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# Higher-order symmetries and conservation laws of multi-dimensional Gordon-type equations

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**Abstract.** In this paper a class of multi-dimensional Gordon-type equations are analysed using a multiplier and homotopy approach to construct conservation laws. The main focus is the analysis of the classical versions of the Gordon-type equations and obtaining higher-order variational symmetries and corresponding conserved quantities. The results are extended to the multi-dimensional Gordon-type equations with the two-dimensional Klein–Gordon equation in particular yielding interesting results.

**Keywords.** Conservation laws; multipliers; multi-dimensional Gordon-type equations.

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## 1. Introduction

The higher-order multipliers and conservation laws of the canonical form of the Gordon-type equations,

$$u_{XT} - k(u) = 0, \tag{1}$$

can be recast in the well-known classical form,

$$u_{tt} - u_{xx} - k(u) = 0. \tag{2}$$

The purpose for doing this is to gain an insight into extending the results to the multi-dimensional Gordon-type equations which are somewhat cumbersome if one was to pursue a canonical form. That is, the multi-dimensional Gordon-type equations are best considered as

$$u_{tt} - \Delta u - k(u) = 0, \tag{3}$$

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where $\Delta$ denotes the Laplacian. This equation is of the classical form rather than the canonical form.

Equation (1) has been extensively studied in terms of their symmetries and variational properties [1]. In particular, the sine-Gordon equation $u_{XT} - \sin u = 0$ has been shown to have higher-order variational symmetries, $\mathcal{X} = Q\partial_u$. For example,

$$\mathcal{X}_1 = \left(u_{XXX} + \frac{1}{2}u_X^3\right)\partial_u,$$

$$\mathcal{X}_2 = \left(u_{XXXXX} + \frac{5}{2}u_X^2 u_{XXX} + \frac{5}{2}u_X u_{XX}^2 + \frac{3}{8}u_X^5\right)\partial_u,$$

$$\mathcal{X}_3 = \left(u_{TTT} + \frac{1}{2}u_T^3\right)\partial_u. \tag{4}$$

Of the variational symmetries in (4), the first two lead to the corresponding higher-order conserved densities,

$$\Phi_1^T = -\frac{1}{2}u_{XX}^2 + \frac{1}{8}u_X^4, \qquad \Phi_2^T = \frac{1}{2}u_{XXX}^2 - \frac{5}{4}u_X^2 u_{XX}^2 + \frac{1}{16}u_X^6. \tag{5}$$

In a similar way, we investigate the existence of higher-order variational symmetries for other classes of Gordon-type equations with dimension one and two, and we find possible conserved densities via the multiplier method outlined below. We conclude with some interesting and unexpected results.

We apply the multiplier approach that leads to a large class of interesting and higher-order conserved flows that would not have been obtained by variational techniques such as Noether's theorem. In particular, we obtain higher-order multipliers.

We present some of the definitions and notations below. Intrinsic to a Lie algebraic treatment of differential equations is the universal space $\mathcal{A}$ (see [1]). The space $\mathcal{A}$ is the vector space of all differential functions of all finite orders and forms an algebra. Consider an $r$th-order system of partial differential equations of $n$ independent variables $x = (x^1, x^2, \ldots, x^n)$ and $m$ dependent variables $u = (u^1, u^2, \ldots, u^m)$,

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \qquad \mu = 1, \ldots, \tilde{m}, \tag{6}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first-, second-, ..., $r$th-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to $x^i$ given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \qquad i = 1, \ldots, n, \tag{7}$$

where the summation convention is used whenever appropriate. A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{8}$$

along the solutions of (6). It can be shown that every admitted conservation law arises from multipliers $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{9}$$

holds identically (that is, off the solution space) for some current $\Phi$. The conserved vector may then be obtained by the homotopy operator (see [2–4]). Other works on symmetries and conservation laws can be found in [5–8].

**Definition 1.** The Euler operator for each dependent variable $u^\alpha$ is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{10}$$

In most literature, a variational problem consists of finding the extrema (maxima or minima) of a functional,

$$\mathcal{L}[u] = \int_\Omega L(x, u_{(n)})\mathrm{d}x,$$

in some class of functions $u = f(x)$ defined over $\Omega$ where $\Omega \subset X$ is an open connected subset with smooth boundary $\partial\Omega$ (we consider the Euclidean space with $X = R^n$). The integrand $L(x, u_{(n)})$, called the Lagrangian of the variational problem $\mathcal{L}$, is a smooth function of $x$, $u$ and various derivatives of $u$ [1].

**Definition 2.** The Lie–Bäcklund or generalized operator is given by

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha}, \qquad \xi^i, \eta^\alpha \in \mathcal{A}. \tag{11}$$

This operator is an abbreviated form of the following infinite formal sum:

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}\zeta^\alpha_{i_1\ldots i_s}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \tag{12}$$

where the additional coefficients are determined uniquely by the prolongation formulae,

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij},$$

$$\zeta^\alpha_{i_1\ldots i_s} = D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s > 1. \tag{13}$$

**Definition 3.** A Lie–Bäcklund operator $X$ of the form (11) is called a variational symmetry if it leaves invariant the functional $\mathcal{L}[u] = \int L(x, u_{(n)})\mathrm{d}x$. Variational partial differential equations are partial differential equations that admit Lagrangians.

The following theorem defines the condition under which a symmetry is variational.

**Theorem 1.** *For variational partial differential equations,* $E = 0$, *where* $E \in \mathcal{A}$, *an evolutionary vector field* $\mathcal{X} = Q\partial_u$ *is a variational symmetry if and only if* $\mathcal{X}E + A\mathcal{F}_Q E = 0$, *where* $A\mathcal{F}_Q$ *is the adjoint Fréchet derivative on* $Q$ [1].

## 2. Applications

We now construct symmetries and conservation laws for the classes of eq. (2) discussed above with particular emphasis on the main eq. (3). Most of the tedious calculations have been omitted.

### 2.1 (1+1) *Gordon-type equations*

We now consider some special cases of eq. (2) in classical form. In each case, we list one or two of the higher-order multipliers and conserved densities which arise as a consequence of the transformation to the classical form. Other multipliers and conserved densities exist and may be found in [9].

2.1.1 *The Gordon-type equation:* $u_{xx} - u_{tt} - \sin u = 0$. In converting the equation $u_{xx} - u_{tt} - \sin(u) = 0$ into $u_{XT} - \sin u$, we use the transformations,

$$X = \frac{1}{2}(x - t), \qquad T = \frac{1}{2}(x + t).$$

In the reverse direction we use the transformations,

$$x = X + T, \qquad t = T - X,$$

to obtain higher-order symmetries for the classical equation which we call $\bar{\mathcal{X}}$. We shall now illustrate the method using $\mathcal{X}_1 = (u_{XXX} + \frac{1}{2}u_X^3)\partial_u$ listed in (4). Since

$$\begin{aligned} u_X &= u_x x_X + u_t t_X = u_x - u_t, \\ u_{XX} &= u_{xx} x_X + u_{xt} t_X - u_{tx} x_X - u_{tt} t_X = u_{xx} - 2u_{xt} + u_{tt}, \\ u_{XXX} &= u_{xxx} x_X + u_{xxt} t_X - 2(u_{xtx} x_X + u_{xtt} t_X) + u_{ttx} x_X + u_{ttt} t_X \\ &= u_{xxx} - 3u_{xxt} + 3u_{xtt} - u_{ttt}, \end{aligned}$$

the equivalent of $\mathcal{X}_1$ is

$$\bar{\mathcal{X}}_1 = \left( u_{xxx} - 3u_{xxt} + 3u_{xtt} - u_{ttt} + \frac{1}{2}(u_x - u_t)^3 \right) \partial_u,$$

with the multiplier,

$$\mathcal{Q}_1 = \frac{1}{2}(u_x - u_t)^3 + u_{xxx} - u_{ttt} - 3u_{txx} + 3u_{ttx},$$

leading to a conserved vector,

$$\begin{aligned} \Phi_1^t = \frac{1}{8} \big( & u_t^4 + 4u_{tt}^2 - 3u_t^3 u_x + 4u_{ttt} u_x - 4\cos(u)\, u_x^2 + 8u_{tt} (\sin(u) - u_{xt}) \\ & - 16\sin(u)\, u_{xt} + 3uu_x^2 u_{xt} - 4u_x u_{xtt} + 4uu_{xttt} \\ & + u_t^2 \left(-4\cos(u) + 3u_x^2 + 3u\,(u_{xt} - u_{xx})\right) + 8\sin(u)\, u_{xx} \\ & - 3uu_x^2 u_{xx} + 8u_{xt} u_{xx} - 4u_{xx}^2 - 4u_x u_{xxt} - 12uu_{xxtt} + 4u_x u_{xxx} \\ & + u_t \big(-u_x^3 + u_x\,(8\cos(u) - 6u(u_{xt} - u_{xx})) \\ & \qquad - 8\,(u_{xtt} - 2u_{xxt} + u_{xxx})\big) \\ & + 12uu_{xxxt} - 4uu_{xxxx} \big), \end{aligned}$$

$$\begin{aligned}\Phi_1^x &= \frac{1}{8}\big(-4u_{tt}^2 - 4uu_{tttt} - u_t^3u_x - 8u_{ttt}u_x + 4\cos uu_x^2 + u_x^4 \\ &\quad + u_t^2\left(4\cos u + 3u_x^2 - 3u\left(u_{tt} - u_{xt}\right)\right) + 16\sin uu_{xt} + 3uu_x^2u_{xt} \\ &\quad + u_{tt}\left(-8\sin u - 3uu_x^2 + 8u_{xt}\right) + 16u_xu_{xtt} + 12uu_{xttt} - 8\sin uu_{xx} \\ &\quad - 8u_{xt}u_{xx} + 4u_{xx}^2 - 8u_xu_{xxt} - 12uu_{xxtt} \\ &\quad + u_t\big(4u_{ttt} - 3u_x^3 + u_x\left(-8\cos u + 6u\left(u_{tt} - u_{xt}\right)\right) \\ &\quad - 4\left(u_{xtt} + u_{xxt} - u_{xxx}\right)\big) + 4uu_{xxxt}\big).\end{aligned}$$

2.1.2 *The Gordon-type equation:* $u_{xx} - u_{tt} - u = 0$. The multiplier,

$$Q_{2A} = -u_{xxx} + u_{ttt} + 3u_{txx} - 3u_{ttx}, \tag{14}$$

leads to the conserved vector,

$$\begin{aligned}\Phi_{2A}^t &= \frac{1}{2}\big(u_t^2 - u_{tt}^2 - u_{ttt}u_x + u_x^2 + 2u_{tt}u_{xt} + u_xu_{xtt} - 2u_{xt}u_{xx} + u_{xx}^2 \\ &\quad + u_xu_{xxt} - 2u_t\big(u_x - u_{xtt} + 2u_{xxt} - u_{xxx}\big) - u_xu_{xxx} \\ &\quad + u\big(-2u_{tt} + 4u_{xt} - u_{xttt} - 2u_{xx} + 3u_{xxtt} - 3u_{xxxt} + u_{xxxx}\big)\big),\\ \Phi_{2A}^x &= \frac{1}{2}\big(-u_t^2 + u_{tt}^2 + 2u_{ttt}u_x - u_x^2 - 2u_{tt}u_{xt} - 4u_xu_{xtt} + 2u_{xt}u_{xx} \\ &\quad - u_{xx}^2 + 2u_xu_{xxt} + u_t\big(-u_{ttt} + 2u_x + u_{xtt} + u_{xxt} - u_{xxx}\big) \\ &\quad + u\big(2u_{tt} + u_{tttt} - 4u_{xt} - 3u_{xttt} + 2u_{xx} + 3u_{xxtt} - u_{xxxt}\big)\big).\end{aligned}$$

The multiplier,

$$Q_{2B} = u_{xxx} - u_{ttt} - 3u_{txx} + 3u_{ttx} + e^x, \tag{15}$$

leads to the conserved vector,

$$\begin{aligned}\Phi_{2B}^t &= \frac{1}{2}\big(-u_t^2 + u_{tt}^2 + u_{ttt}u_x - u_x^2 - 2u_{tt}u_{xt} - u_xu_{xtt} + 2u_{xt}u_{xx} - u_{xx}^2 \\ &\quad - u_xu_{xxt} + u_xu_{xxx} - 2u_t\big(e^x - u_x + u_{xtt} - 2u_{xxt} + u_{xxx}\big) \\ &\quad + u\big(2u_{tt} - 4u_{xt} + u_{xttt} + 2u_{xx} - 3u_{xxtt} + 3u_{xxxt} - u_{xxxx}\big)\big),\\ \Phi_{2B}^x &= \frac{1}{2}\big(u_t^2 - u_{tt}^2 + 2e^xu_x - 2u_{ttt}u_x + u_x^2 + 2u_{tt}u_{xt} + 4u_xu_{xtt} - 2u_{xt}u_{xx} \\ &\quad + u_{xx}^2 - 2u_xu_{xxt} + u_t\big(u_{ttt} - 2u_x - u_{xtt} - u_{xxt} + u_{xxx}\big) \\ &\quad - u\big(2e^x + 2u_{tt} + u_{tttt} - 4u_{xt} - 3u_{xttt} + 2u_{xx} + 3u_{xxtt} - u_{xxxt}\big)\big).\end{aligned}$$

2.1.3 *The Gordon-type equation:* $u_{xx} - u_{tt} - e^u = 0$. The multiplier,

$$Q_3 = u_{xxx} - u_{ttt} - 3u_{txx} + 3u_{ttx} - \frac{1}{2}(u_x - u_t)^3,$$

yields the conserved vector,

$$
\begin{aligned}
\Phi_3^t = \frac{1}{8}\big( & -u_t^4 + 4u_{tt}^2 + 3u_t^3 u_x + 4u_{ttt}u_x - 4e^u u_x^2 + 8u_{tt}\big(e^u - u_{xt}\big) \\
& - 16e^u u_{xt} - 3uu_x^2 u_{xt} - 4u_x u_{xtt} + 4uu_{xttt} \\
& - u_t^2\big(4e^u + 3u_x^2 + 3u\big(u_{xt} - u_{xx}\big)\big) + 8e^u u_{xx} + 3uu_x^2 u_{xx} \\
& + 8u_{xt}u_{xx} - 4u_{xx}^2 - 4u_x u_{xxt} - 12uu_{xxtt} + 4u_x u_{xxx} \\
& + u_t\big(u_x^3 + u_x\big(8e^u + 6u\big(u_{xt} - u_{xx}\big)\big) - 8\big(u_{xtt} - 2u_{xxt} + u_{xxx}\big)\big) \\
& + 12uu_{xxxt} - 4uu_{xxxx}\big), \\
\Phi_3^x = \frac{1}{8}\big( & -4u_{tt}^2 - 4uu_{tttt} + u_t^3 u_x - 8u_{ttt}u_x + 4e^u u_x^2 - u_x^4 \\
& + u_t^2\big(4e^u - 3u_x^2 + 3u\big(u_{tt} - u_{xt}\big)\big) + 16e^u u_{xt} - 3uu_x^2 u_{xt} \\
& + u_{tt}\big(-8e^u + 3uu_x^2 + 8u_{xt}\big) + 16u_x u_{xtt} + 12uu_{xttt} - 8e^u u_{xx} \\
& - 8u_{xt}u_{xx} + 4u_{xx}^2 - 8u_x u_{xxt} - 12uu_{xxtt} \\
& + u_t\big(4u_{ttt} + 3u_x^3 + u_x\big(-8e^u - 6u\big(u_{tt} - u_{xt}\big)\big) \\
& \quad - 4\big(u_{xtt} + u_{xxt} - u_{xxx}\big)\big) + 4uu_{xxxt}\big).
\end{aligned}
$$

### 2.2 (1+2) *Gordon-type equations*

For the multi-dimensional Gordon-type eq. (3), higher-order symmetries and multipliers and the corresponding conserved quantities may be determined for $k(u) = u$ only because for other forms of $k(u)$ the underlying calculations produce negative results. This seems to be a consequence of the underlying differential operator being linear only if $k(u) = u$ (see Proposition 5.22 in [1]). In what follows we first assume a form of multiplier for the equation

$$
u_{xx} + u_{yy} - u_{tt} - u = 0, \tag{16}
$$

and secondly we take a formal approach (the multiplier method) for finding multipliers of the equation.

From the multiplier (14), if we assume a multiplier of eq. (16) to be

$$
Q_A = -u_{xxx} - u_{yyy} + u_{ttt} + 3u_{txx} + 3u_{tyy} - 3u_{ttx} - 3u_{tty},
$$

we obtain the conserved vector,

$$
\begin{aligned}
\Phi_A^t = \frac{1}{2}\big( & u_t^2 - u_{tt}^2 - u_{ttt}u_y + u_y^2 + 2u_{tt}u_{yt} + u_y u_{ytt} - 2u_{yt}u_{yy} + u_{yy}^2 \\
& + u_y u_{yyt} - u_y u_{yyy} - u_{ttt}u_x + u_{yyt}u_x + u_x^2 + 2u_{tt}u_{xt} - 2u_{yy}u_{xt} \\
& + u_x u_{xtt} - u_x u_{xyy} - 2u_{yt}u_{xx} + 2u_{yy}u_{xx} - 2u_{xt}u_{xx} + u_{xx}^2 \\
& + u_y u_{xxt} + u_x u_{xxt} - u_y u_{xxy} - u_x u_{xxx} \\
& + u_t\big(-2u_y + 2u_{ytt} - 4u_{yyt} + 2u_{yyy} - 2u_x + 2u_{xtt} + u_{xyy} - 4u_{xxt} \\
& \quad + u_{xxy} + 2u_{xxx}\big) \\
& - u\big(2u_{tt} - 4u_{yt} + u_{yttt} + 2u_{yy} - 3u_{yytt} + 3u_{yyyt} - u_{yyyy} - 4u_{xt} \\
& \quad + u_{xttt} + 2u_{xyyt} + 2u_{xx} - 3u_{xxtt} + 2u_{xxyt} - 2u_{xxyy} + 3u_{xxxt} \\
& \quad - u_{xxxx}\big)\big),
\end{aligned}
$$

$$\begin{aligned}
\Phi_A^x =\ & \frac{1}{2}\big( -u_t^2 + u_{tt}^2 - u_{tt}u_{yy} + 2u_{ttt}u_x - 3u_{ytt}u_x + 2u_{yyt}u_x - u_{yyy}u_x - u_x^2 \\
& - 2u_{tt}u_{xt} + 2u_{yy}u_{xt} - 4u_x u_{xtt} + u_x u_{xyy} - u_{yy}u_{xx} + 2u_{xt}u_{xx} - u_{xx}^2 \\
& + 2u_x u_{xxt} + u_t\big( -u_{ttt} + u_{yyt} + 2u_x + u_{xtt} - u_{xyy} + u_{xxt} - u_{xxx}\big) \\
& + u\big(2u_{tt} + u_{tttt} - u_{yytt} - 4u_{xt} - 3u_{xttt} + 3u_{xytt} - u_{xyyt} + u_{xyyy} \\
& \quad + 2u_{xx} + 3u_{xxtt} - u_{xxyy} - u_{xxxt}\big)\big), \\
\Phi_A^y =\ & \frac{1}{2}\big( -u_t^2 + u_{tt}^2 + 2u_{ttt}u_y - u_y^2 - 2u_{tt}u_{yt} - 4u_y u_{ytt} + 2u_{yt}u_{yy} - u_{yy}^2 \\
& + 2u_y u_{yyt} - 3u_y u_{xtt} - u_{tt}u_{xx} + 2u_{yt}u_{xx} - u_{yy}u_{xx} + 2u_y u_{xxt} \\
& + u_t\big( -u_{ttt} + 2u_y + u_{ytt} + u_{yyt} - u_{yyy} + u_{xxt} - u_{xxy}\big) + u_y u_{xxy} \\
& - u_y u_{xxx} + u\big(2u_{tt} + u_{tttt} - 4u_{yt} - 3u_{yttt} + 2u_{yy} + 3u_{yytt} \\
& \qquad - u_{yyyt} + 3u_{xytt} - u_{xxtt} - u_{xxyt} - u_{xxyy} + u_{xxxy}\big)\big).
\end{aligned}$$

Similarly, if from the multiplier (15) we assume a multiplier of eq. (16) to be

$$\mathcal{Q}_B = u_{xxx} + u_{yyy} - u_{ttt} - 3u_{txx} - 3u_{tyy} + 3u_{ttx} + 3u_{tty} + e^x + e^y,$$

we obtain the conserved vector,

$$\begin{aligned}
\Phi_B^t =\ & \frac{1}{2}\big( -u_t^2 + u_{tt}^2 + u_{ttt}u_y - u_y^2 - 2u_{tt}u_{yt} - u_y u_{ytt} + 2u_{yt}u_{yy} - u_{yy}^2 \\
& - u_y u_{yyt} + u_y u_{yyy} + u_{ttt}u_x - u_{yyt}u_x - u_x^2 - 2u_{tt}u_{xt} + 2u_{yy}u_{xt} \\
& - u_x u_{xtt} + u_x u_{xyy} + 2u_{yt}u_{xx} - 2u_{yy}u_{xx} + 2u_{xt}u_{xx} - u_{xx}^2 - u_y u_{xxt} \\
& - u_x u_{xxt} + u_y u_{xxy} + u_x u_{xxx} \\
& - u_t\big(2e^x + 2e^y - 2u_y + 2u_{ytt} - 4u_{yyt} + 2u_{yyy} - 2u_x + 2u_{xtt} + u_{xyy} \\
& \quad - 4u_{xxt} + u_{xxy} + 2u_{xxx}\big) \\
& + u\big(2u_{tt} - 4u_{yt} + u_{yttt} + 2u_{yy} - 3u_{yytt} + 3u_{yyyt} - u_{yyyy} - 4u_{xt} \\
& \quad + u_{xttt} + 2u_{xyyt} + 2u_{xx} - 3u_{xxtt} + 2u_{xxyt} - 2u_{xxyy} + 3u_{xxxt} \\
& \quad - u_{xxxx}\big)\big), \\
\Phi_B^x =\ & \frac{1}{2}\big(u_t^2 - u_{tt}^2 + u_{tt}u_{yy} + 2e^x u_x + 2e^y u_x - 2u_{ttt}u_x + 3u_{ytt}u_x - 2u_{yyt}u_x \\
& + u_{yyy}u_x + u_x^2 + 2u_{tt}u_{xt} - 2u_{yy}u_{xt} + 4u_x u_{xtt} - u_x u_{xyy} + u_{yy}u_{xx} \\
& - 2u_{xt}u_{xx} + u_{xx}^2 - 2u_x u_{xxt} \\
& + u_t\big(u_{ttt} - u_{yyt} - 2u_x - u_{xtt} + u_{xyy} - u_{xxt} + u_{xxx}\big) \\
& + u\big( -2e^x - 2u_{tt} - u_{tttt} + u_{yytt} + 4u_{xt} + 3u_{xttt} - 3u_{xytt} + u_{xyyt} \\
& \quad - u_{xyyy} - 2u_{xx} - 3u_{xxtt} + u_{xxyy} + u_{xxxt}\big)\big), \\
\Phi_B^y =\ & \frac{1}{2}\big(u_t^2 - u_{tt}^2 + 2e^x u_y + 2e^y u_y - 2u_{ttt}u_y + u_y^2 + 2u_{tt}u_{yt} + 4u_y u_{ytt} \\
& - 2u_{yt}u_{yy} + u_{yy}^2 - 2u_y u_{yyt} + 3u_y u_{xtt} + u_{tt}u_{xx} - 2u_{yt}u_{xx} + u_{yy}u_{xx} \\
& - 2u_y u_{xxt} - u_y u_{xxy} \\
& + u_t\big(u_{ttt} - 2u_y - u_{ytt} - u_{yyt} + u_{yyy} - u_{xxt} + u_{xxy}\big) + u_y u_{xxx} \\
& + u\big( -2e^y - 2u_{tt} - u_{tttt} + 4u_{yt} + 3u_{yttt} - 2u_{yy} - 3u_{yytt} + u_{yyyt} \\
& \quad - 3u_{xytt} + u_{xxtt} + u_{xxyt} + u_{xxyy} - u_{xxxy}\big)\big).
\end{aligned}$$

More formally, suppose

$$\frac{\delta}{\delta u}\left[\mathcal{Q}\left(u_{xx}+u_{yy}-u_{tt}-u\right)\right]=0, \tag{17}$$

where $\mathcal{Q}=\mathcal{Q}(x, y, t, u_x, u_x, u_{xx}, u_{xy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy})$. Although not pursued here, the calculations may include derivatives of $u$ with respect to $t$. Then,

$$\mathcal{Q}\left[\left(u_{xx}+u_{yy}-u_{tt}-u\right)\right]=D_t\Phi^t+D_x\Phi^x+D_y\Phi^y,$$

where $\Phi^t$, $\Phi^x$, $\Phi^y$ is the conserved flow ($\Phi^t$ being the conserved density).

We obtain the following for the multiplier $\mathcal{Q}$:

$$\begin{aligned}\mathcal{Q}=\frac{1}{6}\Bigg\{-3\Bigg(&-\frac{1}{3}qC_4x^3+(aC_4+pyC_4+pC_3+qC_1)x^2\\&+\big(-ny^2C_4+((-b+c)\,C_4-2C_1p-2nC_3)y\\&\quad-4pC_5-2aC_1+(-b+c)\,C_3-2nC_6\\&\quad+(-2C_6+2C_2)q-2\beta C_{11}\big)x\\&+\frac{1}{3}my^3C_4+(nC_1-aC_4+mC_3)y^2\\&+((2C_6-2C_2)p+(b-c)C_1-2aC_3+4nC_5\\&\quad+2mC_6+2C_{11}\alpha)\,y-2pC_8-2nC_7-2aC_2-2qC_{10}\\&+\ (-2c+2b)\,C_5-2\alpha C_{13}-2\beta C_{12}-2mC_9\Bigg)\Bigg\},\end{aligned} \tag{18}$$

where $C_i$, $i=1,2,3,\ldots,13$, are arbitrary constants and

$$\begin{aligned}\alpha&=u_x,\quad \beta=u_y,\\a&=u_{xy},\quad b=u_{xx},\quad c=u_{yy},\\m&=u_{xxx},\ n=u_{xxy},\ p=u_{xyy},\ q=u_{yyy}.\end{aligned}$$

When we solve (18), we obtain the set of multipliers $\mathcal{Q}_i$ together with their conserved densities $\Phi_i^t$.

$$\begin{aligned}\mathcal{Q}_1&=u_x,\\\Phi_1^t&=\frac{1}{2}(-u_tu_x+uu_{xt}),\\\Phi_1^x&=\frac{1}{2}\left(-u^2+u(-u_{tt}+u_{yy})+u_x^2\right),\\\Phi_1^y&=\frac{1}{2}(u_yu_x-uu_{xy}).\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_2&=u_y,\\\Phi_2^t&=\frac{1}{2}(-u_tu_y+uu_{yt}),\\\Phi_2^x&=\frac{1}{2}(u_yu_x-uu_{xy}),\\\Phi_2^y&=\frac{1}{2}\left(-u^2+u_y^2+u(-u_{tt}+u_{xx})\right).\end{aligned}$$

$$
\begin{aligned}
\mathcal{Q}_3 &= xu_y - yu_x,\\
\Phi_3^t &= \frac{1}{2}(u_t(-xu_y + yu_x) + u(xu_{yt} - yu_{xt})),\\
\Phi_3^x &= \frac{1}{2}(yu^2 + u_x(xu_y - yu_x) + u(yu_{tt} - u_y - yu_{yy} - xu_{xy})),\\
\Phi_3^y &= \frac{1}{2}(-xu^2 + u_y(xu_y - yu_x) + u(-xu_{tt} + u_x + yu_{xy} + xu_{xx})).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_4 &= u_{xxx},\\
\Phi_4^t &= \frac{1}{2}(-u_t u_{xxx} + uu_{xxxt}),\\
\Phi_4^x &= \frac{1}{2}(u_x^2 + u_x(u_{xtt} - u_{xyy}) + u_{xx}(-u_{tt} + u_{yy} + u_{xx})\\
&\quad + u(-2u_{xx} - u_{xxtt} + u_{xxyy})),\\
\Phi_4^y &= \frac{1}{2}(u_y u_{xxx} - uu_{xxxy}).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_5 &= u_{yyy},\\
\Phi_5^t &= \frac{1}{2}(-u_t u_{yyy} + uu_{yyyt}),\\
\Phi_5^x &= \frac{1}{2}(u_{yyy}u_x - uu_{xyyy}),\\
\Phi_5^y &= \frac{1}{2}\left(u_y^2 + u_{yy}(-u_{tt} + u_{yy} + u_{xx}) + u_y(u_{ytt} - u_{xxy})\right.\\
&\quad \left.+u(-2u_{yy} - u_{yytt} + u_{xxyy})\right).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_6 &= u_{xyy},\\
\Phi_6^t &= \frac{1}{2}(-u_t u_{xyy} + uu_{xyyt}),\\
\Phi_6^x &= \frac{1}{6}\left(u_y^2 - u_{tt}u_{yy} + u_{yy}^2 + 3u_x u_{xyy} + u_{yy}u_{xx} + u_y(u_{ytt} - u_{yyy} - u_{xxy})\right.\\
&\quad \left.-u(2u_{yy} + u_{yytt} - u_{yyyy} + 2u_{xxyy})\right),\\
\Phi_6^y &= \frac{1}{6}(u_{ytt}u_x - u_{yyy}u_x - 4uu_{xy} - 2u_{tt}u_{xy} + 2u_{yy}u_{xy} - 2uu_{xytt} - uu_{xyyy}\\
&\quad + 2u_{xy}u_{xx} - u_x u_{xxy} + u_y(2u_x + u_{xtt} + 2u_{xyy} - u_{xxx}) + 2uu_{xxxy}).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_7 &= u_{xxy},\\
\Phi_7^t &= \frac{1}{2}(-u_t u_{xxy} + uu_{xxyt}),\\
\Phi_7^x &= \frac{1}{6}(u_{ytt}u_x - u_{yyy}u_x - 4uu_{xy} - 2u_{tt}u_{xy} + 2u_{yy}u_{xy} - 2uu_{xytt} + 2uu_{xyyy}\\
&\quad + 2u_{xy}u_{xx} + 2u_x u_{xxy} + u_y(2u_x + u_{xtt} - u_{xyy} - u_{xxx}) - uu_{xxxy}),\\
\Phi_7^y &= \frac{1}{6}\left(u_x^2 - u_{tt}u_{xx} + u_{yy}u_{xx} + u_{xx}^2 + 3u_y u_{xxy} + u_x(u_{xtt} - u_{xyy} - u_{xxx})\right.\\
&\quad \left.-u(2u_{xx} + u_{xxtt} + 2u_{xxyy} - u_{xxxx})\right).
\end{aligned}
$$

$$\mathcal{Q}_8 = -u_{yx} + xu_{yyy} - yu_{xyy},$$

$$\Phi_8^t = \frac{1}{2}(u_t(-xu_{yyy} + u_{xy} + yu_{xyy}) + u(xu_{yyyt} - u_{xyt} - yu_{xyyt})),$$

$$\Phi_8^x = \frac{1}{12}\big(-2yu_y^2 - 2yu_y u_{ytt} - u_y u_{yy} - 2yu_{yy}^2 + u_{tt}(u_y + 2yu_{yy}) + 2yu_y u_{yyy} + 6xu_{yyy}u_x - 6u_x u_{xy} - 6yu_x u_{xyy} - u_y u_{xx} - 2yu_{yy}u_{xx} + 2yu_y u_{xxy} + u\big(2u_y + u_{ytt} + 4yu_{yy} + 2yu_{yytt} - 7u_{yyy} - 2yu_{yyyy} - 6xu_{xyyy} + 5u_{xxy} + 4yu_{xxyy}\big)\big),$$

$$\Phi_8^y = \frac{1}{12}\big(6xu_y^2 - 6xu_{tt}u_{yy} + 6xu_{yy}^2 + u_{tt}u_x - 2yu_{ytt}u_x - u_{yy}u_x + 2yu_{yyy}u_x + 4yu_{tt}u_{xy} - 4yu_{yy}u_{xy} + 6xu_{yy}u_{xx} - u_x u_{xx} - 4yu_{xy}u_{xx} + 2yu_x u_{xxy} + 2u_y(3xu_{ytt} - 2yu_x - yu_{xtt} - 3u_{xy} - 2yu_{xyy} - 3xu_{xxy} + yu_{xxx}) + u(-12xu_{yy} - 6xu_{yytt} + 2u_x + u_{xtt} + 8yu_{xy} + 4yu_{xytt} + 11u_{xyy} + 2yu_{xyyy} + 6xu_{xxyy} - u_{xxx} - 4yu_{xxxy})\big).$$

$$\mathcal{Q}_9 = u_{xx} - 2xu_{xyy} + 2yu_{xxy} - u_{yy},$$

$$\Phi_9^t = \frac{1}{2}(u_t(u_{yy} + 2xu_{xyy} - u_{xx} - 2yu_{xxy}) + u(-u_{yyt} - 2xu_{xyyt} + u_{xxt}2yu_{xxyt})),$$

$$\Phi_9^x = \frac{1}{6}\big(-2xu_y^2 + 2xu_{tt}u_{yy} - 2xu_{yy}^2 - u_{tt}u_x + 2yu_{ytt}u_x - 2u_{yy}u_x - 2yu_{yyy}u_x - 4yu_{tt}u_{xy} + 4yu_{yy}u_{xy} - 6xu_x u_{xyy} - 2xu_{yy}u_{xx} + 4u_x u_{xx} + 4yu_{xy}u_{xx} + 4yu_x u_{xxy} - 2u_y(xu_{ytt} - xu_{yyy} - 2yu_x - yu_{xtt} + yu_{xyy} - xu_{xxy} + yu_{xxx}) - u(-4xu_{yy} - 2xu_{yytt} + 2xu_{yyyy} + 2u_x + u_{xtt} + 8yu_{xy} + 4yu_{xytt} - 10u_{xyy} - 4yu_{xyyy} - 4xu_{xxyy} + 2u_{xxx} + 2yu_{xxxy})\big),$$

$$\Phi_9^y = \frac{1}{6}\big(u_{tt}(u_y + 4xu_{xy} - 2yu_{xx}) - 2\big(xu_{ytt}u_x - xu_{yyy}u_x - yu_x^2 - yu_x u_{xtt} + 2xu_{yy}u_{xy} + yu_x u_{xyy} - yu_{yy}u_{xx} + 2xu_{xy}u_{xx} - yu_{xx}^2 - xu_x u_{xxy} + yu_x u_{xxx} + u_y(2u_{yy} + 2xu_x + xu_{xtt} + 2xu_{xyy} - u_{xx} - 3yu_{xxy} - xu_{xxx})\big) + u(2u_y + u_{ytt} + 2(u_{yyy} + 4xu_{xy} + 2xu_{xytt} + xu_{xyyy} - 2yu_{xx} - yu_{xxtt} - 5u_{xxy} - 2yu_{xxyy} - 2xu_{xxxy} + yu_{xxxx}))\big).$$

$$\mathcal{Q}_{10} = -xu_{xxy} - xu_{yyy} + yu_{xyy} + yu_{xxx},$$

$$\Phi_{10}^t = \frac{1}{2}(u_t(xu_{yyy} - yu_{xyy} + xu_{xxy} - yu_{xxx}) + u(-xu_{yyyt} + yu_{xyyt} - xu_{xxyt} + yu_{xxxt})),$$

$$\begin{aligned}
\Phi^x_{10} = \frac{1}{6}\big(& y u_y^2 - y u_{tt} u_{yy} + y u_{yy}^2 - x u_{ytt} u_x - 2x u_{yyy} u_x + 3y u_x^2 + 3y u_x u_{xtt} \\
& + 2x u_{tt} u_{xy} - 2x u_{yy} u_{xy} - 3y u_{tt} u_{xx} + 4y u_{yy} u_{xx} - 2x u_{xy} u_{xx} \\
& + 3y u_{xx}^2 - 2x u_x u_{xxy} \\
& + u_y (y u_{ytt} - y u_{yyy} - 2x u_x - x u_{xtt} + x u_{xyy} - y u_{xxy} + x u_{xxx}) \\
& + u(-2y u_{yy} - y u_{yytt} + 3u_{yyy} + y u_{yyyy} + 4x u_{xy} + 2x u_{xytt} \\
& \quad + x u_{xyyy} - 6y u_{xx} - 3y u_{xxtt} + 3u_{xxy} + y u_{xxyy} + x u_{xxxy})\big),
\end{aligned}$$

$$\begin{aligned}
\Phi^y_{10} = \frac{1}{6}\big(& -3x u_y^2 + 3x u_{tt} u_{yy} - 3x u_{yy}^2 + y u_{ytt} u_x - y u_{yyy} u_x - x u_x^2 - x u_x u_{xtt} \\
& - 2y u_{tt} u_{xy} + 2y u_{yy} u_{xy} + x u_x u_{xyy} + x u_{tt} u_{xx} - 4x u_{yy} u_{xx} + 2y u_{xy} u_{xx} \\
& - x u_{xx}^2 - y u_x u_{xxy} + x u_x u_{xxx} \\
& - u_y (3x u_{ytt} - y(2u_x + u_{xtt} + 2(u_{xyy} + u_{xxx}))) \\
& - u(-6x u_{yy} - 3x u_{yytt} + 4y u_{xy} + 2y u_{xytt} + 3u_{xyy} + y u_{xyyy} - 2x u_{xx} \\
& \quad - x u_{xxtt} + x u_{xxyy} + 3u_{xxx} + y u_{xxxy} + x u_{xxxx})\big).
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_{11} &= -x u_{xx} + x^2 u_{xyy} + x u_{yy} + y^2 u_{xxx} - 2xy u_{yxx} - 2y u_{xy}, \\
\Phi^t_{11} &= \frac{1}{2}(-u_t (x u_{yy} - 2y u_{xy} + x^2 u_{xyy} - x u_{xx} - 2xy u_{xxy} + y^2 u_{xxx}) \\
&\quad + u(x u_{yyt} - 2y u_{xyt} + x^2 u_{xyyt} - x u_{xxt} - 2xy u_{xxyt} + y^2 u_{xxxt})),
\end{aligned}$$

$$\begin{aligned}
\Phi^x_{11} = \frac{1}{6}\big(& -2u^2 + x^2 u_y^2 + x^2 u_y u_{ytt} - y u_y u_{yy} + x^2 u_{yy}^2 - x^2 u_y u_{yyy} - 4xy u_y u_x \\
& - 2xy u_{ytt} u_x + 2x u_{yy} u_x + 2xy u_{yyy} u_x + 3y^2 u_x^2 - 2xy u_y u_{xtt} \\
& + 3y^2 u_x u_{xtt} - 4xy u_{yy} u_{xy} - 6y u_x u_{xy} + 2xy u_y u_{xyy} + 3x^2 u_x u_{xyy} \\
& - 3y^2 u_x u_{xyy} - y u_y u_{xx} + x^2 u_{yy} u_{xx} + 3y^2 u_{yy} u_{xx} - 4x u_x u_{xx} \\
& - 4xy u_{xy} u_{xx} + 3y^2 u_{xx}^2 + u_{tt}\big(y u_y - x^2 u_{yy} + x u_x + 4xy u_{xy} - 3y^2 u_{xx}\big) \\
& - x^2 u_y u_{xxy} - 4xy u_x u_{xxy} + 2xy u_y u_{xxx} \\
& + u\big(-2u_{tt} + 2y u_y + y u_{ytt} - u_{yy} - 2x^2 u_{yy} - x^2 u_{yytt} - y u_{yyy} \\
& \quad + x^2 u_{yyyy} + 2x u_x + x u_{xtt} + 8xy u_{xy} + 4xy u_{xytt} - 10x u_{xyy} \\
& \quad - 4xy u_{xyyy} + 5u_{xx} - 6y^2 u_{xx} - 3y^2 u_{xxtt} + 11y u_{xxy} \\
& \quad - 2x^2 u_{xxyy} + 3y^2 u_{xxyy} + 2x u_{xxx} + 2xy u_{xxxy}\big)\big),
\end{aligned}$$

$$\begin{aligned}
\Phi^y_{11} = \frac{1}{6}\big(& 4x u_y u_{yy} + 2x^2 u_y u_x + x^2 u_{ytt} u_x - y u_{yy} u_x - x^2 u_{yyy} u_x - 2xy u_x^2 \\
& + x^2 u_y u_{xtt} - 2xy u_x u_{xtt} - 6y u_y u_{xy} + 2x^2 u_{yy} u_{xy} + 2x^2 u_y u_{xyy} \\
& + 2xy u_x u_{xyy} - 2x u_y u_{xx} - 2xy u_{yy} u_{xx} - y u_x u_{xx} + 2x^2 u_{xy} u_{xx} \\
& - 2xy u_{xx}^2 + u_{tt}\big(-x u_y + y u_x - 2x^2 u_{xy} + 2xy u_{xx}\big) - 6xy u_y u_{xxy} \\
& - x^2 u_x u_{xxy} - x^2 u_y u_{xxx} + 3y^2 u_y u_{xxx} + 2xy u_x u_{xxx} \\
& + u\big(-2x u_y - x u_{ytt} - 2x u_{yyy} + 2y u_x + y u_{xtt} + 6u_{xy} - 4x^2 u_{xy} \\
& \quad - 2x^2 u_{xytt} + 5y u_{xyy} - x^2 u_{xyyy} + 4xy u_{xx} + 2xy u_{xxtt} \\
& \quad + 10x u_{xxy} + 4xy u_{xxyy} - 7y u_{xxx} + 2x^2 u_{xxxy} - 3y^2 u_{xxxy} \\
& \quad - 2xy u_{xxxx}\big)\big).
\end{aligned}$$

$$
\begin{aligned}
\mathcal{Q}_{12} &= yu_{xx} + x^2u_{yyy} - yu_{yy} + y^2u_{yxx} - 2xyu_{yyx} - 2xu_{xy},\\
\Phi^t_{12} &= \frac{1}{2}\left(-u_t\left(-yu_{yy} + x^2u_{yyy} - 2xu_{xy} - 2xyu_{xyy} + yu_{xx} + y^2u_{xxy}\right)\right.\\
&\quad \left. + u\big(-yu_{yyt} + x^2u_{yyyt} - 2xu_{xyt} - 2xyu_{xyyt} + yu_{xxt} + y^2u_{xxyt}\big)\right),\\
\Phi^x_{12} &= \frac{1}{6}\big(-2xyu_y^2 - 2xyu_yu_{ytt} - xu_yu_{yy} - 2xyu_{yy}^2 + 2xyu_yu_{yyy} + 2y^2u_yu_x\\
&\quad + y^2u_{ytt}u_x - 2yu_{yy}u_x + 3x^2u_{yyy}u_x - y^2u_{yyy}u_x + y^2u_yu_{xtt}\\
&\quad + 2y^2u_{yy}u_{xy} - 6xu_xu_{xy} + u_{tt}(xu_y - y(-2xu_{yy} + u_x + 2yu_{xy}))\\
&\quad - y^2u_yu_{xyy} - 6xyu_xu_{xyy} - xu_yu_{xx} - 2xyu_{yy}u_{xx} + 4yu_xu_{xx}\\
&\quad + 2y^2u_{xy}u_{xx} + 2xyu_yu_{xxy} + 2y^2u_xu_{xxy} - y^2u_yu_{xxx}\\
&\quad + u\big(2xu_y + xu_{ytt} + 4xyu_{yy} + 2xyu_{yytt} - 7xu_{yyy} - 2xyu_{yyyy}\\
&\qquad -2yu_x - yu_{xtt} + 6u_{xy} - 4y^2u_{xy} - 2y^2u_{xytt} + 10yu_{xyy} - 3x^2u_{xyyy}\\
&\qquad + 2y^2u_{xyyy} + 5xu_{xxy} + 4xyu_{xxyy} - 2yu_{xxx} - y^2u_{xxxy}\big)\big),\\
\Phi^y_{12} &= \frac{1}{6}\big(-2u^2 + 3x^2u_y^2 + 3x^2u_yu_{ytt} - 4yu_yu_{yy} + 3x^2u_{yy}^2 - 4xyu_yu_x - xu_{yy}u_x\\
&\quad + 2xyu_{yyy}u_x + y^2u_x^2 - 2xyu_yu_{xtt} + y^2u_xu_{xtt} - 6xu_yu_{xy} - 4xyu_yu_{xyy}\\
&\quad - y^2u_xu_{xyy} + 2yu_yu_{xx} + 3x^2u_{yy}u_{xx} + y^2u_{yy}u_{xx} - xu_xu_{xx}\\
&\quad - 4xyu_{xy}u_{xx} + y^2u_{xx}^2 + u_{tt}\left(yu_y - 3x^2u_{yy} + xu_x + 4xyu_{xy} - y^2u_{xx}\right)\\
&\quad - 3x^2u_yu_{xxy} + 3y^2u_yu_{xxy} + 2xyu_xu_{xxy} + 2xyu_yu_{xxx} - y^2u_xu_{xxx}\\
&\quad + u\big(-2u_{tt} + 2yu_y + yu_{ytt} + 5u_{yy} - 6x^2u_{yy} - 3x^2u_{yytt} + 2yu_{yyy}\\
&\qquad + 2xu_x + xu_{xtt} + 8xyu_{xy} + 4xyu_{xytt} + 11xu_{xyy} + 2xyu_{xyyy}\\
&\qquad - u_{xx} - 2y^2u_{xx} - y^2u_{xxtt} - 10yu_{xxy} + 3x^2u_{xxyy} - 2y^2u_{xxyy}\\
&\qquad -xu_{xxx} - 4xyu_{xxxy} + y^2u_{xxxx}\big) - 4xyu_{yy}u_{xy} - 2xyu_{ytt}u_x\big).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_{13} &= -xyu_{xx} - \frac{1}{3}x^3u_{yyy} + \frac{1}{3}y^3u_{xxx} + xyu_{yy} - y^2u_{xy} - xy^2u_{xxy}\\
&\quad + yx^2u_{xyy} + x^2u_{xy},\\
\Phi^t_{13} &= \frac{1}{6}\left(u_t\big(-3xyu_{yy} + x^3u_{yyy} - 3x^2u_{xy} + 3y^2u_{xy} - 3x^2yu_{xyy} + 3xyu_{xx}\right.\\
&\qquad + 3xy^2u_{xxy} - y^3u_{xxx}\big)\\
&\quad + u\big(3xyu_{yyt} - x^3u_{yyyt} + 3x^2u_{xyt} - 3y^2u_{xyt}\\
&\qquad \left. + 3x^2yu_{xyyt} - 3xyu_{xxt} - 3xy^2u_{xxyt} + y^3u_{xxxt}\big)\right),\\
\Phi^x_{13} &= \frac{1}{12}\big(-4yu^2 + 2x^2yu_y^2 + 2x^2yu_yu_{ytt} + x^2u_yu_{yy} - y^2u_yu_{yy} + 2x^2yu_{yy}^2\\
&\quad - 2x^2yu_yu_{yyy} - 4xy^2u_yu_x - 2xy^2u_{ytt}u_x + 4xyu_{yy}u_x - 2x^3u_{yyy}u_x\\
&\quad + 2xy^2u_{yyy}u_x + 2y^3u_x^2 - 2xy^2u_yu_{xtt} + 2y^3u_xu_{xtt} - 4xy^2u_{yy}u_{xy}\\
&\quad - 6y^2u_xu_{xy} + 2xy^2u_yu_{xyy} + 6x^2yu_xu_{xyy} - 2y^3u_xu_{xyy} + x^2u_yu_{xx}\\
&\quad - y^2u_yu_{xx} + 2x^2yu_{yy}u_{xx} + 2y^3u_{yy}u_{xx} - 8xyu_xu_{xx} - 4xy^2u_{xy}u_{xx}\\
&\quad + 2y^3u_{xx}^2 + u_{tt}\big(\left(-x^2 + y^2\right)u_y - 2y\left(x^2u_{yy} - xu_x - 2xyu_{xy} + y^2u_{xx}\right)\big)\\
&\quad - 2x^2yu_yu_{xxy} - 4xy^2u_xu_{xxy} + 2xy^2u_yu_{xxx}\\
&\quad + u\big(-4yu_{tt} - 2\left(x^2 - y^2\right)u_y - x^2u_{ytt} + y^2u_{ytt} - 2yu_{yy} - 4x^2yu_{yy}
\end{aligned}
$$

$$
\begin{aligned}
& - 2x^2yu_{yytt} + 7x^2u_{yyy} - y^2u_{yyy} + 2x^2yu_{yyyy} + 4xyu_x + 2xyu_{xtt} \\
& - 12xu_{xy} + 8xy^2u_{xy} + 4xy^2u_{xytt} + 2x^3u_{xyyy} - 4xy^2u_{xyyy} \\
& + 10yu_{xx} - 4y^3u_{xx} - 2y^3u_{xxtt} - 5x^2u_{xxy} - 4x^2yu_{xxyy} + 2y^3u_{xxyy} \\
& + 4xyu_{xxx} + 2xy^2u_{xxxy} - 20xyu_{xyy} + 11y^2u_{xxy}\big) + 6x^2u_xu_{xy}\big),
\end{aligned}
$$

$$
\begin{aligned}
\Phi^y_{13} = & \frac{1}{12}\big(4xu^2 - 2x^3u_y^2 - 2x^3u_yu_{ytt} + 8xyu_yu_{yy} - 2x^3u_{yy}^2 + 4x^2yu_yu_x \\
& + 2x^2yu_{ytt}u_x + x^2u_{yy}u_x - y^2u_{yy}u_x - 2x^2yu_{yyy}u_x - 2xy^2u_x^2 \\
& + 2x^2yu_yu_{xtt} - 2xy^2u_xu_{xtt} + 6x^2u_yu_{xy} - 6y^2u_yu_{xy} + 4x^2yu_{yy}u_{xy} \\
& + 4x^2yu_yu_{xyy} + 2xy^2u_xu_{xyy} - 4xyu_yu_{xx} - 2x^3u_{yy}u_{xx} - 2xy^2u_{yy}u_{xx} \\
& + x^2u_xu_{xx} - y^2u_xu_{xx} + 4x^2yu_{xy}u_{xx} - 2xy^2u_{xx}^2 \\
& + u_{tt}\big(-2xyu_y + 2x^3u_{yy} - x^2u_x + y^2u_x - 4x^2yu_{xy} + 2xy^2u_{xx}\big) \\
& + 2x^3u_yu_{xxy} - 6xy^2u_yu_{xxy} - 2x^2yu_xu_{xxy} - 2x^2yu_yu_{xxx} \\
& + 2y^3u_yu_{xxx} + 2xy^2u_xu_{xxx} \\
& + u\big(4xu_{tt} - 4xyu_y - 2xyu_{ytt} - 10xu_{yy} + 4x^3u_{yy} + 2x^3u_{yytt} \\
& \quad - 4xyu_{yyy} - 2x^2u_x + 2y^2u_x - x^2u_{xtt} + 12yu_{xy} - 8x^2yu_{xy} \\
& \quad - 4x^2yu_{xytt} - 11x^2u_{xyy} + 5y^2u_{xyy} - 2x^2yu_{xyyy} + 4xy^2u_{xx} \\
& \quad + 2xy^2u_{xxtt} + 20xyu_{xxy} - 2x^3u_{xxyy} + 4xy^2u_{xxyy} + x^2u_{xxx} \\
& \quad - 7y^2u_{xxx} + 4x^2yu_{xxxy} - 2y^3u_{xxxy} - 2xy^2u_{xxxx} + 2xu_{xx} \\
& \quad + y^2u_{xtt}\big)\big).
\end{aligned}
$$

We now select a few of the symmetries above and prove that they are variational using Theorem 1 where $\mathcal{X} = \mathcal{Q}_i\partial_u$.

$$
\begin{aligned}
\mathcal{X}E + \mathcal{A}\mathcal{F}_{\mathcal{Q}_4}E = & \; Q^{xx} + Q^{yy} - Q^{tt} - Q + (-D_{xxx})(u_{xx} + u_{yy} - u_{tt} - u) \\
= & \; u_{xxxxx} + u_{xxxyy} - u_{xxxtt} - u_{xxx} \\
& - u_{xxxxx} - u_{xxxyy} + u_{xxxtt} + u_{xxx} = 0, \\
\mathcal{X}E + \mathcal{A}\mathcal{F}_{\mathcal{Q}_8}E = & \; Q^{xx} + Q^{yy} - Q^{tt} - Q \\
& + \big(D_{xy} + yD_{xyy} - xD_{yyy}\big)(u_{xx} + u_{yy} - u_{tt} - u) \\
= & \; u_{xxxy} + u_{xyyy} - u_{xytt} - u_{xy} + y(u_{xxxyy} + u_{xyyyy} - u_{ttxyy} - u_{xyy}) \\
& - x(u_{xxyyy} + u_{yyyyy} - u_{ttyyy} - u_{yyy}) + 2u_{xyyy} \\
& + xu_{xxyyy} - yu_{xxxyy} - u_{xxxy} + u_{yyyyy} - 2u_{xyyy} - yu_{xyyyy} \\
& - u_{xyyy} - xu_{yyytt} + yu_{xyytt} + u_{xytt} - xu_{yyy} \\
& + yu_{xyy} + u_{xy} = 0, \\
\mathcal{X}E + \mathcal{A}\mathcal{F}_{\mathcal{Q}_{11}}E = & \; Q^{xx} + Q^{yy} - Q^{tt} - Q \\
& + (xD_{xx} + 2yD_{xy} - xD_{yy} - y^2D_{xxx} + 2xyD_{xxy} \\
& \quad - x^2D_{xyy})(u_{xx} + u_{yy} - u_{tt} - u) \\
= & \; x(u_{xxxx} + u_{xxyy} - u_{xxtt} - u_{xx}) \\
& + 2y(u_{xxxy} + u_{xyyy} - u_{ttxyy} - u_{xy}) \\
& - x(u_{xxyy} + u_{yyyyy} - u_{ttyy} - u_{yy}) \\
& - y^2(u_{xxxxx} + u_{xxxyy} - u_{xxxtt} - u_{xxx})
\end{aligned}
$$

$$
\begin{aligned}
&+ 2xy(u_{xxxxy} + u_{xxyyy} - u_{ttxxy} - u_{xxy})\\
&- x^2(u_{xxxyy} + u_{yyyyx} - u_{xyytt} - u_{xyy}) + 4u_{xyy} + 4xu_{xxyy}\\
&+ x^2u_{xxxyy} - 4yu_{xxxyy} - 2xyu_{xxxxy} + xu_{xxyy} - xu_{xxxx}\\
&+ y^2u_{xxxxx} - 2yu_{xxxy} + x^2u_{xyyyy} - 4xu_{xxyy} - 2xyu_{xxyyy}\\
&- xu_{xxyy} + 2yu_{xxxy} + 2u_{xxx} + 2yu_{xxxy} + y^2u_{xxxyy}\\
&- 4u_{xyy} - 2yu_{xyyy} - x^2u_{xyytt} + 2xyu_{xxytt} - xu_{yytt} + xu_{xxtt}\\
&- y^2u_{xxxtt} + 2yu_{xytt} - 2u_{xxx} - x^2u_{xyy} + 2xyu_{xxy} - xu_{yy}\\
&+ xu_{xx} - y^2u_{xxx} + 2yu_{xy} + xu_{yyyy} = 0.
\end{aligned}
$$

Thus, the evolutionary, higher-order symmetries $\mathcal{X}_i = \mathcal{Q}_i\,\partial_u$ are variational.

## 3. Concluding remarks

We studied the classical forms of certain classes of (1+2) Gordon-type equations to determine possible higher-order symmetries. These were obtained by investigating the classical (1+1) Gordon-type equations and, more formally, using the multiplier approach. It turned out that only the (1+2) Klein–Gordon-type equation produced such symmetries which, in fact, were proved to be variational. The corresponding conserved densities were also calculated.

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