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A note on the interplay between symmetries, reduction and conservation laws of Stokes' first problem for third-grade rotating fluids

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Abstract. We investigate the invariance properties, nontrivial conservation laws and interplay between these notions that underly the equations governing Stokes' first problem for third-grade rotating fluids. We show that a knowledge of this leads to a number of different reductions of the governing equations and, thus, a number of exact solutions can be obtained and a spectrum of further analyses may be pursued.

Keywords. Symmetries and conservation laws; third-grade rotating fluids.

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1. Introduction and background

In this article we shall study Stokes' first problem for the unsteady rotating flow of a third-grade fluid. Following the notation and preliminaries in [1], the equations governing the fluid motion can be written in the coupled form as

$$\frac{\partial F}{\partial \tau} + 2i\Omega F = \nu \frac{\partial^2 F}{\partial z^2} + \frac{\alpha_1}{\rho} \frac{\partial^3 F}{\partial z^2 \partial \tau} + \frac{2\beta_3}{\rho} \frac{\partial}{\partial z} \left(\left(\frac{\partial F}{\partial z} \right)^2 \frac{\partial \bar{F}}{\partial z} \right), \tag{1}$$

in which $F(z, \tau) = u + iv$, $\bar{F}(z, \tau) = u - iv$, where u and v are the velocity components, Ω is the constant angular velocity, α_1 and β_3 are the material constants and $\nu = \mu/\rho$ (where ρ is the density and μ is the dynamic viscosity) is the kinematic viscosity. Further, the thermodynamics of the third-grade fluid [2] requires that the material constants and the viscosity satisfy the following conditions:

$$\mu \geq 0, \qquad |\alpha_1| \leq \sqrt{24\mu\beta_3}, \qquad \beta_3 \geq 0.$$

For complete details, the reader is referred to [1]. For the dimensionless form we introduce the following variables:

$$x = \frac{U_0}{\nu} z, \qquad t = \frac{U_0^2}{\nu}\tau, \qquad f = \frac{F}{U_0}, \qquad C = \frac{U_0^2}{\nu}\Omega,$$

where U_0 is the initial velocity. Equation (1) in terms of the dimensionless variables takes the following form:

$$\frac{\partial f}{\partial t} + 2iCf = \frac{\partial^2 f}{\partial x^2} + a\frac{\partial^3 f}{\partial x^2 \partial t} + 2b\left(\left(\frac{\partial f}{\partial x}\right)^2 \frac{\partial \bar{f}}{\partial x}\right), \tag{2}$$

in which

$$a = \frac{\alpha_1 U_0^2}{\rho \nu^2} \quad \text{and} \quad b = \frac{\beta_3 U_0^4}{\rho \nu^3}.$$

We now present some preliminaries. Consider an rth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$,

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}, \tag{3}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, ..., rth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \ldots, n, \tag{4}$$

where the summation convention is used whenever appropriate. A vector $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{5}$$

along the solutions of (3). It can be shown that every admitted conservation law arises from multipliers $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i \tag{6}$$

holds identically (that is, off the solution space) for some vector Φ [3].

We determine the conserved flows by first constructing the multipliers Q_μ which are obtained by noting that the Euler(–Lagrange) operator, $\delta/\delta u^\alpha$, annihilates total divergences, that is, a defining equation for Q_μ would be

$$\frac{\delta}{\delta u^\alpha}\left[Q_\mu G^\mu\right] = 0. \tag{7}$$

The conserved vector Φ is then obtained by a well-known 'homotopy' formula [3].

We present some comments regarding Lie symmetries and conservation laws. The Lie symmetry approach is now an established route for the reduction of differential equations and its advantages in the analysis of nonlinear partial differential equations are vast. The method centres around the algebra of one-parameter Lie groups of transformations that are admitted by the partial differential equation. Once known, the reduction of the partial differential equation is standard and may lead to exact (symmetry invariant) solutions (see [4,5]).

There are a number of reasons to find conserved densities for partial differential equations. Some conservation laws are physical (for example, the conservation of momentum and energy) and others facilitate the analysis of the partial differential equation and predicts integrability. Also, some reasons are related to the numerical solution of partial differential equations. For example, one should check whether the conserved quantities are in fact constant (see [3]). That is, if for example, $u = u(x,t)$, leading to $\Phi^t = \Phi^t(x,t)$ and $\Phi^x = \Phi^x(x,t)$ and $\Phi^x(x,t) \to 0$ as $|x| \to \infty$, then the conserved form $D_t\Phi^t + D_x\Phi^x = 0$ is $\partial_t\Phi^t + \partial_x\Phi^x = 0$ so that

$$\int_{-\infty}^{\infty} (\partial_t\Phi^t + \partial_x\Phi^x)\mathrm{d}x = 0$$

so that differentiating with respect to t leads to

$$\int_{-\infty}^{\infty} \Phi^t\mathrm{d}x = \text{constant}$$

for all solutions of the partial differential equation.

Lastly, the use of symmetry properties of a given system of partial differential equations to construct or generate new conservation laws from known conservation laws has been investigated [6,7].

2. Symmetries and conservation laws

We now study the symmetries, conservation laws and their relationships for the system under investigation.

2.1 *Symmetries and reductions*

In order to do a symmetry analysis of eq. (2) we write $f = u + iv$ so that (2) becomes the system,

$$u_t - 2Cv = u_{xx} + au_{txx} + 2b\frac{\partial}{\partial x}\left(u_x^3 + u_x v_x^2\right),$$
$$v_t + 2Cu = v_{xx} + av_{txx} + 2b\frac{\partial}{\partial x}\left(v_x^3 + u_x^2 v_x\right). \quad (8)$$

The Lie point symmetry generators are given by the standard procedure [5]. That is, suppose

$$X = \mathcal{X}(x,t,u,v)\frac{\partial}{\partial x} + \mathcal{T}(x,t,u,v)\frac{\partial}{\partial t} + \mathcal{U}(x,t,u,v)\frac{\partial}{\partial u} + \mathcal{V}(x,t,u,v)\frac{\partial}{\partial v}$$

Table 1. The commutator table for the generators X_i ($i = 1$–5).

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2		0	0	$-2CX_5$	$2CX_4$
X_3			0	$-X_5$	X_4
X_4				0	0
X_5					0

is an infinitesimal form of the one-parameter Lie group of transformations keeping invariant the combined equations in the system (8) along its solution space. It turns out that the generators form a five-dimensional Lie algebra with basis given by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u},$$
$$X_4 = -\sin(2Ct)\frac{\partial}{\partial v} + \cos(2Ct)\frac{\partial}{\partial u}, \quad X_5 = \cos(2Ct)\frac{\partial}{\partial v} + \sin(2Ct)\frac{\partial}{\partial u}.$$

Whilst the first three are not surprising, X_4 and X_5 are somewhat unexpected. Table 1 is the commutator table. Individually or linear combinations of the generators X_i ($i = 1$–5) lead to a reduction and exact (symmetry invariant) solutions of (8).

(i) We first perform a reduction using $X = (1/2m)\,\partial_t + X_5$, for some constant m, which leads to the characteristic equation

$$\frac{2m\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{\sin(2Ct)} = \frac{\mathrm{d}v}{\cos(2Ct)}$$

for its invariants, viz., $u_* = u + (m/C)\cos(2Ct)$ and $v_* = v - (m/C)\sin(2Ct)$, which will be functions of a single new independent variable $\alpha = x$. With this change of variables, (8) becomes the system of ordinary differential equations,

$$-2v_* = u_*'' + 2b\left(3u_*'^2 u_*'' + v_*'^2 u_*'' + 2u_*' v_*' v_*''\right),$$

$$2u_* = v_*'' + 2b\left(3v_*'^2 v_*'' + u_*'^2 v_*'' + 2u_*' v_*' u_*''\right),$$

or

$$-2v_* = D\left(u_*' + 2b\left(u_*'^3 + u_*' v_*'^2\right)\right),$$
$$2u_* = D\left(v_*' + 2b\left(v_*'^3 + v_*' u_*'^2\right)\right). \quad (9)$$

The right-hand side of system (9) has an Euler–Lagrange form but due to the opposite signs of the left-hand side, a complete Lagrangian does not exist. One may then pursue a partial Lagrangian approach (see [8]) using the function

$$L = \frac{1}{2}u'^2 + \frac{1}{2}v'^2 + \frac{b}{2}\left(u_*'^2 + v_*'^2\right)^2.$$

This particular form of the partial Lagrangian for which $\delta L/\delta u_* = 2v_*$ and $\delta L/\delta v_* = -2u_*$ do not lead to generators for the construction of conserved vectors. Alternatively, since the system admits the translation symmetry $\partial/\partial\alpha$, some reduction of the system is possible.

(ii) The combined generator $Y = \partial_t + k\partial_x - v\partial_u + u\partial_v$, for some constant k, leads to the invariants $\alpha = x - kt$, $u_* = u\sin t + v\cos t$ and $v_* = u\cos t - v\sin t$ so that

$$\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1}\begin{pmatrix} u_* \\ v_* \end{pmatrix}$$

and

$$\begin{pmatrix} u_{(i)} \\ v_{(i)} \end{pmatrix} = A^{-1}\begin{pmatrix} u_*^{(i)} \\ v_*^{(i)} \end{pmatrix},$$

where

$$A = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix}$$

and $u_*^{(i)}$ is u_*', and similarly with higher-order derivatives u_*'' or u_*'''. Also,

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = kA\begin{pmatrix} u_*' \\ v_*' \end{pmatrix} + \begin{pmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{pmatrix}\begin{pmatrix} u_* \\ v_* \end{pmatrix}.$$

Substituting these into (8) leads to the system of ordinary differential equations,

$$\begin{aligned} v_* + 2Cv_* &= u_*'' + 6bu_*'^2u_*'' + k\left(u_*' - au_*'''\right) + 2bu_*''v_*'^2 + av_*'' + 4bu_*'v_*'v_*'', \\ u_* + 2Cu_* &= v_*'' + 6bv_*'^2v_*'' + k\left(v_*' - av_*'''\right) + 2bv_*''u_*'^2 - au_*'' + 4bu_*'v_*'u_*''. \end{aligned}$$

Other reductions like the travelling wave can be pursued. The solution of the respective reduced systems of ordinary differential equations will then involve repeating the symmetry approach or alternative ones like Laplace transforms, homotopy analysis or even some numerical approach.

In terms of f and $\bar{f}$, the generators of eq. (1) have the form

$$\begin{aligned} &\bar{X}_1 = \frac{\partial}{\partial x}, \qquad \bar{X}_2 = \frac{\partial}{\partial t}, \\ &\tilde{X}_3 = -i\bar{X}_3 = f\frac{\partial}{\partial f} - \bar{f}\frac{\partial}{\partial \bar{f}}, \qquad \bar{X}_4 = \mathrm{e}^{-i(2Ct)}\frac{\partial}{\partial f} + \mathrm{e}^{i(2Ct)}\frac{\partial}{\partial \bar{f}}, \\ &\tilde{X}_5 = i\bar{X}_5 = -\mathrm{e}^{-i(2Ct)}\frac{\partial}{\partial f} + \mathrm{e}^{i(2Ct)}\frac{\partial}{\partial \bar{f}}. \end{aligned}$$

2.2 *Conservation laws and invariance*

Here, since (8) is a system of partial differential equations in u and v, the characteristic equation (7) becomes

$$\frac{\delta}{\delta(u,v)}\left(Q_1\left[u_t - 2Cv - \left(u_{xx} + au_{txx} + 2b\frac{\partial}{\partial x}\left(u_x^3 + u_xv_x^2\right)\right)\right]\right.$$
$$\left. +\ Q_2\left[v_t + 2Cu - \left(v_{xx} + av_{txx} + 2b\frac{\partial}{\partial x}\left(v_x^3 + u_x^2v_x\right)\right)\right]\right) = 0. \tag{10}$$

Since Q_1 and Q_2 are chosen to be dependent on derivative up to second-order, the calculations are extremely cumbersome and are done with the aid of a computer. The nontrivial solutions obtained, it turns out, are independent of derivatives. We find that $Q_1 = Q_1(t)$, $Q_2 = Q_2(t)$ and

$$Q_1' = 2CQ_2, \qquad Q_2' = -2CQ_1.$$

If we choose $C = 1/2k$ and Q_2 replaced by kQ_2 for the constant k, then the conditions become $Q_1' = (1/k)Q_2$ and $Q_1 + kQ_2' = 0$. Thus, we require

$$Q_1'' + \frac{1}{k^2}Q_1 = 0.$$

Hence, we find the multipliers to be

$$(Q_1, Q_2) = \left(p\cos\left(\frac{t}{k}\right) + q\sin\left(\frac{t}{k}\right), q\cos\left(\frac{t}{k}\right) - p\sin\left(\frac{t}{k}\right)\right),$$

leading to the nontrivial conserved flow with components, via the homotopy formula,

$$\begin{aligned}
\Phi^x = &-\frac{1}{3k}\left(6bk\left(p\cos\left(\frac{t}{k}\right) + q\sin\left(\frac{t}{k}\right)\right)u_x^3\right.\\
&+\left((ap+3kq)\cos\left(\frac{t}{k}\right) + (-3kp+aq)\sin\left(\frac{t}{k}\right)\right)v_x\\
&+6bk\left(q\cos\left(\frac{t}{k}\right) - p\sin\left(\frac{t}{k}\right)\right)u_x^2 v_x\\
&+6bk\left(q\cos\left(\frac{t}{k}\right) - p\sin\left(\frac{t}{k}\right)\right)v_x^3\\
&+\left((3kp-aq)\cos\left(\frac{t}{k}\right) + (ap+3kq)\sin\left(\frac{t}{k}\right)\right.\\
&\qquad\left.+6bk\left(p\cos\left(\frac{t}{k}\right) + q\sin\left(\frac{t}{k}\right)\right)v_x^2\right)u_x\\
&+2ak\left(\left(p\cos\left(\frac{t}{k}\right) + q\sin\left(\frac{t}{k}\right)\right)u_{xt}\right.\\
&\qquad\left.\left.+\left(q\cos\left(\frac{t}{k}\right) - p\sin\left(\frac{t}{k}\right)\right)v_{xt}\right)\right),\\
\Phi^t = &\frac{1}{3}\left(3\left(p\cos\left(\frac{t}{k}\right) + q\sin\left(\frac{t}{k}\right)\right)u + 3\left(q\cos\left(\frac{t}{k}\right) - p\sin\left(\frac{t}{k}\right)\right)v\right.\\
&-a\left(\left(p\cos\left(\frac{t}{k}\right) + q\sin\left(\frac{t}{k}\right)\right)u_{xx}\right.\\
&\qquad\left.\left.+\left(q\cos\left(\frac{t}{k}\right) - p\sin\left(\frac{t}{k}\right)\right)v_{xx}\right)\right).
\end{aligned}$$

An invariance analysis shows that Q_1 and Q_2 are strictly invariant under X_i for $i = 1, 3, 4, 5$, that is, $X_iQ_1 = 0$ and $X_iQ_2 = 0$, so that the conserved vector (Φ^x, Φ^t) is associated (see [6]) with the generators X_i, for $i = 1, 3, 4, 5$. However,

$$X_2Q_1 = -\frac{p}{k}\sin\left(\frac{t}{k}\right) + \frac{q}{k}\cos\left(\frac{t}{k}\right), \quad X_2Q_2 = -\frac{q}{k}\sin\left(\frac{t}{k}\right) - \frac{p}{k}\cos\left(\frac{t}{k}\right).$$

Making the replacement $P = q/k$ and $Q = -p/k$,

$$X_2 Q_1 = P\cos\left(\frac{t}{k}\right) + Q\sin\left(\frac{t}{k}\right), \quad X_2 Q_2 = Q\cos\left(\frac{t}{k}\right) - P\sin\left(\frac{t}{k}\right)$$

so that (Q_1, Q_2) is ray invariant under X_2.

It can be shown that in the original variable $f = u + iv$,

$$\begin{aligned}\Phi^t &= \frac{p}{2}\left((f + f_{xx})\,\mathrm{e}^{iT} + \left(\bar{f} + \bar{f}_{xx}\right)\mathrm{e}^{-iT}\right) \\ &\quad + \frac{q}{2}\left((f + f_{xx})\,\mathrm{e}^{-i(\pi/2-T)} + \left(\bar{f} + \bar{f}_{xx}\right)\mathrm{e}^{i(\pi/2-T)}\right),\end{aligned}$$

where $T = t/k$. Note that Φ^t is strictly invariant under translation in x and rotation which are linear momentum and 'angular momentum' in the complex plane, respectively. We again note a feature of the conserved density, namely,

$$\int_{-\infty}^{\infty} \Phi^t \mathrm{d}x = \text{constant}.$$

3. Conclusions

We obtained the invariance properties (via the Lie point symmetry generators), nontrivial conservation laws and studied the interplay between these for the equations governing Stokes' first problem for third-grade rotating fluids. We showed that a number of different reductions of the governing systems and, thus, a number of exact solutions, can be obtained and further analyses may be pursued. This procedure may be applied to a number of problems of a similar kind.

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