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Transient heat transfer in longitudinal fins of various profiles with temperature-dependent thermal conductivity and heat transfer coefficient

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Abstract. Transient heat transfer through a longitudinal fin of various profiles is studied. The thermal conductivity and heat transfer coefficients are assumed to be temperature dependent. The resulting partial differential equation is highly nonlinear. Classical Lie point symmetry methods are employed and some reductions are performed. Since the governing boundary value problem is not invariant under any Lie point symmetry, we solve the original partial differential equation numerically. The effects of realistic fin parameters such as the thermogeometric fin parameter and the exponent of the heat transfer coefficient on the temperature distribution are studied.

Keywords. Heat transfer; longitudinal fin; temperature-dependent heat transfer coefficient and thermal conductivity; symmetry analysis; numerical solutions.

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1. Introduction

A search for exact and numerical solutions for models arising in heat flow through extended surfaces continues to be of scientific interest. The literature in this area is immense (see, for example, [1] and references cited therein). Perhaps such interest has been instilled by frequent encounters of fin problems in many engineering applications to enhance heat transfer. In recent years many authors have been interested in the steady-state problems [2–11] describing heat flow in fins of different shapes and profiles. Exact solutions exist when both the thermal conductivity and heat transfer coefficients are constant [2], and even when they are not constant provided thermal conductivity is a differential consequence of the heat transfer coefficient [5,6]. In the heat transfer models describing natural convection, radiation, boiling and condensation, the heat transfer coefficient depends on local temperature. Furthermore, for engineering applications and physical phenomena the thermal conductivity of a fin is assumed to be linearly dependent on temperature (see [12]). The dependency of both thermal conductivity and the heat transfer coefficient on temperature renders the problem highly nonlinear.

The transient heat transfer problems where thermal conductivity is temperature dependent and the heat transfer coefficient which depends on the spatial variable have also attracted some attention (see [12]). Subsequently, symmetry analysts considered the problem in [12] to determine all forms of thermal conductivity and heat transfer coefficients for which the governing equation admits extra symmetries [13–16]. However, only general solutions were constructed. An accurate transient analysis provided insight into the design of fins that would fail in steady-state operations but are sufficient for desired operating periods [17]. Worth noting is the earlier work in [18] wherein the transient problem is considered for a fin of arbitrary profile. However, both thermal conductivity and heat transfer coefficient are considered to be constants.

In this paper we study the heat transfer in longitudinal fins of different profiles. Furthermore, both the thermal conductivity and heat transfer coefficients are temperature dependent. The mathematical formulation is given in §2. We employ symmetry techniques to analyse the resulting model in §3. Due to the non-existence of exact solutions we seek numerical solutions in §4. In §5 we provide concluding remarks.

2. Mathematical formulation

We consider a longitudinal one-dimensional fin with a profile area $A_{\rm p}$. The perimeter of the fin is denoted by P and the length of the fin by L. The fin is attached to a fixed base surface of temperature T_b and extends into a fluid of temperature T_a. The fin profile is given by the function $F(X)$ and the fin thickness at the base is δ_b. The energy balance for a longitudinal fin is given by

$$\rho c_{\rm v}\frac{\partial T}{\partial t} = A_{\rm p}\frac{\partial}{\partial X}\left(F(X)K(T)\frac{\partial T}{\partial X}\right) - P\delta_b H(T)(T-T_a)\,, \quad 0<X<L, \tag{1}$$

where K and H are the non-uniform thermal conductivity and heat transfer coefficients depending on the temperature (see [2,3,6,8]), ρ is the density, $c_{\rm v} = 2c/(\delta_b A_{\rm p})$ is the volumetric heat capacity with c being the specific heat capacity, T is the temperature distribution, $F(X)$ is the fin profile, t is the time and X is the spatial variable. The fin length is measured from the tip to the base as shown in figure 1 (see also [1–3]). An insulated fin at one end with the base temperature at the other implies boundary conditions which is given by [1]

$$T(t,L) = T_b \quad \text{and} \quad \left.\frac{\partial T}{\partial X}\right|_{X=0} = 0, \tag{2}$$

and initially the fin is kept at the temperature of the fluid (the ambient temperature),

$$T(0,X) = T_a.$$

The schematic representation of a fin with arbitrary profile is given in figure 1.

Introducing the dimensionless variables and the dimensionless numbers,

$$x=\frac{X}{L},\quad \tau=\frac{k_a t}{\rho c_{\rm v} L^2},\quad \theta=\frac{T-T_a}{T_b-T_a},\quad k=\frac{K}{k_a},\quad h=\frac{H}{h_b},\quad \mathcal{M}^2=\frac{2Ph_bL^2}{A_{\rm p}k_a}$$

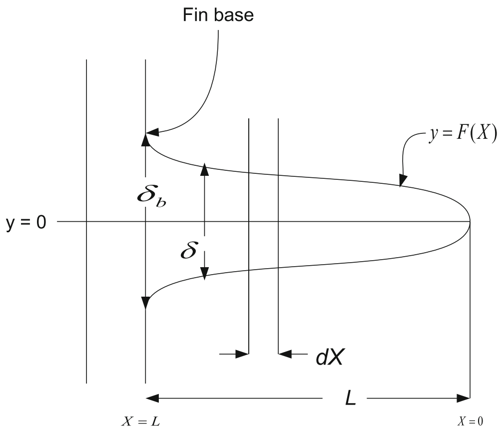

Figure 1. Schematic representation of a longitudinal fin with arbitrary profile $F(X)$.

and

$$f(x) = \frac{2}{\delta_b} F(X), \tag{3}$$

reduces eq. (1) to

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(f(x) k(\theta) \frac{\partial \theta}{\partial x} \right) - \mathcal{M}^2 h(\theta)\theta, \quad 0 < x < 1, \tag{4}$$

and the initial and boundary conditions become

$$\theta(0, x) = 0, \ 0 \leq x \leq 1; \quad \theta(\tau, 1) = 1, \ \tau > 0; \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0, \ \tau \geq 0. \tag{5}$$

Here $\mathcal{M}$ is the thermogeometric fin parameter, δ_b is the fin thickness at the base, δ is the fin thickness, θ is the dimensionless temperature, x is the dimensionless spatial variable, $f(x)$ is the dimensionless fin profile, τ is the dimensionless time, k is the dimensionless thermal conductivity, k_a is the thermal conductivity of the fin at the ambient temperature, h is the dimensionless heat transfer coefficient and h_b is the heat transfer coefficient at the fin base. For most industrial applications the heat transfer coefficient may be given as the power law [2,19],

$$H(T) = h_b \left(\frac{T - T_a}{T_b - T_a} \right)^n, \tag{6}$$

where the exponent n and h_b are constants. The constant n may vary between -6.6 and 5. However, in most practical applications it lies between -3 and 3 [19]. If the heat transfer coefficient is given by eq. (6), then the hypothetical boundary condition (that is, insulation) at the tip of the fin is taken into account [19]. If the tip is not assumed to be insulated then the problem becomes overdetermined (see also [20]). This boundary condition is realized for sufficiently long fins [19]. Also, the heat transfer through the outermost edge of the fin is negligible compared to that which passes through the side [20]. The exponent n represents laminar film boiling or condensation when $n = -1/4$, laminar natural convection when $n = 1/4$, turbulent natural convection when $n = 1/3$, nucleate boiling when $n = 2$, radiation when $n = 3$. $n = 0$ implies a constant heat transfer coefficient. Exact solutions may be constructed for the steady-state one-dimensional differential equation describing temperature distribution in a straight fin when the thermal conductivity is a constant and $n = -1, 0, 1$ and 2 [19].

The thermal conductivity of the fin may be assumed to vary linearly with the temperature for many engineering applications [2,12], that is,

$$K(T) = k_a[1 + \beta(T - T_a)],$$

where β is the thermal conductivity gradient. The one-dimensional transient heat conduction equation is then given by

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left[f(x)(1 + B\theta)\frac{\partial \theta}{\partial x}\right] - \mathcal{M}^2\theta^{n+1}, \quad 0 < x < 1, \tag{7}$$

where the thermal conductivity parameter $B = \beta(T_b - T_a)$ is non-zero.

3. Classical Lie point symmetry analysis

In brief, the symmetry of a differential equation is an invertible transformation of the dependent and independent variables that does not change the original differential equation. Symmetries depend continuously on a parameter and form a group; the one-parameter group of transformations. This group can be determined algorithmically. The theory and applications of Lie groups may be obtained in excellent references such as [21–23].

We omit further theoretical discussions but list the Lie point symmetries admitted by the fin models of different profiles in table 1. The time translation admitted and listed in table 1 reduces eq. (7) to a steady-state problem. The nonlinearity of eq. (7) is reduced when $n = -1$ since the term involving $\mathcal{M}$ has no dependent or even independent variables. This also leads to extra Lie point symmetries being admitted.

3.1 *Symmetry reductions: Some illustrative examples*

The obtained symmetries may be used to reduce the number of variables of the governing equation by one. Symmetries reduce a 1+1-dimensional partial differential equation to an ordinary differential equation. The reduced equation may or may not be exactly solvable.

3.1.1 *Rectangular case.* The symmetry generator $\mathcal{Y}_1$ implies the steady-state heat transfer. This case will be studied in detail elsewhere. The linear combination of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ leads to a travelling wave solution of the form

$$\theta = G(\gamma), \quad \gamma = x \pm a\tau,$$

Table 1. Classical Lie point symmetries admitted by eq. (7).

Fin profile $f(x)$	Parameter n	Symmetries
Rectangular	Arbitrary	$\mathcal{Y}_1 = \frac{\partial}{\partial \tau}, \ \mathcal{Y}_2 = \frac{\partial}{\partial x}$
$f(x) = 1$	$n = -1$	$\mathcal{Y}_3 = \frac{1}{B}\left[(1 + B\theta)\frac{\partial}{\partial \theta} + Bx\frac{\partial}{\partial x} + B\tau\frac{\partial}{\partial \tau}\right]$
Triangular	Arbitrary	$\mathcal{Y}_1 = \frac{\partial}{\partial \tau}$
$f(x) = x$	$n = -1$	$\mathcal{Y}_2 = -\frac{1}{B^2}\left[(1 + B\theta)\frac{\partial}{\partial \theta} + 2Bx\frac{\partial}{\partial x} + B\tau\frac{\partial}{\partial \tau}\right]$
Concave parabolic $f(x) = x^2$	Arbitrary	$\mathcal{Y}_1 = \frac{\partial}{\partial \tau}, \ \mathcal{Y}_2 = -x\frac{\partial}{\partial x}$
Convex parabolic	Arbitrary	$\mathcal{Y}_1 = \frac{\partial}{\partial \tau}$
$f(x) = \sqrt{x}$	$n = -1$	$\mathcal{Y}_2 = -\frac{1}{2B^2}\left[3(1 + B\theta)\frac{\partial}{\partial \theta} + 4Bx\frac{\partial}{\partial x} + 3B\tau\frac{\partial}{\partial \tau}\right]$

where a is a constant representing the wave speed and G satisfies the ordinary differential equation,

$$(1 + BG)G'' + B(G')^2 \pm aG' - \mathcal{M}^2 G^{n+1} = 0. \tag{8}$$

The prime indicates the derivative with respect to γ. We observe that the initial and boundary conditions (5) do not reduce to two boundary condition for $G(\gamma)$. Note that eq. (8) with arbitrary n may be integrated to quadratures since it admits a translation of γ. On the other hand, eq. (8) with $n = -1$ admits a two-dimensional non-Abelian Lie subalgebra spanned by the base vectors

$$\Gamma_1 = \frac{\partial}{\partial \gamma} \quad \text{and} \quad \Gamma_2 = \frac{1 + BG}{B}\frac{\partial}{\partial G} + \gamma\frac{\partial}{\partial \gamma}.$$

Since the symmetry Lie algebra is two-dimensional, eq. (8) with $n = -1$ is integrable [24]. This non-commuting pair of symmetries lead to the canonical forms

$$\varphi = BG + 1 \quad \text{and} \quad \omega = BG + \gamma + 1.$$

The corresponding canonical forms of the vectors are given by

$$\mathbf{v_1} = \frac{\partial}{\partial \omega} \quad \text{and} \quad \mathbf{v_2} = \varphi\frac{\partial}{\partial \varphi} + \omega\frac{\partial}{\partial \omega}.$$

Writing $\omega = \omega(\varphi)$ and considering $a > 0$ transforms eq. (8) into

$$\omega'' = \frac{\omega' - 1}{\varphi}\left[1 + a(\omega' - 1) - \mathcal{M}^2 B(\omega' - 1)^2\right], \tag{9}$$

where the prime indicates the derivative with respect to φ. Equation (9) is not linearizable since the Lie criterion for liberalization (see [24]) is not satisfied. However, three cases arise for the exact (invariant) solution.

(a) $\omega' - 1 = 0$ leads to a trivial solution which is not related to the original problem. Therefore we ignore it.
(b) If the sum of the terms in the bracket in (9) vanishes, then we obtain in terms of the original variables the general exact solution,

$$\theta = \frac{2\mathcal{M}^2}{-a \pm \sqrt{a^2 + 4\mathcal{M}^2 B}} \left\{ x + a\tau - \frac{a \pm \sqrt{a^2 + 4\mathcal{M}^2 B}}{2\mathcal{M}^2 B} + c_1 \right\},$$

where c_1 is an arbitrary constant. Note that this solution does not satisfy the initial and boundary conditions (5).
(c) The solution for the entire eq. (9) is given in terms of quadratures. We omit such a solution in this paper.

The admitted symmetry generator $\mathcal{Y}_3$ listed in table 1 leads to the functional form of the invariant solution,

$$\theta = \frac{1}{B}\left(\tau G(\gamma) - 1\right), \quad \gamma = \frac{x}{\tau},$$

and G satisfies the ordinary differential equation,

$$GG'' + (G')^2 + \gamma G' - G - \mathcal{M}^2 = 0,$$

which has no Lie point symmetries. Furthermore, we again find that the initial and boundary conditions (5) do not reduce to two boundary conditions for $G(\gamma)$.

3.1.2 *Triangular case.* The symmetry generator $\mathcal{Y}_2$ leads to the reductions

$$\theta = \frac{1}{B}[\tau G(\gamma) - 1] \quad \text{with} \quad \gamma = \frac{\sqrt{x}}{\tau},$$

where G satisfies the nonlinear ordinary differential equation,

$$\frac{1}{4}GG'' + \left(\frac{1}{2\gamma} - \frac{G}{4\gamma} - \gamma\right)G'\frac{1}{4}G'^2 - G - \mathcal{M}^2 = 0.$$

The initial and boundary conditions (5) do not reduce to two boundary conditions for $G(\gamma)$.

3.1.3 *Concave parabolic case.* The symmetry generator $\mathcal{Y}_2$ yields the functional form of the invariant solution

$$\theta = G(\gamma) \quad \text{with} \quad \gamma = x\mathrm{e}^{\tau},$$

where G satisfies the nonlinear ordinary differential equation,

$$\gamma\frac{\mathrm{d}G}{\mathrm{d}\gamma} = \frac{\mathrm{d}}{\mathrm{d}\gamma}\left[\gamma^2(1 + BG)\frac{\mathrm{d}G}{\mathrm{d}\gamma}\right] - \mathcal{M}^2 G^{n+1}. \tag{10}$$

Again we observe that the initial and boundary conditions (5) do not reduce to two boundary conditions for $G(\gamma)$.

3.1.4 *Convex parabolic case.* The vector field $\mathcal{Y}_2$ leads to the reductions

$$\theta = \frac{1}{B}\,(\tau G(\gamma) - 1) \quad \text{with} \quad \gamma = \frac{x^{1/4}}{\tau^{1/3}},$$

where G satisfies the nonlinear ordinary differential equation,

$$G - \frac{1}{3}\gamma G' = \frac{1}{2}\gamma^5 GG' + \frac{1}{16}\gamma^4 (G')^2 - \frac{3}{16\gamma} GG' + \frac{1}{16}\gamma^4 GG'' - \mathcal{M}^2,$$

which has no Lie point symmetries. Furthermore, the initial and boundary conditions (5) again do not reduce to two boundary conditions for $G(\gamma)$.

Since the original boundary value problem is not invariant under any Lie point symmetry (because the initial and boundary conditions are not invariant), we reconsider the original partial differential equation subject to the prescribed initial and boundary conditions and determine the numerical solutions in the next section.

4. Numerical results

In this section numerical solutions are obtained for eq. (7) subject to the conditions (5) by using the in-built function `pdepe` in MATLAB. We study heat flow in longitudinal fins of different profiles.

4.1 *Rectangular case*

The solutions for this case are depicted in figures 2, 3, 4 and 5. We observe in figure 2 that the temperature increases with an increase in time. We find that the solution profiles for $\tau = 0.5, 0.75, 2$ and $\tau = 0.1, 0.2, 0.3$ indicate a decrease in the temperature as we move away from the base of the fin. The solutions seem to converge to a steady-state solution as time evolves. Furthermore, we notice that the temperature at the tip of the fin increases with time. The effects of thermogeometric fin parameter on temperature distribution are shown in figures 3 and 4. We notice that temperature decreases with increasing values of $\mathcal{M}$.

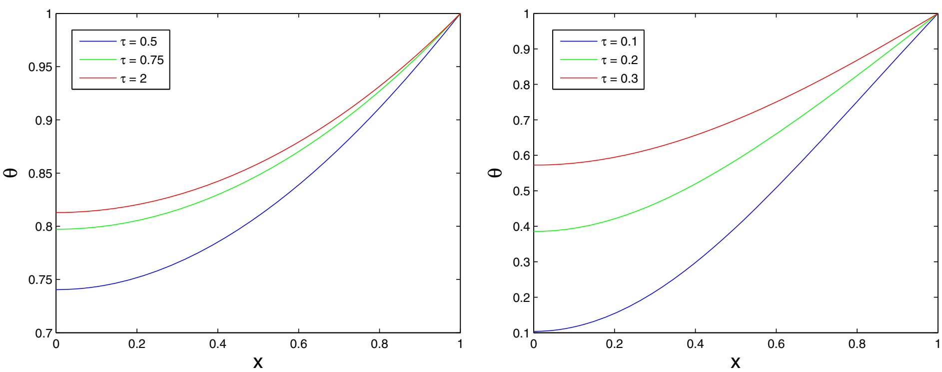

Figure 2. Temperature distribution in a rectangular fin with $B = n = 1$ and $\mathcal{M} = 1$ for varying time.

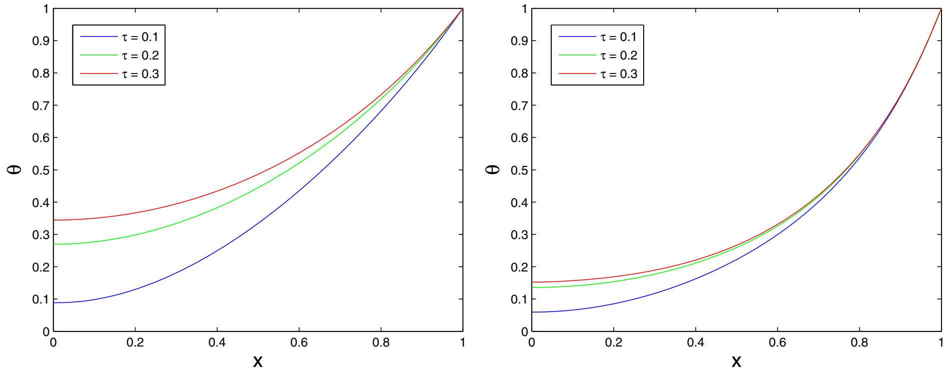

Figure 3. Temperature distribution in a rectangular fin with $B = n = 1$ and with $\mathcal{M} = 3$ (left) and $\mathcal{M} = 6$ (right).

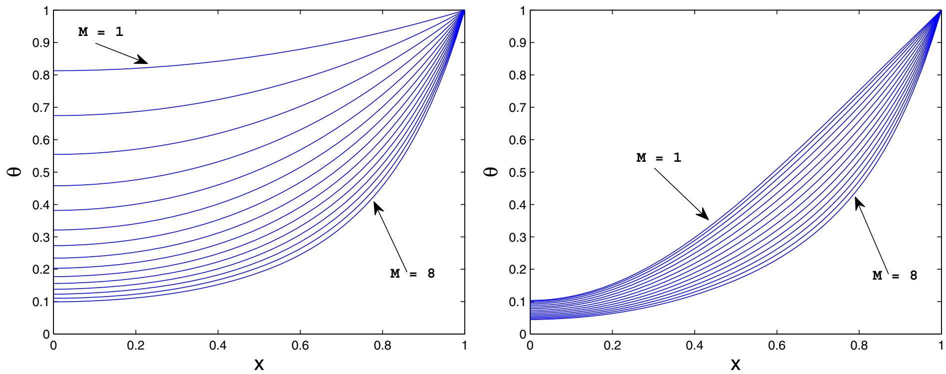

Figure 4. Temperature distribution in a longitudinal rectangular fin for varying values of thermogeometric parameter. Here $B = n = 1$ at $\tau = 2.5$ (left) and $\tau = 0.1$ (right).

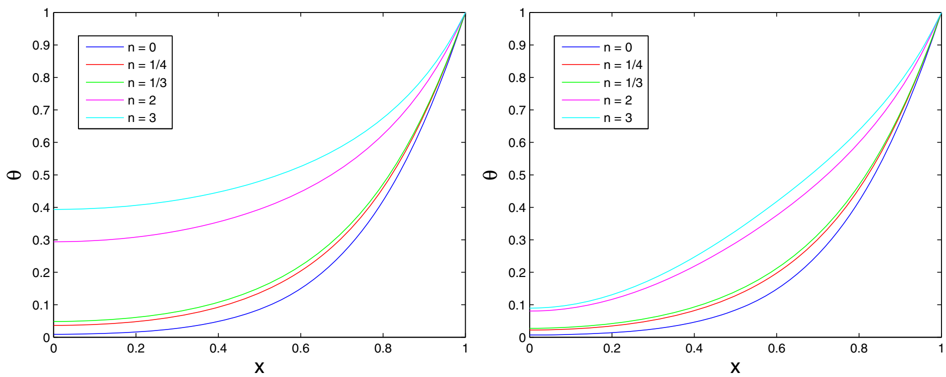

Figure 5. Temperature distribution in a longitudinal rectangular fin for fixed values of B, $\mathcal{M}$ and τ, and varying values of n. Here $B = 1$ and $\mathcal{M} = 6$ at $\tau = 2.5$ (left) and $\tau = 0.1$ (right).

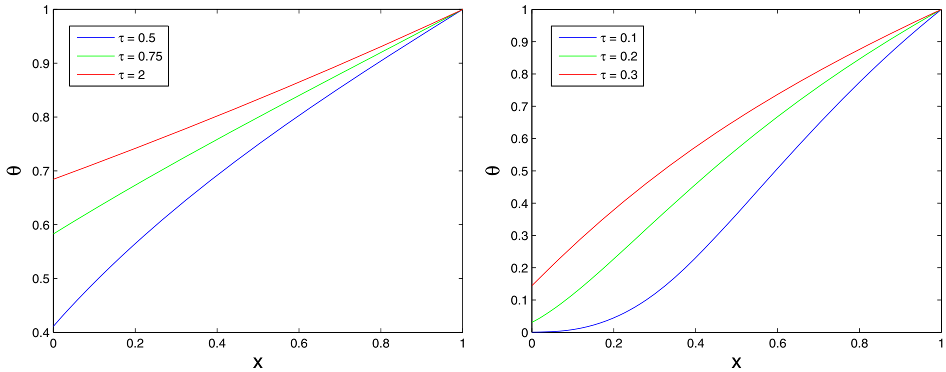

Figure 6. Graphical representation of the numerical solutions for heat transfer in a longitudinal triangular fin with $B = n = 1$ and $\mathcal{M} = 1$.

This asserts the fact that heat transfer through longer fins results in decreased temperatures particularly toward the tip of the fin. Also, the temperature at the tip stays lower at smaller time scales. In figure 5 we note that the temperature increases with increasing values of n.

4.2 *Triangular case*

The solutions for this case are depicted in figures 6, 7 and 8. In figure 6 we observe that at larger time values $\tau = 0.5, 0.75, 2$, the case $\mathcal{M} = 1$, where the derivative condition at the origin is not maintained, indicates that the problem is in fact not physically valid [25]. The effect of the value of fin parameter has been commented on by Yeh and Liaw who discovered the possible occurrence of thermal instability when they considered a steady-state one-dimensional heat conduction equation; they also revealed instances where the problem is not physically valid [25]. Their model indicated that for given values of $\mathcal{M}$ and n the condition of an insulated fin tip may not be satisfied. In our case this seems to only

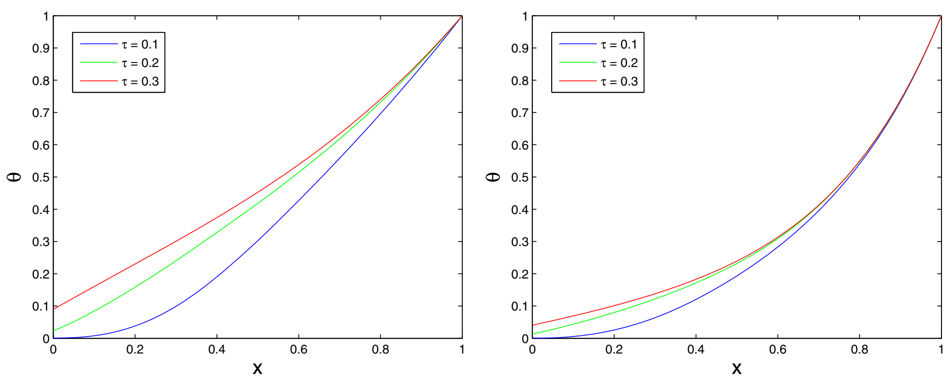

Figure 7. Graphical representation of the numerical solutions for heat transfer in a longitudinal triangular fin with $B = n = 1$ and with $\mathcal{M} = 3$ (left) and $\mathcal{M} = 6$ (right).

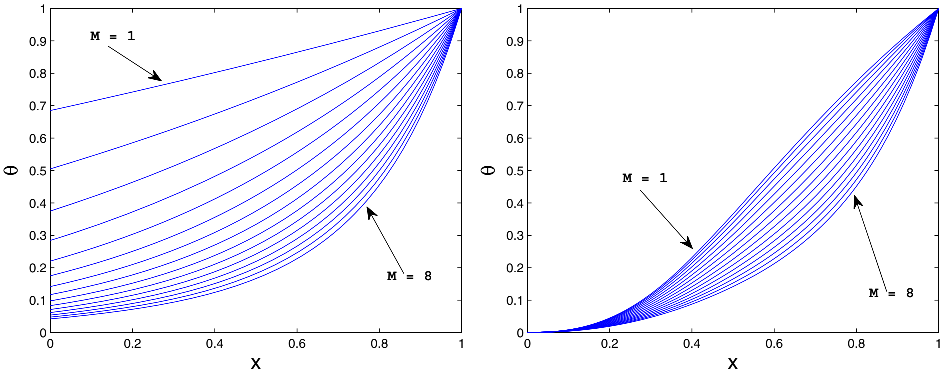

Figure 8. Graphical representation of the numerical solutions for heat transfer in a longitudinal triangular fin with $B = n = 1$ at $\tau = 2.5$ (left) and $\tau = 0.1$ (right) for varying values of $\mathcal{M}$.

occur for small values of $\mathcal{M}$ whereas at larger values of $\mathcal{M}$ or for longer fins this behaviour does not occur. In fact, the relationship between $\mathcal{M}$ and n plays an important role in the stability of the heat transfer in fins and with small values of $\mathcal{M}$ we find that a condition for maintaining stability will become stricter (see [25]). The critical values at which the solution does not maintain the adiabatic condition will be the subject of future research. Figures 7 and 8 show the effects of the thermogeometric fin parameter.

4.3 *Concave parabolic case*

The solutions for this case are depicted in figures 9, 10 and 11. Similar results and observations as in §4.2 are obtained. Again we find that for small values of $\mathcal{M}$ the condition of an insulated fin tip is not maintained, whereas at larger values of $\mathcal{M}$ the solutions adhere to the condition. It seems possible, as in the case of the triangular profile, that for small

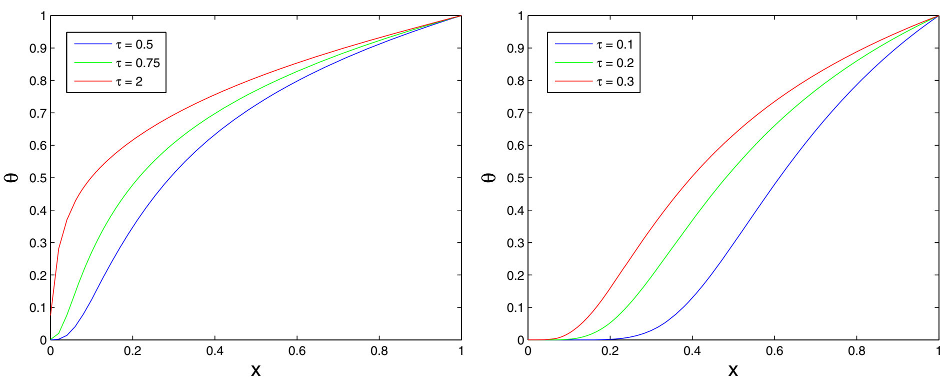

Figure 9. Plots of temperature profile in a longitudinal concave parabolic fin with $B = n = 1$ and $\mathcal{M} = 1$.

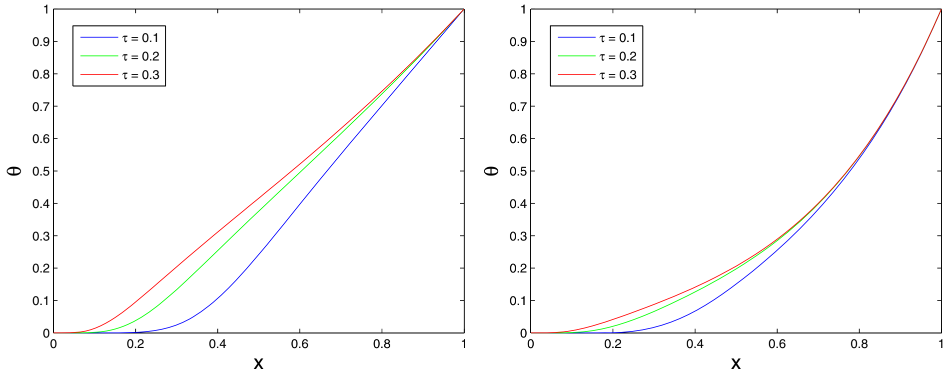

Figure 10. Plots of temperature profile in a longitudinal concave parabolic fin with $B = n = 1$ and $\mathcal{M} = 3$ (left) and $\mathcal{M} = 6$ (right).

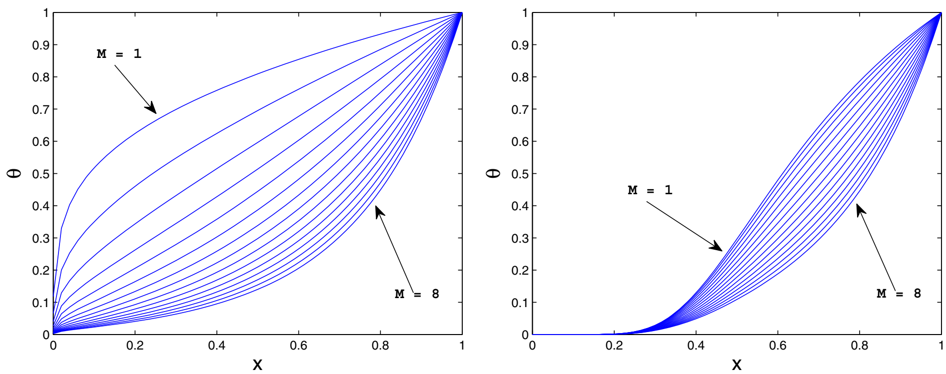

Figure 11. Plots of temperature profile in a longitudinal concave parabolic fin with $B = n = 1$ at $\tau = 2.5$ (left) and $\tau = 0.1$ (right) for varying values of $\mathcal{M}$.

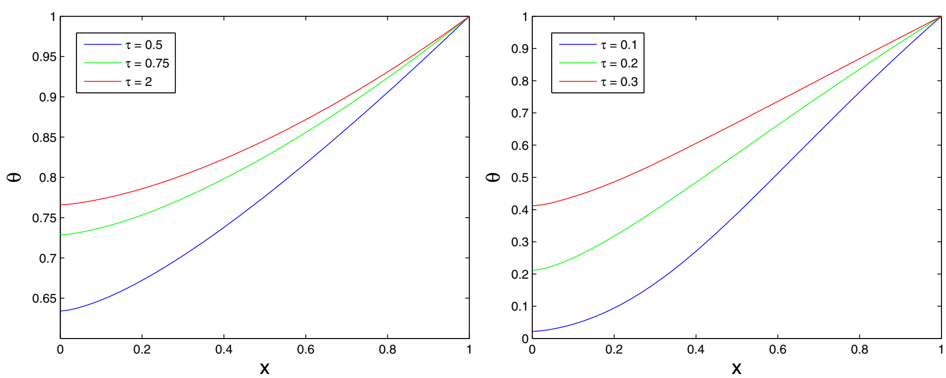

Figure 12. Plots of the numerical solutions for heat flow in a longitudinal convex parabolic fin with $B = n = 1$ and $\mathcal{M} = 1$.

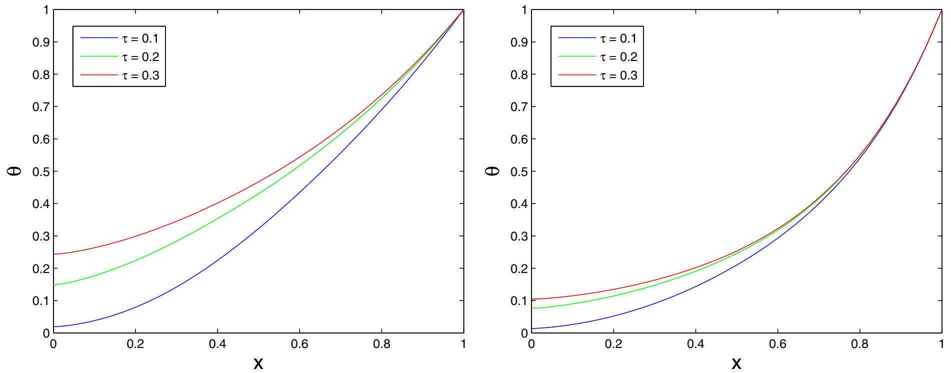

Figure 13. Plots of the numerical solutions for heat flow in a longitudinal convex parabolic fin with $B = n = 1$ and with $\mathcal{M} = 3$ (left) and $\mathcal{M} = 6$ (right).

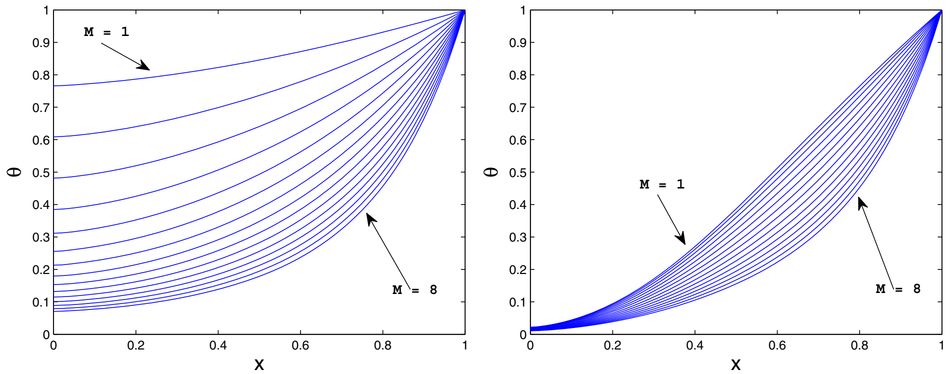

Figure 14. Plots of the numerical solutions for heat flow in a longitudinal convex parabolic fin with $B = n = 1$ at $\tau = 2.5$ (left) and $\tau = 0.1$ (right) for varying values of $\mathcal{M}$.

values of $\mathcal{M}$ the solution is not physically valid and may be related to the occurrence of thermal instability as discussed in [25].

4.4 *Convex parabolic case*

The solutions for this case are depicted in figures 12, 13 and 14. Similar results and observations as in §4.1 are obtained.

5. Concluding remarks

The transient heat transfer in a longitudinal fin of various profiles was studied. The dependence of the thermal conductivity and heat transfer coefficients on the temperature rendered the problem highly nonlinear. This is significant in the study and determination of solutions for fin problems, because as far as we know solutions for the transient heat transfer in a fin exists only when the heat transfer coefficient depends on the spatial variable (see [12]). Classical symmetry analysis resulted in some reductions of the original governing equation. We found that for the cases considered in this paper the initial and boundary conditions were not invariant under the admitted Lie point symmetries. Furthermore, we obtained a general solution for the equation describing heat transfer in a longitudinal triangular fin (but the initial and boundary conditions were not satisfied). Hence we sought numerical solutions.

Perhaps an interesting observation is that for prolonged periods of time the temperature profile indicates that the adiabatic condition cannot be maintained for the heat transfer in longitudinal triangular and concave parabolic fins. However, the behaviour is corrected when the values of the thermogeometric fin parameter increases (that is, for longer or thinner fins). Note that for the fins with arbitrary profile $\mathcal{M} = (\mathrm{Bi})^{1/2}\, E$, where $\mathrm{Bi} = (h_b \delta_b)/k_a$ is the Biot number and $E = 2L/\delta_b$ is the extension factor [1]. The thermogeometric fin parameter also increases when the Biot number is increased. This may be practical in a confined region where the length of the fin cannot be increased. We can thus deduce that the influence of the thermogeometric parameter and the exponent n is very likely related to thermal instability; in our case this was observed for the triangular and concave profiles. Critical values of the thermogeometric fin parameter for which the heat transfer in fins of a certain profile are unstable, along with the importance of the fin tip temperature are fascinating areas to explore further.

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