr}u_{tt} - 2r^2\sin\theta u_{tr}u_{tt} + uu_{tt\phi\phi}\\
&+ \cos\theta uu_{tt\theta} + \sin\theta uu_{tt\theta\theta} - 7r\sin\theta uu_{ttr}\\
&+ 3r^2\sin\theta u_r u_{ttr} - 2r^2\sin\theta uu_{ttrr} + 7r\sin\theta uu_{ttt}\\
&+ 3rt\sin\theta u_r u_{ttt} - u_t(u_{t\phi\phi} + \cos\theta u_{t\theta}\\
&+ \sin\theta(u_{t\theta\theta} + r(2u_{tr} + ru_{trr} + 6u_{tt} + ru_{ttr} + 3tu_{ttt})))\\
&+ r^2\sin\theta uu_{tttr} - 3rt\sin\theta uu_{tttr} + 3rt\sin\theta uu_{tttt}),
\end{aligned}$$

$$\mathcal{T}^\theta = \frac{1}{2}\sin\theta(u_\theta(2u_{tt} + ru_{ttr} + tu_{ttt}) - u(2u_{tt\theta} + ru_{ttr\theta} + tu_{ttt\theta})),$$

$$\mathcal{T}^\phi = \frac{1}{2}(u_\phi(2u_{tt} + ru_{ttr} + tu_{ttt}) - u(2u_{tt\phi} + ru_{ttr\phi} + tu_{ttt\phi})).$$

(ii) The case $m = k$ where k is an arbitrary constant.

$$\mathcal{X}_1^2 = u_{\phi tt}\partial_u,$$

$$\begin{aligned}
\mathcal{T}^t = \frac{1}{6}(&-u_{\phi\phi\phi}u_t - \cos\theta u_{\theta\phi}u_t - \sin\theta u_{\theta\theta\phi}u_t + 2k\sin\theta u_{r\phi}u_t - 2r\sin\theta u_{r\phi}\\
&+ 2kr\sin\theta u_{rr\phi}u_t - r^2\sin\theta u_{rr\phi}u_t + 2\cos\theta u_\theta u_{t\phi} + 2r\sin\theta u_\phi u_{tt}\\
&+ 2\sin\theta u_{\theta\theta}u_{t\phi} - 4k\sin\theta u_r u_{t\phi} + 4r\sin\theta u_r u_{t\phi} - 4kr\sin\theta u_{rr}u_{t\phi}\\
&+ 2r^2\sin\theta u_{rr}u_{t\phi} - 2r\sin\theta u_t u_{t\phi} - u_\phi u_{t\phi\phi} + 2uu_{t\phi\phi\phi} - 2r\sin\theta u_\phi u_{tr}\\
&+ 2\cos\theta uu_{t\theta\phi} - \sin\theta u_\phi u_{t\theta\theta} + 2\sin\theta uu_{t\theta\theta\phi} + 2k\sin\theta u_\phi u_{tr} - \cos\theta u_\phi\\
&- 4r^2\sin\theta u_{t\phi}u_{tr} - 4k\sin\theta uu_{tr\phi} + 4r\sin\theta uu_{tr\phi} + 2r^2\sin\theta u_t u_{tr\phi}\\
&+ 2kr\sin\theta u_\phi u_{trr} - r^2\sin\theta u_\phi u_{trr} - 4kr\sin\theta uu_{trr\phi} + 2r^2\sin\theta uu_{trr\phi}\\
&- 4r\sin\theta uu_{tt\phi} - 3r^2\sin\theta u_r u_{tt\phi} + 2r^2\sin\theta u_\phi u_{ttr} - r^2\sin\theta uu_{ttr\phi}),
\end{aligned}$$

$$\mathcal{T}^r = \frac{1}{2} r\sin\theta((-2k + r)u_r u_{tt\phi} - ru_t u_{tt\phi} + u((2k - r)u_{ttr\phi} + ru_{ttt\phi})),$$

$$\mathcal{T}^\theta = \frac{1}{2}\sin\theta(u_\theta u_{tt\phi} - uu_{tt\theta\phi}),$$

$$\begin{aligned}
\mathcal{T}^\phi = \frac{1}{6}(&u_{\phi\phi}u_{tt} + \cos\theta u_\theta u_{tt} + \sin\theta u_{\theta\theta}u_{tt} - 2k\sin\theta u_r u_{tt} + 2r\sin\theta u_r u_{tt}\\
&- 2kr\sin\theta u_{rr}u_{tt} + r^2\sin\theta u_{rr}u_{tt} - 2r^2\sin\theta u_{tr}u_{tt} + 3u_\phi u_{tt\phi}
\end{aligned}$$

$$+\cos\theta uu_{tt\theta}+\sin\theta uu_{tt\theta\theta}-2k\sin\theta uu_{ttr}+2r\sin\theta uu_{ttr}-u_t(u_{t\phi\phi}$$
$$+\cos\theta u_{t\theta}+\sin\theta(u_{t\theta\theta}-2(k-r)u_{tr}+r((-2k+r)u_{trr}-2ru_{ttr})))$$
$$-2kr\sin\theta uu_{ttrr}+r^2\sin\theta uu_{ttrr}-2r\sin\theta uu_{ttt}-2r^2\sin\theta uu_{tttr}).$$

(iii) The case $m=t$.

$$\mathcal{X}_1^3=(tu_{t\phi\phi}+ru_{r\phi\phi}+u_{\phi\phi})\partial_u,$$

$$\begin{aligned}\mathcal{T}^t=\frac{1}{6}(&tu_{\phi\phi}^2+tuu_{\phi\phi\phi\phi}+t\cos\theta uu_{\theta\phi\phi}+t\sin\theta uu_{\theta\theta\phi\phi}+6r^2\sin\theta uu_{r\phi\phi}\\&+2rt\sin\theta uu_{r\phi\phi}-2t^2\sin\theta uu_{r\phi\phi}-3r^3\sin\theta u_ru_{r\phi\phi}+3r^3\sin\theta uu_{rr\phi\phi}\\&+r^2t\sin\theta uu_{rr\phi\phi}-2rt^2\sin\theta uu_{rr\phi\phi}-2rt\sin\theta uu_{t\phi\phi}-3r^2t\sin\theta u_{t\phi\phi}\\&+u_{\phi\phi}(t\cos\theta u_\theta+\sin\theta(tu_{\theta\theta}+(-3r^2+2rt-2t^2)u_r+rt((r-2t)u_{rr}\\&-2(u_t+ru_{tr}))))+tu_\phi(-u_{\phi\phi\phi}-\cos\theta u_{\theta\phi}-\sin\theta(u_{\theta\theta\phi}+2(r-t)u_{r\phi}\\&+r((r-2t)u_{rr\phi}-2(u_{t\phi}+ru_{tr\phi}))))+r^2t\sin\theta uu_{tr\phi\phi}),\end{aligned}$$

$$\begin{aligned}\mathcal{T}^r=\frac{1}{6}r(&u_{\phi\phi}^2+uu_{\phi\phi\phi\phi}+\cos\theta uu_{\theta\phi\phi}+\sin\theta uu_{\theta\theta\phi\phi}-4r\sin\theta uu_{r\phi\phi}\\&+10t\sin\theta uu_{r\phi\phi}+3r^2\sin\theta u_ru_{r\phi\phi}-6rt\sin\theta u_ru_{r\phi\phi}-2r^2\sin\theta u_{r\phi\phi}\\&+4rt\sin\theta uu_{rr\phi\phi}-3r^2\sin\theta u_{r\phi\phi}u_t+4r\sin\theta uu_{t\phi\phi}+3rt\sin\theta u_ru_{t\phi\phi}\\&-6t^2\sin\theta u_ru_{t\phi\phi}-3rt\sin\theta u_tu_{t\phi\phi}+u_{\phi\phi}(\cos\theta u_\theta+\sin\theta(u_{\theta\theta}+(5r\\&-8t)u_r+r((r-2t)u_{rr}-5u_t-2ru_{tr})))-u_\phi(u_{\phi\phi\phi}+\cos\theta u_{\theta\phi}\\&+\sin\theta(u_{\theta\theta\phi}+2(r-t)u_{r\phi}+r((r-2t)u_{rr\phi}-2(u_{t\phi}+ru_{tr\phi}))))\\&+r^2\sin\theta uu_{tr\phi\phi}-3rt\sin\theta uu_{tr\phi\phi}+6t^2\sin\theta uu_{tr\phi\phi}+3rt\sin\theta uu_{tt\phi\phi}),\end{aligned}$$

$$\mathcal{T}^\theta=\frac{1}{2}\sin\theta(u_{\phi\phi}u_\theta+u_\theta(ru_{r\phi\phi}+tu_{t\phi\phi})-u(u_{\theta\phi\phi}+ru_{r\theta\phi\phi}+tu_{t\theta\phi\phi})),$$

$$\begin{aligned}\mathcal{T}^\phi=\frac{1}{6}(&-ru_{\phi\phi\phi}u_r-r\cos\theta u_{\theta\phi}u_r-r\sin\theta u_{\theta\theta\phi}u_r+2ru_{\phi\phi}u_{r\phi}-tu_{\phi\phi\phi}u_t\\&+2r\sin\theta u_{\theta\theta}u_{r\phi}+2r^2\sin\theta u_ru_{r\phi}-2rt\sin\theta u_ru_{r\phi}+2r^3\sin\theta u_{r\phi}u_{rr}\\&-4r^2t\sin\theta u_{r\phi}u_{rr}-r^3\sin\theta u_ru_{rr\phi}+2r^2t\sin\theta u_ru_{rr\phi}+2tu_{\phi\phi}u_{t\phi}\\&-t\cos\theta u_{\theta\phi}u_t-t\sin\theta u_{\theta\theta\phi}u_t-4r^2\sin\theta u_{r\phi}u_t-2rt\sin\theta u_{r\phi}u_t\\&+2t^2\sin\theta u_{r\phi}u_t-r^2t\sin\theta u_{rr\phi}u_t+2rt^2\sin\theta u_{rr\phi}u_t+2r\cos\theta u_\theta u_{r\phi}\end{aligned}$$

$$+ 2t\cos\theta u_\theta u_{t\phi} + 2t\sin\theta u_{\theta\theta}u_{t\phi} + 2r^2\sin\theta u_r u_{t\phi} + 4rt\sin\theta u_r u_{t\phi}$$

$$- 4t^2\sin\theta u_r u_{t\phi} + 2r^2 t\sin\theta u_{rr}u_{t\phi} - 4rt^2\sin\theta u_{rr}u_{t\phi}$$

$$- 2rt\sin\theta u_t u_{t\phi} - 4r^3\sin\theta u_{r\phi}u_{tr} - 4r^2 t\sin\theta u_{t\phi}u_{tr} + 2r^3\sin\theta u_{tr\phi}$$

$$+ 2r^2 t\sin\theta u_t u_{tr\phi} + u_\phi(4u_{\phi\phi} + \cos\theta u_\theta + \sin\theta u_{\theta\theta} + 2ru_{r\phi\phi}$$

$$- r\cos\theta u_{r\theta} - r\sin\theta u_{r\theta\theta} - 3r^2\sin\theta u_{rr} + 4rt\sin\theta u_{rr}$$

$$- r^3\sin\theta u_{rrr} + 2r^2 t\sin\theta u_{rrr} + 2tu_{t\phi\phi} - t\cos\theta u_{t\theta} - t\sin\theta u_{t\theta\theta}$$

$$+ 4r^2\sin\theta u_{tr} - 2rt\sin\theta u_{tr} + 2t^2\sin\theta u_{tr} + 2r^3\sin\theta u_{trr}$$

$$+ (2rt^2 - r^2 t)\sin\theta u_{trr} + 2rt\sin\theta u_{tt} + 2r^2 t\sin\theta u_{ttr}) - u(2u_{\phi\phi\phi}$$

$$- \cos\theta u_{\theta\phi} - \sin\theta u_{\theta\theta\phi} - 6r\sin\theta u_{r\phi} + 6t\sin\theta u_{r\phi} + ru_{r\phi\phi\phi}$$

$$- 2r\cos\theta u_{r\theta\phi} - 2r\sin\theta u_{r\theta\theta\phi} - 9r^2\sin\theta u_{rr\phi} + 14rt\sin\theta u_{rr\phi}$$

$$- 2r^3\sin\theta u_{rrr\phi} + 4r^2 t\sin\theta u_{rrr\phi} + 6r\sin\theta u_{t\phi} + tu_{t\phi\phi\phi}$$

$$- 2t\cos\theta u_{t\theta\phi} - 2t\sin\theta u_{t\theta\theta\phi} + 14r^2\sin\theta u_{tr\phi} + (4t^2 - 4rt)$$

$$\times \sin\theta u_{tr\phi} + 4r^3\sin\theta u_{trr\phi} - 2r^2 t\sin\theta u_{trr\phi} + 4rt^2\sin\theta u_{trr\phi}$$

$$+ 4rt\sin\theta u_{tt\phi} + 4r^2 t\sin\theta u_{ttr\phi})).$$

(iv) The case $m = e^t$.

$$\mathcal{X}_1^4 = u_{\phi\phi\phi}\partial_u,$$

$$\mathcal{T}^t = -\frac{1}{2}r^2\sin\theta(u_{\phi\phi\phi}u_r - uu_{r\phi\phi\phi}),$$

$$\mathcal{T}^r = \frac{1}{2}r\sin\theta(u_{\phi\phi\phi}((-2e^t + r)u_r - ru_t) + u((2e^t - r)u_{r\phi\phi\phi} + ru_{t\phi\phi\phi})),$$

$$\mathcal{T}^\theta = \frac{1}{2}\sin\theta(u_{\phi\phi\phi}u_\theta - uu_{\theta\phi\phi\phi}),$$

$$\mathcal{T}^\phi = \frac{1}{2}({u_{\phi\phi}}^2 + u_{\phi\phi}(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} - 2(e^t - r)u_r + r((-2e^t + r)u_{rr}$$

$$- 2(u_t + ru_{tr})))) - u_\phi(\cos\theta u_{\theta\phi} + \sin\theta(u_{\theta\theta\phi} - 2(e^t - r)u_{r\phi}$$

$$+ r((-2e^t + r)u_{rr\phi} - 2(u_{t\phi} + ru_{tr\phi})))) + u(\cos\theta u_{\theta\phi\phi}$$

$$+ \sin\theta(u_{\theta\theta\phi\phi} - 2(e^t - r)u_{r\phi\phi} + r((-2e^t + r)u_{rr\phi\phi}$$

$$- 2(u_{t\phi\phi} + ru_{tr\phi\phi}))))).$$

(v) The case $m = m(t)$ where $m(t)$ is an arbitrary function of t.

$$\mathcal{X}_1^5 = u_{\phi\phi\phi}\partial_u,$$

$$\mathcal{T}^t = -\frac{1}{2}r^2\sin\theta(u_{\phi\phi\phi}u_r - uu_{r\phi\phi\phi}),$$

$$\mathcal{T}^r = \frac{1}{2}r\sin\theta(u_{\phi\phi\phi}((r-2m(t))u_r - ru_t) + (2m(t)-r)uu_{r\phi\phi\phi} + ru_{t\phi\phi\phi})),$$

$$\mathcal{T}^\theta = \frac{1}{2}\sin\theta(u_{\phi\phi\phi}u_\theta - uu_{\theta\phi\phi\phi}),$$

$$\begin{aligned}\mathcal{T}^\phi = \frac{1}{2}&({u_{\phi\phi}}^2 + u_{\phi\phi}(\cos\theta u_\theta + \sin\theta(u_{\theta\theta} + 2(r-m(t))u_r\\
&+ r((r-2m(t))u_{rr} - 2(u_t + ru_{tr}))))\\
&- u_\phi(\cos\theta u_{\theta\phi} + \sin\theta(u_{\theta\theta\phi} + 2(r-m(t))u_{r\phi}\\
&+ r((r-2m(t))u_{rr\phi} - 2(u_{t\phi} + ru_{tr\phi}))))\\
&+ u(\cos\theta u_{\theta\phi\phi} + \sin\theta(u_{\theta\theta\phi\phi} + 2(r-m(t))u_{r\phi\phi}\\
&+ r((r-2m(t))u_{rr\phi\phi} - 2(u_{t\phi\phi} + ru_{tr\phi\phi}))))).\end{aligned}$$

5. Discussion and conclusion

We have considered the classical wave equation in some Lorentzian spacetime backgrounds with a point in mind that the wave equation there may naturally inherit nonlinearities from the geometry. In this connection, we have considered the Vaidya metric for which a special case is the Papapetrou metric. We have given some symmetry reductions to show how the wave equation there can be either solved or reduced to ordinary differential equations using the method of invariants. Also, some conservation laws were constructed. In his book, Ibragimov [10] suggests that in three-dimensional flat space the linear wave equation admits a 16-dimensional Lie algebra of point symmetries excluding the infinite symmetry. In this study we show that the wave equations admit fewer symmetries when it is solved on a curved manifold. A special case to note is $m = 0$ when the Vaidya manifold is supposedly flat. Manifolds that are flat need not to lead to the wave equation admitting the maximal 16-dimensional Lie algebra of point symmetries. Finally, some higher-order symmetries and associated conservation laws were presented. Solving or analysing the nonlinear wave equation in curved spacetime background using the invariance approach may provide some insight in geometry or relativity for different manifolds.

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