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Exact solutions of the generalized Lane–Emden equations of the first and second kind

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Abstract. In this paper we discuss the integrability of the generalized Lane–Emden equations of the first and second kinds. We carry out their Noether symmetry classification. Various cases for the arbitrary functions in the equations are obtained for which the equations have Noether point symmetries. First integrals of such cases are obtained and also reduction to quadrature of the corresponding Lane–Emden equations are presented. New cases are found.

Keywords. Lane–Emden equations of the first and second kinds; Noether point symmetries; first integrals.

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1. Introduction

In the study of stellar structure the Lane–Emden equation,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2}{x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + y^r = 0, \tag{1}$$

where r is a constant, models the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. This equation was proposed by Lane [1] and studied in detail by Emden [2]. Fowler [3,4] considered a generalization of eq. (1), called Emden–Fowler equation [5], where the last term is replaced by $x^{\nu-1}y^r$.

It has been reported that only for $r = 0$, 1 and 5 does eq. (1) have exact solutions (see, for example Chandrasekhar [6], Davis [7], Datta [8] and Wrubel [9]). Usually, for $r = 5$, only a one-parameter family of solutions is presented [10].

Several methods such as numerical, perturbation, Adomian decomposition, homotopy analysis, power series, differential transformation and variational approach were used to solve eq. (1) (see, for example, the works of Shang *et al* [11], Dehghan and Shakeri [12], Ramos [13], Marzban *et al* [14], Ertürk [15], Horedt [16,17], Bender [18], Lima [19],

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Roxburgh and Stockman [20], Adomian *et al* [21], Shawagfeh [22], Burt [23], Wazwaz [24] and Liao [25]).

Other applications of the Lane–Emden equation (1) can be found in [26–29].

For an isothermal gaseous sphere where the temperature remains constant, Emden [2] also studied the equation,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2}{x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{\beta y} = 0, \tag{2}$$

where β is a constant. Recently, Momoniat and Harley [30] obtained an approximate implicit solution for eq. (2) with $\beta = 1$.

The modified Emden equation,

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0,$$

was investigated for first integrals by Leach [31]. Moreover, transformation properties of a more general Emden–Fowler equation were considered in Mellin *et al* [5]. A review paper by Wong [32] contains more than 140 references concerning the Emden–Fowler equation. Some work on symmetries and solutions of Lane–Emden-type equations can be found in the works of Kara and Mahomed [33,34], Berkovich [35], Govinder and Leach [36] and Bozhkov and Martins [37,38]. For the applications of symmetries (Lie group analysis) to differential equations, the interested reader is referred to [39–41].

Goenner and Havas [42] and Goenner [43] studied the so-called generalized Lane–Emden equation of the first kind,

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha\frac{\mathrm{d}y}{\mathrm{d}x} + \beta x^{\nu} y^n = 0, \tag{3}$$

and generalized Lane–Emden equation of the second kind,

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha\frac{\mathrm{d}y}{\mathrm{d}x} + \beta x^{\nu} \mathrm{e}^{ny} = 0, \tag{4}$$

where α, β, ν and n are constants (see also Harley and Momoniat [44,45]). In Goenner [43], symmetries of eq. (3) were uncovered to explain integrability of (3) for certain values of the parameters considered in Goenner and Havas [42]. Recently, Khalique and Ntsime [46] obtained exact solutions of the Lane–Emden-type equation,

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + n\frac{\mathrm{d}y}{\mathrm{d}x} + x^{\nu} f(y) = 0, \tag{5}$$

for some particular values of the function $f(y)$.

In this paper we study the more generalized versions of eqs (3) and (4), namely, the generalized Lane–Emden equation of the first kind,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{n}{t}\frac{\mathrm{d}u}{\mathrm{d}t} + f(t)u^p = 0, \tag{6}$$

and the generalized Lane–Emden equation of the second kind,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{n}{t}\frac{\mathrm{d}u}{\mathrm{d}t} + g(t)\mathrm{e}^{qu} = 0, \tag{7}$$

where n, p, q are real constants and $f(t)$ and $g(t)$ are arbitrary real-valued functions. We carry out the Noether symmetry classification and integrability of (6) and (7).

Here we emphasize an important fact about the usefulness of a Noether point symmetry for a second-order ordinary differential equation, namely, that a single Noether point symmetry enables complete integration to quadratures in contrast to a single Lie point symmetry which, in general, may not lead to double reduction.

The paper is structured as follows. In §2 we briefly recall the preliminaries of the Noether point symmetry approach. We provide the Noether symmetry classification of eqs (6) and (7) for various functions $f(t)$ and $g(t)$, which is done for the first time here in §3 and 4, respectively. Furthermore, in §3 and 4 we determine the double reductions of eqs (6) and (7) for the functions for which (6) and (7) have Noether point symmetries. Concluding remarks are presented in §5.

2. Preliminaries

Here we recall some definitions and theorems concerning Noether point symmetries (see [46]).

The first prolongation of the vector field,

$$X = \tau(t,u)\frac{\partial}{\partial t} + \xi(t,u)\frac{\partial}{\partial u}, \tag{8}$$

is given by

$$X^{[1]} = X + (\xi_t + (\xi_u - \tau_t)\dot{u} - \tau_u\dot{u}^2)\frac{\partial}{\partial \dot{u}}, \tag{9}$$

where dot denotes total time derivative with respect to t.

Suppose the second-order ordinary differential equation,

$$\ddot{u} = M(t,u,\dot{u}), \tag{10}$$

has a first-order Lagrangian $L(t,u,\dot{u})$. This means that eq. (10) is equivalent to the Euler–Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = 0. \tag{11}$$

Definition 1 (Noether point symmetry). The vector field X is called a Noether point symmetry generator corresponding to a first-order Lagrangian $L(t,u,\dot{u})$ of eq. (10) if there exists a gauge function $B(t,u)$ such that

$$X^{[1]}(L) + D(\tau)L = D(B), \tag{12}$$

where D is the total differentiation operator defined by [41]

$$D = \frac{\partial}{\partial t} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \cdots . \tag{13}$$

The importance of the existence of a Noether point symmetry is given in the following theorems.

Theorem 1 [47]. *If X is a Noether point symmetry generator corresponding to a Lagrangian $L(t, u, \dot{u})$ of eq.* (10) *then*

$$I = \tau L + (\xi - \dot{u}\tau)\frac{\partial L}{\partial \dot{u}} - B, \tag{14}$$

is a Noether first integral of eq. (10) *associated with the operator X.*

Proof. See, for example [48,49]. □

Theorem 2. *The Noether first integral I associated with the Noether point symmetry X satisfies*

$$X^{[1]} I = 0, \tag{15}$$

that is, X is a Lie point symmetry generator of the first integral I of eq. (10).

Proof. See, for example [49–52]. □

Theorem 3. *If for a Lagrangian $L(t, u, \dot{u})$ of eq.* (10) *there corresponds a Noether point symmetry generator, then eq.* (10) *is solvable by means of quadratures.*

Proof. One can deduce this from Theorems 1 and 2. □

3. Noether symmetries and integration of the generalized Lane–Emden equation of the first kind

Consider eq. (6), namely,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{n}{t}\frac{\mathrm{d}u}{\mathrm{d}t} + f(t)u^p = 0.$$

We consider two cases separately.

3.1 $p \neq -1$

It can easily be verified that the standard Lagrangian of this equation is

$$L = \frac{1}{2}t^n \dot{u}^2 - t^n f(t)\frac{u^{p+1}}{p+1}. \tag{16}$$

The insertion of L from (16) into eq. (12) and separation with respect to powers of $\dot{u}$ yields a linear overdetermined system of four partial differential equations.

$$\tau_u = 0,$$
$$\xi_u = \frac{1}{2}(\tau_t - nt^{-1}\tau), \quad (17)$$
$$B_u = t^n \xi_t - \tau_u t^n f(t)\frac{u^{p+1}}{p+1}, \quad (18)$$
$$B_t = -n\tau t^{n-1} f(t)\frac{u^{p+1}}{p+1} - t^n \tau \dot{f}(t)\frac{u^{p+1}}{p+1} - \xi t^n f(t) u^p$$
$$-\tau_t t^n f(t)\frac{u^{p+1}}{p+1}. \quad (19)$$

The solution of the above system gives

$$\tau = a(t),$$
$$\xi = \frac{1}{2}[\dot{a} - nt^{-1}a]u + b(t), \quad (20)$$
$$B = \frac{1}{4}t^n\left[\ddot{a} - n\left(\frac{a}{t}\right)^{\cdot}\right]u^2 + \dot{b}t^n u + c(t) \quad (21)$$

and

$$-n\tau t^{n-1} f(t)\frac{u^{p+1}}{p+1} - t^n \tau \dot{f}(t)\frac{u^{p+1}}{p+1} - \xi t^n f(t) u^p - \tau_t t^n f(t)\frac{u^{p+1}}{p+1}$$
$$= \frac{1}{4}(\ddot{a})^{\cdot} t^n u^2 + \frac{1}{2}nt^{n-2}\dot{a}u^2 - \frac{1}{2}nt^{n-3}au^2 - \frac{1}{4}n^2 t^{n-1}\left(\frac{a}{t}\right)^{\cdot} u^2$$
$$+ \ddot{b}t^n u + \dot{b}nt^{n-1}u + \dot{c}. \quad (22)$$

The analysis of eq. (22) results in seven cases for the function $f(t)$. For each case we present the Noether point symmetry, gauge function, first integral and the solution to the corresponding equation in table 1.

3.2 $p = -1$

We now consider the case when $p = -1$. In this case the generalized Lane–Emden equation of the first kind (6) becomes

$$\frac{d^2u}{dt^2} + \frac{n}{t}\frac{du}{dt} + \frac{f(t)}{u} = 0, \quad (23)$$

which has a natural Lagrangian,

$$L = \frac{1}{2}t^n\dot{u}^2 - t^n f(t)\ln u. \quad (24)$$

Following the above procedure we obtain seven cases for the function $f(t)$ which gives Noether point symmetries. The results are summarized in table 2.

Table 1. Functions, Noether symmetries, gauge functions, first integrals and solutions of the generalized Lane–Emden equation of the first kind for the case $p \neq -1$.

Case	Function, Noether symmetry, gauge function, first integral, solution
1	$f(t) = \alpha$ a constant. We retrieve the two cases obtained in [53]
2	$f(t) = \alpha t^{-2n}$, $n \neq 1$, α a constant $X = t^n \frac{\partial}{\partial t}$, $B = 0$ $I = \frac{1}{2} t^{2n} \dot{u}^2 + \alpha \frac{u^{p+1}}{p+1}$ $\int \frac{\mathrm{d}u}{\pm\sqrt{2C_1 - 2\alpha(p+1)^{-1} u^{p+1}}} = \frac{t^{1-n}}{1-n} + C_2$, where C_1, C_2 are constants
3	$f(t) = \alpha t^{\frac{1}{2}(np-n-p-3)}$, $n \neq 1$, α a constant $X = 2t \frac{\partial}{\partial t} + (1-n) u \frac{\partial}{\partial u}$, $B = 0$ $I = t^{n+1} \dot{u}^2 + \frac{2\alpha}{p+1} t^{\frac{1}{2}(p+1)(n-1)} u^{p+1} - (1-n) t^n u \dot{u}$ $\int \frac{\mathrm{d}y}{\pm\sqrt{C_1 - \alpha(p+1)^{-1} y^{p+1} + (1-n)^2 y^2/4}} = \ln C_2 t$, where $y = u t^{\frac{1}{2}(n-1)}$
4	$f(t) = \alpha t^{np+n-p-3}$, $n \neq 1$, α a constant $X = t^{2-n} \frac{\partial}{\partial t} + (1-n) t^{1-n} u \frac{\partial}{\partial u}$, $B = \frac{1}{2}(n-1)^2$ $I = \frac{1}{2}(n-1)^2 u^2 + \frac{1}{2} t^2 \dot{u}^2 + \frac{\alpha}{p+1} t^{(p+1)(n-1)} u^{p+1} - (1-n) t u \dot{u}$ $\int \frac{\mathrm{d}y}{\pm\sqrt{2C_1 - 2\alpha(p+1)^{-1} y^{p+1}}} = \frac{t^{n-1}}{n-1} + C_2$, where $y = u t^{n-1}$
5	$f(t) = \alpha t^{-2}$, $n = 1$ and α a constant $X = t \frac{\partial}{\partial t}$, $B = 0$ $I = \frac{1}{2} t^2 \dot{u}^2 + \alpha \frac{u^{p+1}}{p+1}$ $\int \frac{\mathrm{d}u}{\pm\sqrt{2C_1 - 2\alpha(p+1)^{-1} u^{p+1}}} = \ln C_2 t$
6	$f(t) = \alpha t^{-2} (\ln t)^{-\frac{1}{2}(p+3)}$, $n = 1$ and α a constant $X = t \ln t \frac{\partial}{\partial t} + \frac{1}{2} u \frac{\partial}{\partial u}$, $B = 0$ $I = \frac{1}{2} t^2 \ln t \dot{u}^2 + \frac{\alpha}{p+1} (\ln t)^{-\frac{1}{2}(p+1)} u^{p+1} - \frac{1}{2} u \dot{u} t$ $\int \frac{\mathrm{d}y}{\pm\sqrt{2C_1 - 2\alpha(p+1)^{-1} y^{p+1} + y^2/4}} = \ln(\ln t) + C_2$, where $y = u(\ln t)^{-\frac{1}{2}}$
7	$f(t) = \alpha t^{-2} (\ln t)^{-(p+3)}$, $n = 1$ and α a constant $X = t(\ln t)^2 \frac{\partial}{\partial t} + (\ln t) u \frac{\partial}{\partial u}$, $B = \frac{1}{2} u^2$ $I = \frac{1}{2} u^2 + \frac{1}{2} t^2 (\ln t)^2 \dot{u}^2 + \frac{\alpha}{p+1} (\ln t)^{-(p+1)} u^{p+1} - (\ln t) u \dot{u} t$ $\int \frac{\mathrm{d}y}{\pm\sqrt{2C_1 - 2\alpha(p+1)^{-1} y^{p+1}}} = -(\ln t)^{-1} + C_2$, where $y = u(\ln t)^{-1}$

Table 2. Functions, Noether symmetries, gauge functions, first integrals and solutions of the generalized Lane–Emden equation of the first kind for the case $p = -1$.

Case	Function, Noether symmetry, gauge function, first integral, solution
1	$f(t) = \alpha$ a constant. This case is subsumed in Case 2 below
2	$f = \alpha t^{-n-1},\ n \neq 1, \alpha$ a constant $X = t\frac{\partial}{\partial t} + \frac{1}{2}(1-n)u\frac{\partial}{\partial u},\ B = \frac{\alpha}{2}(n-1)\ln t$ $I = \frac{1}{2}t^{n+1}\dot{u}^2 + \frac{(n-1)}{2}\dot{u}ut^n + \alpha\ln u + \frac{\alpha(n-1)}{2}\ln t$ $\int \frac{\mathrm{d}y}{\pm\sqrt{2C_1 - 2\alpha\ln y + (1-n)^2y^2/4}} = \ln C_2 t$, where $y = ut^{\frac{1}{2}(n-1)}$
3	$f(t) = \alpha t^{-2n},\ n \neq 1$ and α a constant $X = t^n\frac{\partial}{\partial t},\ B = 0$ $I = \frac{1}{2}t^{2n}\dot{u}^2 + \alpha\ln u$ $\int \frac{\mathrm{d}u}{\pm\sqrt{2C_1 - 2\alpha\ln u}} = \frac{t^{1-n}}{1-n} + C_2$
4	$f(t) = \alpha t^{-2},\ n \neq 1$ and α a constant $X = t^{2-n}\frac{\partial}{\partial t} + (1-n)t^{1-n}u\frac{\partial}{\partial u},\ B = \frac{1}{2}(n-1)^2 + \alpha(n-1)\ln t$ $I = \frac{1}{2}(n-1)^2u^2 + \frac{1}{2}t^2\dot{u}^2 + \alpha\ln u + (n-1)tu\dot{u} + \alpha(n-1)\ln t$ $\int \frac{\mathrm{d}y}{\pm\sqrt{2C_1 - 2\alpha\ln y}} = \frac{t^{n-1}}{n-1} + C_2$, where $y = ut^{n-1}$
5	$f(t) = \alpha t^{-2}, n = 1$ and α a constant $X = t\frac{\partial}{\partial t},\ B = 0$ $I = \frac{1}{2}t^2\dot{u}^2 + \alpha\ln u$ $\int \frac{\mathrm{d}u}{\pm\sqrt{2C_1 - 2\alpha\ln u}} = \ln C_2 t$
6	$f(t) = \alpha t^{-2}(\ln t)^{-1},\ n = 1$ and α a constant $X = t\ln t\frac{\partial}{\partial t} + \frac{1}{2}u\frac{\partial}{\partial u},\ B = -\frac{\alpha}{2}\ln(\ln t)$ $I = \frac{1}{2}t^2\dot{u}^2\ln t + \alpha\ln u - \frac{1}{2}tu\dot{u} - \frac{\alpha}{2}\ln(\ln t)$ $\int \frac{\mathrm{d}y}{\pm\sqrt{2C_1 - 2\alpha\ln y + y^2/4}} = \ln(\ln t) + C_2$, where $y = u(\ln t)^{-\frac{1}{2}}$
7	$f(t) = \alpha t^{-2}(\ln t)^{-2},\ n = 1$ and α a constant. $X = t(\ln t)^2\frac{\partial}{\partial t} + (\ln t)u\frac{\partial}{\partial u},\ B = \frac{1}{2}u^2 - \alpha\ln(\ln t)$ $I = \frac{1}{2}u^2 + \frac{1}{2}\dot{u}^2t^2(\ln t)^2 + \alpha\ln u - \alpha\ln(\ln t) - (\ln t)u\dot{u}t$ $\int \frac{\mathrm{d}y}{\pm\sqrt{2C_1 - 2\alpha\ln y}} = -(\ln t)^{-1} + C_2$, where $y = u(\ln t)^{-1}$

Table 3. Functions, Noether symmetries, gauge functions, first integrals and solutions of the generalized Lane–Emden equation of the second kind.

Case	Function, Noether symmetry, gauge function, first integral, solution
1	$g(t) = \lambda, \lambda$ a constant. In this case we recover the results obtained in [53]
2	$g(t) = \lambda t^{-2n},\ n \neq 1$ and λ is a constant $X = t^n \frac{\partial}{\partial t}, B = 0$ $I = \frac{1}{2} t^{2n} u'^2 - \frac{\lambda}{q} \mathrm{e}^{qu}$ $\int \frac{\mathrm{d}u}{\pm\sqrt{2C_1 - 2\mathrm{e}^{qu}/q}} = \frac{t^{1-n}}{1-n} + C_2$
3	$g(t) = \lambda t^{\beta-1}, n = 1, \beta$ and λ are constants $X = t\frac{\partial}{\partial t} - \frac{(\beta+1)}{q}\frac{\partial}{\partial u}, B = 0$ $I = \frac{1}{2} t^2 \dot{u}^2 + \frac{\lambda}{q} t^{\beta+1} \mathrm{e}^{qu} + \frac{(\beta+1)}{q} t\dot{u}$ $\int \frac{\mathrm{d}y}{\pm\sqrt{1 + 2C_1 q^2 (\beta+1)^{-2} - 2\lambda q(\beta+1)^{-2} y^{\beta+1}}} = \ln C_2 t,\ y = t\exp\left(\frac{qu}{\beta+1}\right)$

4. Noether symmetries and integration of the generalized Lane–Emden equation of the second kind

In this section we consider the integrability of the generalized Lane–Emden equation of the second kind, namely,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \frac{n}{t}\frac{\mathrm{d}u}{\mathrm{d}t} + g(t)\mathrm{e}^{qu} = 0, \tag{25}$$

which has a standard Lagrangian

$$L = \frac{1}{2} t^n \dot{u}^2 - \frac{t^n}{q} g(t)\mathrm{e}^{qu}.$$

Carrying out similar procedure as for the generalized Lane–Emden equation of the first kind, we obtain three cases of $g(t)$ for which the Noether point symmetries exist. The results are summarized in table 3.

5. Concluding remarks

We have completely classified the Noether point symmetries of the generalized Lane–Emden equations of the first and second kinds with respect to their standard Lagrangians. We obtained various cases of functions $f(t)$ and $g(t)$ which resulted in Noether point symmetries. For each of these cases we obtained the first integral and performed reduction to quadrature of the corresponding Lane–Emden equation. To the best of our knowledge, these results are obtained for the first time here. We also made use of the important fact that an available single Noether point symmetry of a scalar second-order ordinary differential equation results in complete reduction to quadrature. This is not, in general, true for a single Lie point symmetry.

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