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Charged fluids with symmetries

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Abstract. We investigate the role of symmetries for charged perfect fluids by assuming that spacetime admits a conformal Killing vector. The existence of a conformal symmetry places restrictions on the model. It is possible to find a general relationship for the Lie derivative of the electromagnetic field along the integral curves of the conformal vector. The electromagnetic field is mapped conformally under particular conditions. The Maxwell equations place restrictions on the form of the proper charge density.

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1. Introduction

It is possible to introduce many types of symmetries on the manifold which restrict the behaviour of quantities associated with the curvature of spacetime. Such symmetries act on quantities associated with the gravitational field, the connection coefficients, the Ricci scalar, the Ricci tensor and the Einstein tensor. Of fundamental interest is the symmetry associated with a conformal Killing vector. Conformal symmetries act directly on the metric tensor field and generate constants of the motion along null geodesics for massless particles. Conformal symmetries arise in various physical applications. The existence of conformal symmetries in relativistic cosmological models, with restrictions on the matter content and fluid four-velocity, have been extensively investigated by Coley and Tupper [1–5]. Yavuz and Yilmaz [6] considered the effect of conformal symmetries in the description of string clouds and string sources. Yilmaz *et al* [7] presented topological defect solutions in spherically symmetric spacetimes admitting conformal vectors. Esculpi and Aloma [8] showed that it is possible to generate anisotropic charged fluid spheres, with a barotropic equation of state in relativistic astrophysics, consistent with a conformal symmetry.

It is necessary to specify the form of the matter field when studying the properties of symmetries. Most treatments of conformal symmetries involve energy–momentum tensors comprising perfect or imperfect matter distributions. The presence of electromagnetic field is likely to produce new effects which are not present in results pertaining to neutral matter. Consequently, in this investigation we consider some of the consequences of imposing a conformal Killing vector on the electromagnetic field tensor and the role of Maxwell's equations.

2. Conformal symmetries

Manifolds with structure may admit groups of transformations which preserve this structure. A conformal motion preserves the metric up to a factor and maps null geodesics conformally. A conformal Killing vector **X** is given by

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\psi g_{ab}, \tag{1}$$

where $\psi = \psi(x^a)$ is the conformal factor. The existence of a conformal Killing vector **X** is subject to the integrability condition

$$\mathcal{L}_{\mathbf{X}} C^a{}_{bcd} = 0 \tag{2}$$

which indicates that the Weyl tensor is conformally invariant. Equation (2) is identically satisfied for conformally flat spacetimes. In Robertson–Walker spacetimes, the full conformal group was found by Maartens and Maharaj [9] by integrating (1) directly; for other cases see particular examples given in Stephani *et al* [10]. However, the existence of solutions to (1), which also satisfy the Einstein field equations, are difficult to prove in general.

We can determine the effect of conformal symmetry (1) on the quantities associated with the curvature. We can establish the following results:

$$\mathcal{L}_{\mathbf{X}} \Gamma^a{}_{bc} = \psi_{,c}\delta^a_b + \psi_{,b}\delta^a_c - g_{bc}\psi^{,a}, \tag{3a}$$

$$\mathcal{L}_{\mathbf{X}} R_{ab} = -2\psi_{;ab} - g_{ab}\Box\psi, \tag{3b}$$

$$\mathcal{L}_{\mathbf{X}} R = -2\psi R - 6\Box\psi, \tag{3c}$$

$$\mathcal{L}_{\mathbf{X}} G_{ab} = 2g_{ab}\Box\psi - 2\psi_{;ab}, \tag{3d}$$

where

$$\Box\psi = g^{ab}\psi_{;ab}.$$

Then, by taking the Lie derivative $\mathcal{L}_{\mathbf{X}}$ of the Einstein equations

$$R_{ab} - \frac{1}{2}R g_{ab} = T_{ab} \tag{4}$$

it is possible to study the dynamics of the fluid with a conformal symmetry.

For the fluid four-vector we can show that

$$\mathcal{L}_{\mathbf{X}} u_a = \psi u_a + v_a, \tag{5}$$

where $u^a v_a = 0$. This result holds in general as first established by Maartens *et al* [11]. For an inheriting conformal vector we have

$$\mathcal{L}_{\mathbf{X}} u_a = \psi u_a. \tag{6}$$

Hence inheriting conformal Killing vectors map fluid flow lines onto fluid flow lines. The physical significance of assumption (6) has been extensively investigated by Maartens *et al* [11] and others. As a consequence of (1) and (6) we observe that

$$\mathcal{L}_{\mathbf{X}} h_{ab} = 2\psi h_{ab} \tag{7}$$

so that the inheriting vector **X** is a conformal motion of the projection tensor. If **X** is an inheriting conformal vector, then we can establish the following relations:

$$\mathcal{L}_{\mathbf{X}} \dot{u}_a = \psi_{,a} + u_a \dot{\psi}, \tag{8a}$$

$$\mathcal{L}_{\mathbf{X}} \Theta = -\psi \Theta + 3\dot{\psi}, \tag{8b}$$

$$\mathcal{L}_{\mathbf{X}} \sigma_{ab} = \psi \sigma_{ab}, \tag{8c}$$

$$\mathcal{L}_{\mathbf{X}} \omega_{ab} = \psi \omega_{ab}. \tag{8d}$$

We observe that the inheriting vector **X** is a conformal motion of both the shear and the vorticity; however the acceleration and expansion are not conformally mapped in general. These equations restrict the forms of the kinematical quantities for an inheriting conformal symmetry **X**.

3. Electromagnetic field

The electromagnetic contribution to the matter content is given by

$$E_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{de} F^{de}. \tag{9}$$

Here the components of the skew-symmetric electromagnetic field tensor **F** may be given in terms of a four-potential **A** as follows:

$$F_{ab} = A_{b;a} - A_{a;b}.$$

The electromagnetic field tensor **F** satisfies the Maxwell's equations

$$F^{ab}{}_{;b} = J^a, \tag{10a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{10b}$$

where **J** represents the current density. We can define

$$J^a = \sigma u^a,$$

where σ is the proper charge density. The Maxwell equations are the basic equations of the electromagnetic field in a curved space.

We are guided by the decomposition of the fluid four-velocity given by (5). Consequently we define $A^a A_a = A^2$ where A is the magnitude of the four-potential **A**. In general we can express the Lie derivative of **A** as follows:

$$\mathcal{L}_{\mathbf{X}} A_a = \alpha A_a + B_a, \tag{11}$$

where $A^a B_a = 0$ and α is an arbitrary function. Then we can prove that

$$\alpha = -\psi + A^{-1}\mathcal{L}_{\mathbf{X}} A \tag{12}$$

which relates the function α to the conformal factor ψ. It is useful to establish the quantity

$$\mathcal{L}_{\mathbf{X}} F_{ab} = 2\alpha F_{ab} + 2\alpha_{[,a} A_{b]} + 2B_{[b,a]} \tag{13}$$

so that the field tensor **F** is not mapped conformally in general. If α is constant and $B_a = 0$ then **X** is a conformal motion for the field tensor.

We can evaluate the Lie derivative of **E**, along the integral curves of a conformal Killing vector, as follows:

$$\mathcal{L}_{\mathbf{X}} E_{ab} = -2\psi E_{ab} + F^c{}_a \mathcal{L}_{\mathbf{X}} F_{cb} + F^c{}_b \mathcal{L}_{\mathbf{X}} F_{ca} - \tfrac{1}{2} g_{ab} F^{de} \mathcal{L}_{\mathbf{X}} F_{de}, \tag{14}$$

where we have used (1) and (9). To simplify (14) we generate the expression

$$F^c{}_b \mathcal{L}_{\mathbf{X}} F_{ca} = 4g^{cf} A_{[b,f]} \left(\alpha_{[,c} A_{a]} + \alpha A_{[a,c]} + B_{[a,c]}\right)$$

and the result

$$F^{de} \mathcal{L}_{\mathbf{X}} F_{de} = 4A^{[e,d]} \left(\alpha_{[,d} A_{e]} + \alpha A_{[e,d]} + B_{[e,d]}\right).$$

The above expressions and (13), with some simplification, enable us to write eq. (14) in the compact form

$$\begin{aligned} \mathcal{L}_{\mathbf{X}} E_{ab} &= 2(\psi - \alpha) E_{ab} + 4g^{cf} A_{[a,f]} \left(\alpha_{[,c} A_{b]} + B_{[b,c]}\right) \\ &\quad + 4g^{cf} A_{[b,f]} \left(\alpha_{[,c} A_{a]} + B_{[a,c]}\right) \\ &\quad - 2g_{ab} A^{[e,d]} \left(\alpha_{[,d} A_{e]} + B_{[e,d]}\right), \end{aligned} \tag{15}$$

where α is given by (12) and ψ is the conformal factor. Thus, we have found the Lie derivative of the electromagnetic energy–momentum tensor **E** along the conformal Killing vector **X**. We emphasize that this is a general result as we have not restricted the form of **A**. We have utilized the decomposition (11) of $\mathcal{L}_{\mathbf{X}} A_a$ to express $\mathcal{L}_{\mathbf{X}} E_{ab}$ in the compact form (15). We have not seen this Lie property of the decomposition of the four-potential **A** applied to the electromagnetic energy–momentum tensor previously. The condition (6) places a severe constraint on **A**. Maartens *et al* [11], using a different notation, found the Lie derivative along **X** of **E** for the special case when the electric field vanishes. Using (15) we are now in a position to perform the analogue of earlier symmetry treatments of perfect fluid and imperfect fluids extended to a charged fluid. This proposed study involves the role of (11) on the the Einstein–Maxwell field equations. From (15) we observe that **E** is not mapped conformally in general. When $B_a = 0$ and α is a constant we have

$$\mathcal{L}_{\mathbf{X}} E_{ab} = 2(\psi - \alpha) E_{ab}.$$

Hence **E** is mapped conformally when $\mathcal{L}_{\mathbf{X}} A_a = \alpha A_a$ and α is a constant quantity.

We would expect that the presence of a conformal symmetry will lead to further constraints through Maxwell's equation. We take the Lie derivative $\mathcal{L}_{\mathbf{X}}$ of the Maxwell eq. (10a) to get

$$\sigma\psi - \mathbf{X}(\sigma) = u_a\left(\mathcal{L}_{\mathbf{X}} J^a\right). \tag{16}$$

We can consider (16) as a first-order differential equation in the proper charge density σ. Thus the conformal symmetry and the Maxwell eq. (10a) do place additional restrictions on the electromagnetic field. The condition (16) is satisfied in particular spacetimes. There exist several exact spherically symmetric solutions of the Einstein–Maxwell equations modelling static charged spheres in which the four-potential **A** is parallel to the fluid vector **u** for which (16) holds. A sample of such solutions is given by Komathiraj and Maharaj [12] and Thirukkanesh and Maharaj [13]. Now take the Lie derivative $\mathcal{L}_{\mathbf{X}}$ of the Maxwell eq. (10b) to get

$$(\mathcal{L}_{\mathbf{X}} F_{ab})_{,c} + (\mathcal{L}_{\mathbf{X}} F_{bc})_{,a} + (\mathcal{L}_{\mathbf{X}} F_{ca})_{,b} = 0. \tag{17}$$

With the help of (13) we can show that this equation can be written as

$$\begin{aligned} &\alpha_{,c} F_{ab} + \alpha_{,a} F_{bc} + \alpha_{,b} F_{ca} \\ &\quad + \alpha_{,a}(A_{b,c} - A_{c,b}) + \alpha_{,b}(A_{c,a} - A_{a,c}) + \alpha_{,c}(A_{a,b} - A_{b,a}) = 0 \end{aligned}$$

which is identically satisfied. Remarkably the conformal symmetry and the Maxwell eq. (10b) do not place additional restrictions on the electromagnetic field. This is true irrespective of the form of the conformal Killing vector **X** and we have not restricted **A** in any way.

4. Discussion

In this analysis we have studied the role of symmetries for charged perfect fluids by assuming that spacetime admits a conformal symmetry. The conformal Killing vector places particular restrictions on the dynamical behaviour of the model and the gravitational field. We found an explicit relationship for the Lie derivative of the electromagnetic field along the integral curves of the conformal vector. It should be noted that this is a general result and has been established without restricting the conformal vector **X** or the four-potential **A** in any way. The electromagnetic field is mapped conformally under particular restrictions:

$$\mathcal{L}_{\mathbf{X}} E_{ab} = 2(\psi - \alpha) E_{ab} \iff \mathcal{L}_{\mathbf{X}} A_a = \alpha A_a \quad \text{and} \quad \alpha_{,a} = 0.$$

The first set of Maxwell eq. (10a) places additional restrictions on the form of the proper charge density in the presence of a conformal symmetry. There is a relationship between the conformal symmetry **X** and the divergence of the field tensor **F**. The conformal symmetry and the second set of Maxwell eq. (10b) do not generate further restrictions on the electromagnetic field.

In future, we need to investigate the influence of the complete Einstein–Maxwell system of field equations with a conformal symmetry. Such a study will comprise an extension of the pioneering work of Maartens *et al* [11] containing uncharged matter. This would involve a superposition of matter fields for charged and uncharged matter. The Einstein equations with both uncharged fluid matter content and electromagnetic contribution are given by

$$G_{ab} = (\mu + p) u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} + E_{ab}, \tag{18}$$

where μ is the energy density, p is the isotropic pressure, q^a is the heat flux vector and π_{ab} is the anisotropic stress tensor. Taking the Lie derivative of (18) yields

$$\begin{aligned}\mathcal{L}_{\mathbf{X}} E_{ab} + u_a u_b \mathcal{L}_{\mathbf{X}}\mu + h_{ab}\mathcal{L}_{\mathbf{X}} p + 2\psi(\mu u_a u_b + p h_{ab}) + 2(\mu + p) u_{(a} v_{b)} \\ + \mathcal{L}_{\mathbf{X}}\pi_{ab} + 2\left(q^{-1}\mathcal{L}_{\mathbf{X}} q + 2\psi\right) u_{(a} q_{b)} + 2q_{(a} v_{b)} + 2u_{(a} w_{b)} \\ = 2\Box\psi g_{ab} - 2\psi_{;ab}, \end{aligned} \quad (19)$$

where we have set $q^2 = q^a q_a$ and $\mathcal{L}_{\mathbf{X}} q^a = \beta q^a + w^a$ with $q^a w_a = 0$. Now contracting (19) with $u^a u^b$, h^{ab}, $u^a h^b{}_c$, $h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}$, q^b, $q^a u^b$ and $q^a q^b$ in turn will yield equations governing the conformal evolution of a charged self-gravitating fluid. This line of inquiry in future work will produce dynamical results which are likely to extend earlier treatments with uncharged matter. In addition, the special requirement of an inheriting conformal symmetry (6), mapping fluid flow lines conformally, should produce results similar to earlier treatments for uncharged matter.

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