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Isometric embeddings in cosmology and astrophysics

GARETH AMERY*, JOTHI MOODLEY and JAMES PAUL LONDAL
Astrophysics and Cosmology Research Unit, School of Mathematical Sciences,
University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa
*Corresponding author. E-mail: AmeryG1@ukzn.ac.za

Abstract. Recent interest in higher-dimensional cosmological models has prompted some significant work on the mathematical technicalities of how one goes about embedding spacetimes into some higher-dimensional space. We survey results in the literature (existence theorems and simple explicit embeddings); briefly outline our work on global embeddings as well as explicit results for more complex geometries; and provide some examples. These results are contextualized physically, so as to provide a foundation for a detailed commentary on several key issues in the field such as: the meaning of 'Ricci equivalent' embeddings; the uniqueness of local (or global) embeddings; symmetry inheritance properties; and astrophysical constraints.

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1. Introduction

Extensions to general relativity often involve extra dimensions. Inspired by string theory [1], D-brane theory [2] and Horava–Witten theory [3], there has been a great deal of interest in extra-dimensional effective theories, evidenced by phenomenological models such as the Randall–Sundrum [4], Arkani–Hamed–Dimopoulos–Dvali [5] and Dvali–Gabadadze–Porrati [6] scenarios, which attempt to tackle problems such as the mass-hierarchy problem, the origins of the primordial spectrum, the inflationary field and dark energy. Gauss–Bonnet gravity [7] takes a complementary approach, building a natural five-dimensional theory by considering additional terms in the action that only contribute for more than four dimensions [8]. Induced matter theory [9] attempts to describe energy–momentum on the brane in terms of purely geometric effects in higher dimensions.

This higher-dimensional physics is complemented by the recent proof of several existence theorems for both local [10–14] and global embeddings [15]. In purely geometric embedding theory, a given spacetime (or 'brane') is embedded in a higher-dimensional spacetime (or 'bulk'), and the bulk field equations solved. This is dual to the braneworld perspective in which an ansatz is made for the bulk, leading to projected effects on the brane. However, there are several potential points of confusion when interpreting these results, which this article aims to clarify. Difficulties in obtaining explicit solutions motivates the study of constraints that may be imposed. In particular, we discuss possible astrophysical constraints, and the inheritance properties of the (conformal) Killing geometry.

In §2 we provide background material, briefly describe the extent of known solutions, and discuss several important existence theorems. In particular, we comment on the meaning of 'Ricci-equivalent' embeddings and on the uniqueness of solutions to the embedding equations. These issues are crucial in adequately evaluating the import of the existence theorems. As examples, we embed cosmic string spacetimes in various ways. In §3 we discuss the possible constraints arising from astrophysical embeddings. In §4 we describe how the (conformal) Killing geometry of static spherically symmetric (SSS) spacetimes is inherited by the five-dimensional Einstein space into which it is embedded. Finally, in §5 we summarize our results and present some potentially profitable future directions.

We consistently adopt the following notational conventions: Roman lower case indices label coordinates $(0, ..., n-1)$ of the embedded space; Roman upper case indices label the spatial coordinates $(1, ..., n-1)$ of this space; Greek indices label the coordinates $(0, 1, ..., m-1;\ m > n)$ of the embedding space and Greek upper case the extra-dimensional components $(n, n+1, ..., m-1)$. A tilde may be used to denote quantities pertaining to the five-dimensional space, and an overbar denotes quantities derived from the n-dimensional component of the higher-dimensional metric. A prime and an overdot are used to denote partial differentiation with respect to the coordinates r and y, respectively. The lower-dimensional space is referred to as the embedded space and the higher-dimensional space as the embedding space.

2. Isometric embeddings

A series of theorems addresses the problem of locally embedding an n-dimensional pseudo-Riemannian manifold with an analytic metric into another pseudo-Riemannian manifold. The Janet–Cartan theorem states that one can construct a local isometric embedding into a flat ($R_{abcd} = 0$) pseudo-Riemannian manifold with a dimension $d \leq n(n+1)/2$ [10]. The Campbell–Magaard theorem guarantees a local isometric embedding into a $(n+1)$-dimensional Ricci-flat ($R_{ab} = 0$) manifold [11]. There exist several applications of this theorem [17,18]. Dahia and Romero [12] provided existence results for embeddings into any given non-degenerate $(n+1)$-dimensional pseudo-Riemannian manifold. The embedding will be unique for all Ricci equivalent tensors **S** and **R**: $R_{ab}(x^\gamma) = (\partial f^i/\partial x^a)(\partial f^j/\partial x^b)S_{ij}(x'^\kappa)$, where $f(x^\gamma) = x'^k$. They further demonstrated that we may choose the embedding space to be an $(n+1)$-dimensional Einstein space. In subsequent papers, they considered the case in which the energy–momentum tensor of the embedding space is singular [13], and demonstrated that the embedding is well-defined with respect to stability and causality [14]. It is crucially important to note that the defining diffeomorphism for Ricci equivalence reduces to a coordinate transformation (and the embedding is truly isometric) only for Einstein embedding spaces. Furthermore, while the solution for any given scenario is unique, one could choose to embed into (say) any Einstein space.

Causal and topological issues lead one to consider the existence of global embeddings. Moodley and Amery [15] recently established that one may promote local existence results to global ones. However, the difficulties with the local case are inherited, by construction;

and the uniqueness problem is exacerbated – there could be holes/singularities in the bulk, away from the brane.

The existence of a local isometric embedding is equivalent to the existence of solutions to the Gauss–Codazzi–Ricci (GCR) equations (essentially the field equations with specified initial/boundary data on the brane):

$$\widetilde{R}^4_i = \frac{\epsilon}{\phi}\tilde{g}^{jk}(\nabla_j\Omega_{ik} - \nabla_i\Omega_{jk}), \tag{1}$$

$$\widetilde{G}^4_4 = -\frac{1}{2}\tilde{g}^{ik}\tilde{g}^{jm}(\widetilde{R}_{ijkm} + \epsilon(\Omega_{ik}\Omega_{jm} - \Omega_{jk}\Omega_{im})), \tag{2}$$

$$\widetilde{R}_{ab} = \bar{R}_{ab} + \epsilon a^{jm}(\Omega_{jm}\Omega_{ab} - 2\Omega_{am}\Omega_{bj}) - \frac{\epsilon}{\phi}\frac{\partial\Omega_{ab}}{\partial y} + \frac{1}{\phi}\nabla_a\nabla_b\phi, \tag{3}$$

where the n-dimensional embedded metric $\mathrm{d}s^2 = g_{ik}(x^a)\,\mathrm{d}x^i\mathrm{d}x^k$ and the $(n+1)$-dimensional embedding metric is (in Gaussian coordinates)

$$\mathrm{d}\tilde{s}^2 = \tilde{g}_{\alpha\beta}(x^i, y)\mathrm{d}x^\alpha\mathrm{d}x^\beta = a_{ab}(x^i, y)\,\mathrm{d}x^a\mathrm{d}x^b + \epsilon\phi^2\,\mathrm{d}y^2, \tag{4}$$

with $\epsilon = \pm 1$, $\phi = \phi(x^a, y)$ an arbitrary function (that is henceforth and w.l.o.g. set equal to 1), and $a_{ab}(x^i, 0) = g_{ab}(x^i)$. Here, $\bar{R}_{ab}$ is the Ricci tensor obtained from a_{ab}, and the extrinsic curvature takes the form $\Omega_{ab} = -((1/2\phi)(\partial a_{ab}/\partial y))$. For $\det|a_{ab}| \neq 0$, one may employ repeated appeals to the Cauchy–Kowalewskaja theorem to prove that there exists a solution to the above system; and to provide a calculation technique: one solves (1) and (2) on the brane (generating an initial/boundary condition for Ω_{ab}), and uses (3) to propagate into the bulk [12].

We shall principally consider embeddings into a vacuum bulk. This is motivated by the isometric existence theorem for Einstein spaces described above, computational simplicity, and by physical importance: five-dimensional Einstein spaces are of direct relevance to the Randall–Sundrum scenarios [4], Gauss–Bonnet gravity [7], induced matter theory [9], and type-IIA supergravity [16]. Exact solutions to the GCR equations are notoriously hard to obtain for branes with complex geometry: most are either for cosmological spacetimes embedded into Einstein spaces [17–19], or for non-trivial ${}^{(4)}R$ into 'stacking' product topologies with non-trivial curvature/energy–momentum (for example, [20]). However, we recently established embeddings of the global monopole metric into Minkowski space [21], and a Vaidya-class metric into an Einstein space [22].

We now provide an example of how one may tackle embedding problems. We shall assume a general form for the embedding metric and solve the propagation eq. (3). The Gauss and Codazzi equations then impose constraints that limit the solution space and/or the class of spacetimes that may be so embedded. Similar techniques have been used to generate energetically rigid (Killing vector preserving) warped local embeddings for spherically symmetric spacetimes (Einstein Universe and Schwarzschild–de Sitter) [23], and the global monopole exterior [21].

Let us consider the spacetime of a four-dimensional cosmic string exterior:

$$\mathrm{d}s^2 = \mathrm{e}^{\hat{A}(r)}[-\,\mathrm{d}t^2 + \,\mathrm{d}z^2] + \,\mathrm{d}r^2 + \mathrm{e}^{\hat{B}(r)}\,\mathrm{d}\theta^2,$$

satisfying the vacuum Einstein field equations:

$$-R_{11} = 0 = R_{00} = \left[-\frac{1}{2}\hat{A}'' - \frac{1}{2}(\hat{A}')^2 - \frac{1}{4}\hat{A}'\hat{B}'\right] e^{\hat{A}},$$
$$0 = R_{22} = \hat{A}'' + \frac{1}{2}\hat{B}'' + \frac{1}{2}(\hat{A}')^2 + \frac{1}{4}(\hat{B}')^2,$$
$$0 = R_{33} = \left[\frac{1}{2}\hat{B}'' + \frac{1}{4}(\hat{B}')^2 + \frac{1}{2}\hat{A}'\hat{B}'\right] e^{\hat{B}}. \quad (5)$$

Solutions to this system must have the form of either a conical geometry [24]:

$$ds^2 = a_1[-dt^2 + dz^2] - dr^2 + (a_2 r + a_3)^2 d\theta^2, \quad (6)$$

or a special case of the Kasner metric [24,25]:

$$ds^2 = (b_1 r + b_2)^{4/3}[-dt^2 + dz^2] - dr^2 + (b_1 r + b_2)^{-2/3} d\theta^2, \quad (7)$$

where a_i and b_i are arbitrary constants. We note that, since the spacetime is Ricci-flat, it is known to have an embedding into an Einstein $\tilde{g}_{\alpha\beta}$, with $a_{ij} = e^{\sqrt{\frac{-2\epsilon\Lambda}{3}}y} g_{ij}$ [26]. We shall regain this result in a special case and also generate alternate Ricci-flat embeddings.

In Gauss normal form (with $\phi = 1$) we make the metric ansatz

$$a_{ij} = \bar{g}_{ij} = \text{diag}[-e^A, e^A, 1, e^B], \quad (8)$$

where $A(y, r) = \hat{A}(r) + \bar{A}(y)$, and similarly for B. We calculate $\bar{R}_{ij}$ as

$$\bar{R}_{00} = R_{00}e^{\bar{A}(y)}, \quad \bar{R}_{11} = R_{11}e^{\bar{A}(y)}, \quad \bar{R}_{22} = R_{22}, \quad \bar{R}_{33} = R_{33}e^{\bar{B}(y)}, \quad (9)$$

and note that they vanish by eq. (5). Using our ansatz and (9) in the propagation eq. (3), and setting $\Lambda = 0$, we find that $\bar{A}$ satisfies the first and third equations of (5), and hence certainly permits solutions of forms (6) and (7), with $r \to y$, and constants $\bar{a}_i, \bar{b}_i$. For the initial condition $a_{ij}(0, r) = g_{ij}(r)$ to be satisfied, we require that $\bar{A}(0) = 0 = \bar{B}(0)$. With

$$\bar{\Omega}_{ij}(0, r) = \Omega_{ij} = \text{diag}[-ag_{00}, ag_{11}, 0, bg_{33}],$$

and $a = -(1/2)\dot{\bar{A}}(0)$, $b = -(1/2)\dot{\bar{B}}(0)$ constants, the Gauss and Codazzi equations yield the additional constraints

$$a\hat{A}' + \frac{b}{2}\hat{B}' = 0, \quad 2a[a + 2b] = 0. \quad (10)$$

Suppose we wish to embed a conical spacetime (6) with $\bar{A}(y)$, $\bar{B}(y)$ also having the conical form, and hence, from the initial conditions, $\bar{a}_1 = 1$ and $\bar{a}_3 = \pm 1$. Then equations (10) imply that either $\bar{a}_2 = 0$ or $a_2 = 0$. The former case yields

$$\tilde{g}_{\alpha\beta} = \text{diag}[-a_1, a_1, 1, (a_2 r + a_3)^2, \epsilon],$$

which is a stacking of the original conical geometry, $M^{(\text{conical})}_{(r)} \times Y_{(y)}$, and regains the results of [26]. The latter yields

$$\tilde{g}_{\alpha\beta} = \text{diag}[-a_1, a_1, 1, (a_3)^2(\bar{a}_2 y \pm 1)^2, \epsilon],$$

which is a special case of a conical geometry embedded into what is essentially another $M^{(\text{conical})}_{(y)} \times R_{(r)}$, where the bracketed subscript indicates the functional dependence. Similarly, if we wish to embed a conical spacetime (6) with $\bar{A}(y)$, $\bar{B}(y)$ having the Kasner form

(7), then $\bar{b}_2 = 1$, and eqs (10) imply that $a_1 = -(a_3)^2/6$ and $a_2 = 0$. Hence, we have a (special) conical geometry embedded into $M^{(\text{Kasner})}_{(y)} \times R_{(r)}$. Other permutations may be similarly constructed. These examples serve also to stress how one can very easily embed into a more complicated geometry. For physical/philosophical reasons, however, one might prefer to consider the simplest embedding geometries.

3. Astrophysical constraints

There exist many different classes of higher-dimensional gravitational models. Broadly speaking, they may be grouped into several categories: Kaluza–Klein compactified models (for example, ref. [7]); induced matter models [9]; asymptotically flat bulks (for example, ref. [27]); and δ function branes (with or without energy–momentum in the bulk) [28]. The first three are simply related to local isometric embeddings, while the last is a little more subtle [13]. However, all of them are typically posited in a global language, and one must take care in moving from a local to global picture since there is a lot of freedom in model choice [15] – for example, one can globally construct Z_2-symmetric models from non-Z_2-symmetric local embeddings: for the global monopole exterior ($r > r_c$), we may embed as [21]

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + K^{-1}\,\mathrm{d}r^2 + (r - \alpha y)^2\,[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2] + \epsilon\,\mathrm{d}y^2,$$

which has $r =$ constant surfaces with an energy–momentum singularity at positive $y = r/\alpha$. However, for $y < r/\alpha$, we can transform to five-dimensional Minkowski space, which may then be used to generate many Z_2-symmetric global embeddings.

If one recalls that the motivation (at least in part) for non-compact extra dimensions is to deal with the hierarchy problem, it is instructive to consider how these types of models achieve matter localization, and whether astrophysical constraints are likely to be obtainable. Kaluza–Klein models are naturally purely geometric/topological constructs, that are periodic instead of Z_2-symmetric, and the confinement is achieved by compactification. They do not, in themselves, offer obvious routes to new astrophysics, but may do so in the context of a different higher-dimensional gravitational theory (such as Einstein–Gauss–Bonnet gravity [7]). Induced matter gravity typically supposes that the bulk is globally empty and that four-dimensional energy–momentum is purely a consequence of five-dimensional geometry. They therefore need not be Z_2-symmetric, since there are no bulk matter flows.

While the first two categories avoid the issue, the δ-function and asymptotically flat approaches, naturally achieve matter confinement, but at a price: they are subject to quite severe constraints unless we allow significant energy–momentum in the bulk near the brane. We phrase our discussion in terms of the Shiromiza–Maeda–Sasaki [28] approach to the Randall–Sundrum [4] type brane world models, with

$$T^{(5)}_{\alpha\beta} = T^{(5)}_{\alpha\beta}\,\delta(y), \quad T^{(5)}_{44} = -2\xi, \quad T^{(5)}_{4i} = 0,$$

where ξ is the vacuum energy and $T^{(5)}_{ij}$ the four-dimensional energy–momentum. We shall provide only the bare minimum of details we require, referring the reader to [28,29] for more information. The five-dimensional field equations are

$$G^{(5)}_{\alpha\beta} = -\Lambda\,\tilde{g}_{\alpha\beta} + \kappa T^{(5)}_{\alpha\beta}.$$

The Gauss equation gives the curvature tensor for the brane in terms of the five-dimensional curvature and the extrinsic curvature. To evaluate the four-dimensional projected field equations on the brane (i.e. at $y = 0$), we therefore must first determine Ω_{ij}. The Lanczos junction conditions [30] give the discontinuity in the extrinsic curvature on the brane: $[\Omega_{ij}]_{\pm} = \kappa(T_{ij} - [g_{ij}/(n-1)]T)$. As we expect there to be no flow through the brane we impose Z_2-symmetry on the bulk $g(x^i, y) = g(x^i, -y)$ so that the extrinsic curvature is given by

$$-\Omega^{-}_{ij} = \Omega^{+}_{ij} = \frac{\kappa}{2}\left(T_{ij} - \frac{g_{ij}}{3}T\right) = \frac{1}{2}\left(R_{ij} - \frac{g_{ij}}{6}R\right). \tag{11}$$

The projected field equations so obtained contain both high-energy corrections and a Weyl term encoding global contributions from the five-dimensional graviton. Hence, the Weyl fluid cannot be specified from data on the brane. Treatments of this open system have included weak field analyses, Taylor expansion about the brane, or the imposition of appropriate initial conditions on a four-dimensional space-like surface, with boundary conditions at the brane [31]. For our purposes it is sufficient to note that one can partially determine the Weyl fluid on the brane using the Bianchi identities since the matter content of the brane sources the Weyl fluid. If one assumes that the matter on the brane is a perfect fluid (or minimally coupled scalar field), then we may obtain an effective total fluid formalism [29], in which there is an unspecified Weyl anisotropic stress. This is significant as, to successfully embed (apparently perfect fluid) astrophysical models, we typically need to include an effective anisotropic term $\Pi^{\mathrm{tot}}_{ij} = \Pi^{*}_{ij}$ on the brane, so as to relax the constraints imposed by eq. (11). In particular with $\Lambda^{(4)} = 0$, and assuming a fluid with anisotropic stress in (11), we obtain

$$p = -\frac{-24\epsilon\Lambda - 12\epsilon\mu + 2\mu^2 + 3\Pi^i_j\Pi^j_i}{6(\mu + 6\epsilon)}. \tag{12}$$

As Π^i_j is unspecified it can be used to generate any effective equation of state that we require. If we assume a bulk such that there is no anisotropic stress, eq. (12) becomes

$$p = -\frac{-24\epsilon\Lambda - 12\epsilon\mu + 2\mu^2}{6(\mu + 6\epsilon)}. \tag{13}$$

No known static spherically symmetric (SSS) solution has this equation of state. (Indeed there is only one SSS solution that has a non-parametric equation of state [32].) However, in the spirit of Stephani's constructions [33], one may apply the Oppenheimer–Volkoff [34] technique to the equation of state (13) and match to the Schwarzschild exterior solution to obtain [26]

$$\mathrm{e}^{\nu} = \left(1 - \frac{2m}{r_{\mathrm{b}}}\right)\frac{(\mu + 6\epsilon)^2}{\sqrt{\mu^2 + 12\epsilon\mu + 6\epsilon\Lambda}},$$

$$\mathrm{e}^{\lambda} = \frac{1 + r\mu'\frac{12\Lambda\epsilon - 36 + 12\epsilon\mu + \mu^2}{(6\epsilon+\mu)(6\Lambda\epsilon + 12\epsilon\mu + \mu^2)}}{1 - r^2\frac{\mu^2 - 6\mu\epsilon - 12\Lambda\epsilon}{3(\mu + 6\epsilon)}}, \tag{14}$$

(where r_b is the boundary of the star and m is a constant related to the mass of the star) subject to the consistency equation

$$\mu = \frac{1 - r^2 \frac{\mu^2 - 6\mu\epsilon - 12\Lambda\epsilon}{3(\mu+6\epsilon)}}{1 + r\mu' \frac{12\Lambda\epsilon - 36 + 12\epsilon\mu + \mu^2}{(6\epsilon+\mu)(6\Lambda\epsilon + 12\epsilon\mu + \mu^2)}} (\lambda'/r - 1/r^2) + 1/r^2. \tag{15}$$

We note that this solution may be inflationary (negative pressure), but is, therefore, not astrophysically realistic. Moreover, the consistency eq. (15) imposes strong constraints: for example, dust ($p = 0$) implies that $\mu = 0 = \Lambda$. So this class of solutions does not include Z_2-symmetric Einstein embedded dust solutions, although it is possible to have a Minkowski dust solution embedded into a Z_2-symmetric space if one abandons a bulk with $T^{(5)}_{\alpha\beta} = 0$ [26]. Further analysis would involve the solution of eq. (15) and the investigation of four-dimensional solutions satisfying (12).

If we restrict our attention to high-energy density (very curved) brane solutions with $\mu \gg 1, |\Lambda|$, then (12) and (13) become $p \approx -\frac{\mu}{3} - \frac{\Pi^i_j \Pi^j_i}{2\mu}$ and $p \approx -\frac{\mu}{3}$, respectively. We thus regain (for this special case) the result of Deruelle and Katz [27] obtained by Taylor expanding into a bulk that is approximately an Einstein space near the brane. We note that, similarly to [27], if the geometry of the bulk is freely chosen, reasonable equations of state may be regained via (12). The price, however, is that, near the brane, the Weyl anisotropic stress must be of the order of the brane energy – this is a significant deviation from the empty AdS.

4. Killing geometry inheritance

In this section we discuss the inheritance properties of the (conformal) Killing geometry when embedding static spherically symmetric (SSS) spacetimes with metric

$$g_{ik} = \mathrm{diag}(-\mathrm{e}^{2\nu(r)}, \mathrm{e}^{2\lambda(r)}, r^2, r^2 \sin^2\theta), \tag{16}$$

into a vacuum bulk. These spacetimes are typically used to provide astrophysical models, and an analysis of their embedding properties facilitates the confrontation between higher-dimensional cosmological models and astrophysics. For a SSS four-dimensional spacetime, embedded into an empty five-dimensional Einstein space, it may be shown that the embedding space metric may take the form

$$\tilde{g}_{ij} = a_{ij} = A(y,r) g_{00}\, \delta^0_i \delta^0_j + B(y,r) g_{CD}\, \delta^C_i \delta^D_j, \tag{17}$$

where, in general, $A \neq B$, and the functions of y, r are unlikely to be separable. The embedding must obey the initial conditions $A(0,r) = 1 = B(0,r)$, as well as two conditions (arising from the extrinsic curvature) on the first partials with respect to the coordinate y [26].

A conformal Killing vector (CKV) $\mathbf{X}$ is defined by the action of the Lie derivative on the metric tensor field: $\mathcal{L}_\mathbf{X}\mathbf{g} = 2\psi\mathbf{g}$, with the conformal factor $\psi(\mathbf{x})$. We shall also be concerned with the special cases: $\psi = 0$, a Killing vector field (KV); and $\psi_{,a} = 0 \neq \psi$, a homothetic Killing vector field (HKV). CKVs generate isometries along null geodesics and are useful in simplifying the field/embedding equations, in classification schema [35], and in applications to, for instance, perturbation theory [36] and singularity theorems [37]. HKVs scale distances by a constant factor, preserve the null geodesic affine parameter, and are related to self-similarity [35]. Proper KVs characterize the (continuous) symmetry properties of pseudo-Riemannian spaces in an invariant fashion: they generate first integrals along time-like geodesics via Noether's theorem, and may be used to investigate the physical properties of a spacetime, via the structure of their isometry group [35].

A spacetime's Killing geometry may be reflected in the symmetry properties of the extrinsic curvature, assuming an embedding into some given higher-dimensional spacetime [26,38]: if Ω_{ab} are generated from intrinsic quantities and their derivatives only then we have 'intrinsic rigidity', and $\mathcal{L}_{\mathbf{X}}\Omega_{ab} = 0$, for $\mathbf{X}$ a KV (so, SSS implies that the extrinsic curvature in the bulk depends only on r, y); if we further require that the extrinsic curvature is a function of the Ricci tensor and the hypersurface metric only, then we have 'energetic rigidity'.

There do exist some results on the inheritance properties of 'decomposable metrics' having the form

$$g_{\alpha\beta} = g_{ab}(x^c)\,\delta^a_\alpha\delta^b_\beta + g_{\Upsilon\Xi}(x^\Pi)\,\delta^\Upsilon_\alpha\delta^\Xi_\beta, \tag{18}$$

corresponding to the product spacetimes $g_{\alpha\beta} = g_{ab} \otimes g_{\Upsilon\Xi}$. In Gaussian coordinates, a $(n+1)$-dimensional decomposable spacetime is obtained by setting $a_{ij}(y, \mathbf{x}) = g_{ij}(\mathbf{x})$ in (4). For decomposable spacetimes it may be shown [39] that KVs for the constituent metrics are KVs of the full metric; the full metric possesses a HKV iff each of the constituent metrics do; and that a necessary condition for the full metric to admit a proper CKV with conformal factor ψ is that the constituent metrics admit gradient CKVs $\psi_{,a}$ and $\psi_{,\Upsilon}$.

For one extra dimension, the metric (18) is a generalization of a stacking ($M \times AdS^1$) embedding – e.g. ref. [20] – but does not include more general warped Lorentzian manifolds – $(M, \mathbf{g})$ with $M = O \times S$, $\mathbf{g} = \mathbf{g}_1 \otimes Y\mathbf{g}_2$ for $(O, \mathbf{g}_1)$, $(S, \mathbf{g}_2)$ submanifolds, and Y a function defined on O – as are typical in brane world scenarios [4]. By appropriate redefinitions and conformal transformations, one might take them to a metric conformal to (18). However, while a conformal transformation takes a CKV to a CKV, the KVs are generally mapped to CKVs. Moreover, there exist metrics of interest not conformal to a decomposable metric: in particular, the metric (17) for a SSS spacetime embedded in an (empty) five-dimensional Einstein space.

The conformal geometry for SSS spacetimes is well known [40]. The CKVs satisfy the equations

$$\begin{aligned}
&X^{(4)}_{0,0} - {}^{(4)}\Gamma^1_{00}X^{(4)}_1 = \varphi g_{00},\\
&X^{(4)}_{0,C} + X^{(4)}_{C,0} - 2\,{}^{(4)}\Gamma^0_{01}X^{(4)}_0\delta^1_C = 0,\\
&X^{(4)}_{C,D} + X^{(4)}_{D,C} - 2\,{}^{(4)}\Gamma^E_{CD}X^{(4)}_E = 2\varphi g_{CD},
\end{aligned} \tag{19}$$

and the Killing vectors ($\varphi = 0$) are

$$\begin{aligned}
Y^{(0)}_i &= (\mathrm{e}^{2\nu}, 0, 0, 0),\\
Y^{(1)}_i &= (0, 0, 0, r^2\sin^2\theta),
\end{aligned}$$

$$Y_i^{(2)} = (0, 0, r^2 \sin\phi, r^2 \sin\theta \cos\theta \cos\phi),$$
$$Y_i^{(3)} = (0, 0, -r^2 \cos\phi, r^2 \sin\theta \cos\theta \sin\phi),$$

where the bracketed superscript is a label, not a tensorial index.

We now formulate the equations to be solved: suppose that $\mathbf{X}^{(5)}$ is a CKV for the embedding spacetime (4) with conformal factor $\psi(y, \mathbf{x})$. Writing the Lie derivative in terms of partials, and using the connections calculated from the metric (4) we obtain the defining equations for a CKV in the five-dimensional space:

$$X_{4,4}^{(5)} = \epsilon\psi(y, \mathbf{x}), \tag{20}$$

$$2X_{(4,i)}^{(5)} - a^{lk} a_{ki,4} X_l^{(5)} = 0, \tag{21}$$

$$2X_{(i,j)}^{(5)} - a^{kn}[a_{in,j} + a_{nj,i} - a_{ij,n}]X_k^{(5)} - \epsilon a_{ij,4} X_4^{(5)} = 2\psi(y, \mathbf{x}) a_{ij}. \tag{22}$$

We shall treat the systems (20)–(22) for a KV ($\psi = 0$) in the five-dimensional spacetime, given the solution to (19) for a KV ($\varphi = 0$) in the four-dimensional spacetime.

4.1 *SSS inheritance*

Restricting our attention to KVs and using (17) we see that eq. (20) yields $X_{4,4}^{(5)} = 0$, so that $X_4^{(5)}$ is a function of r only. In fact, when considering inherited SSS KVs, we may, as is shown in §4.1.3, without loss of generality, set $X_4^{(5)} = 0$. Hence, our inherited KVs are hypersurface-like.

We may similarly rewrite eq. (21) as the four equations

$$X_{0,4}^{(5)} + X_{4,0}^{(5)} = \frac{\dot{A}}{A} X_0^{(5)}, \tag{23}$$

$$X_{C,4}^{(5)} + X_{4,C}^{(5)} = \frac{\dot{B}}{B} X_C^{(5)}, \tag{24}$$

and (22) as the ten equations

$$2X_{0,0}^{(5)} - 2\frac{A}{B}\,{}^{(4)}\Gamma^1_{00} X_1^{(5)} + \frac{A'}{B} g^{11} g_{00} X_1^{(5)} + \epsilon \dot{A} g_{00} X_4^{(5)} = 0, \tag{25}$$

$$X_{0,C}^{(5)} + X_{C,0}^{(5)} - 2\,{}^{(4)}\Gamma^0_{01} X_0^{(5)} \delta^1_C - \frac{A'}{A} X_0^{(5)} \delta^1_C = 0, \tag{26}$$

and

$$X_{C,D}^{(5)} + X_{D,C}^{(5)} - 2\,{}^{(4)}\Gamma^E_{CD} X_E^{(5)}$$
$$-\frac{1}{B}\left[B_{,D}\delta^E_C + B_{,C}\delta^E_D - B_{,F} g^{EF} g_{CD}\right] X_E^{(5)} + \epsilon \dot{B} g_{CD} X_4^{(5)} = 0. \tag{27}$$

The system of eqs (20), (23)–(27) must be solved to obtain a five-dimensional CKV. We consider three cases:

4.1.1 $^{(4)}R_{ij} = 0$. When the four-dimensional spacetime is Ricci-flat, it can be shown that $A = B = 1$ [26], and so the embedding space is decomposable. The results obtained by Apostolopoulos and Carot [39] can be applied here.

4.1.2 $A(y, r) = B(y, r)$. When $A(y, r) = B(y, r)$, the five-dimensional metric has the form $\tilde{g}_{ij} = a_{ij} = A(y)g_{ij}$, and the Killing equations are

$$\dot{X}_4^{(5)} = 0, \tag{28}$$

$$X_{i,4}^{(5)} + X_{4,i}^{(5)} = \frac{\dot{A}}{A} X_i^{(5)}, \tag{29}$$

$$X_{i,j}^{(5)} + X_{j,i}^{(5)} - 2^{(4)}\Gamma^k_{ij} X_k^{(5)} + \epsilon \dot{A} g_{ij} X_4^{(5)} = 0. \tag{30}$$

By setting $X_4^{(5)} = 0$, eq. (29) implies that $X_i^{(5)} = f_i(\mathbf{X})A(y)$ where f_i are unknown functions. Substituting this expression into (30) gives $f_i = X_i^{(4)}$, a four-dimensional KV. Thus, we have $\mathbf{X}^{(5)} = (A(y)\mathbf{X}^{(4)}, 0)$ for $\mathbf{X}^{(4)}$, a KV of the embedded SSS spacetime.

4.1.3 $^{(4)}R = {}^{(4)}R(r)$. Now we investigate the Killing geometry of the embedding space for the general case in which $^{(4)}R$ may depend on r, and $A(y, r) \neq B(y, r)$. We show how the five-dimensional spacetime inherits the four-dimensional Killing geometry. Suppose that

$$\mathbf{X}^{(5)} = (P(x^i, y)\mathbf{X}^{(4)},\ H(x^i, y)),$$

where $\mathbf{X}^{(4)}$ is a KV for the embedded SSS spacetime, and P and H are unknown functions. We obtain the following set of equations that must be solved for P and H:

$$\dot{H} = 0, \quad H_{,C} + \dot{P} X_C^{(4)} - \frac{\dot{B}}{B} P X_C^{(4)} = 0,$$

$$H_{,0} + \dot{P} X_0^{(4)} - \frac{\dot{A}}{A} P X_0^{(4)} = 0, \quad P_{,0} X_0^{(4)} + \epsilon \frac{\dot{A}}{2} g_{00} H = 0,$$

$$P_{,C} X_0^{(4)} + P_{,0} X_C^{(4)} - \frac{A'}{A} P X_0^{(4)} \delta^1_C = 0,$$

and

$$P_{,D} X_C^{(4)} + P_{,C} X_D^{(4)} - \frac{B'}{B}[\delta^1_D \delta^2_C + \delta^1_C \delta^2_D] P X_2^{(4)}$$
$$- \frac{B'}{B}[\delta^1_D \delta^3_C + \delta^1_C \delta^3_D] P X_3^{(4)} + \epsilon \dot{B} g_{CD} H = 0. \tag{31}$$

Here we have used the fact that, for all four KVs in the SSS spacetime, $X_1^{(4)} = 0$ and $X_{0,0}^{(4)} = 0$. Setting $C = 1 = D$ in system (31) yields $H = 0$. (Note also that setting $P = 0$

similarly implies that $H = 0$, so there is no KV in y-direction. This is a yet more general result, independent of any assumptions of inheritance.) Then the above system becomes

$$\left[\dot{P} - \frac{\dot{B}}{B}P\right] X_C^{(4)} = 0, \quad \left[\dot{P} - \frac{\dot{A}}{A}P\right] X_0^{(4)} = 0,$$
$$P_{,0}X_0^{(4)} = 0, \quad P_{,C}X_0^{(4)} + P_{,0}X_C^{(4)} - \frac{A'}{A}PX_0^{(4)}\delta_C^1 = 0,$$
$$P_{,2}X_2^{(4)} = 0, \quad P_{,3}X_3^{(4)} = 0,$$
$$P_{,2}X_3^{(4)} + P_{,3}X_2^{(4)} = 0, \quad \left[P' - \frac{B'}{B}P\right] X_2^{(4)} = 0,$$
$$\left[P' - \frac{B'}{B}P\right] X_3^{(4)} = 0.$$

For $\mathbf{X}^{(4)} = \mathbf{Y}^{(0)}$ we have

$$\dot{P} = \frac{\dot{A}}{A}P, \quad P' = \frac{A'}{A}P \quad \text{and} \quad P_{,0} = P_{,2} = P_{,3} = 0.$$

This is easily solved to obtain $P(y, r) = k_0 A(y, r)$, $k_0 \in \mathbb{R}$. Thus,

$$\mathbf{X}^{(5)} = (k_0 A(y, r)\mathbf{Y}^{(0)}, 0).$$

For the other three KVs we have $P_{,0} = P_{,2} = P_{,3} = 0$ and

$$\dot{P} = \frac{\dot{B}}{B}P, \quad P' = \frac{B'}{B}P,$$

which yields the solution $P(y, r) = k_1 B(y, r)$, $k_1 \in \mathbb{R}$, and so

$$\mathbf{X}^{(5)} = (k_1 B(y, r)\mathbf{X}^{(4)}, 0),$$

with $\mathbf{X}^{(4)} = \mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}, \mathbf{Y}^{(3)}$. We have thus demonstrated the manner in which the bulk inherits the Killing geometry of the embedded four-dimensional hypersurface. This is to be expected, since the embedding is constructed to be energetically rigid.

4.2 *General considerations*

We conclude this section with some general considerations: First we recall (see §4.1.3) that there does not exist any KV in the y-direction. This is sensible as such a KV would result in y-independence for Ω_{ij} and hence $\tilde{g}_{\alpha\beta}$, which contradicts the assumed form (17) for the metric, and yields a non-Einsteinian stacking. Secondly, using eq. (23) with $X_4^{(5)}$ a function of r only, and eq. (26) with $C = 1$, we find that $X_0^{(5)} = k_0 e^{2\nu} A$.

Next, we note that, by taking $X_1^{(5)} = 0$, in eq. (27) with $C = 1 = D$, we obtain $X_4^{(5)} = 0$ (as is the case for the inherited vectors above). Similarly, taking $X_4^{(5)} = 0$ in the same equation and in eq. (24) with $C = 1$, yields $X_1^{(5)} = m(y)e^{\lambda}B^{1/2}$, with $m(y) = n(r)B^{1/2}$. If B is separable, then we can arrange that the right-hand side is indeed a function of y only. However, this is not generally true, and we are forced to set $X_1^{(5)} = 0$. Hence, for the general SSS case, $X_1^{(5)} = 0 \Longleftrightarrow X_4^{(5)} = 0$.

Finally, we observe that the KVs calculated above exhaust the possibilities for KVs that are hypersurface-like. This is reasonable, as we have imposed the SSS requirement on the $y =$ constant hypersurfaces, and may be verified by direct (and laborious) computation.

The calculation of the remaining (conformal) Killing geometry requires the solution of vastly more complicated equations. These calculations represent a continuing programme of research. We intend (with other collaborators) to employ the techniques of Lie symmetry analysis to attack these problems, both to establish the (conformal) Killing geometry of a bulk with an embedded SSS hypersurface as a problem in its own right, and also as an experimental platform from which we hope to draw insight into the relationship between Lie symmetry reduction and (C)KV inheritance.

5. Conclusion

In this paper we have provided a survey of known embedding existence theorems, highlighting their limitations with regard to the non-isometry of Ricci-equivalent embeddings, and the non-uniqueness in the choice of embedding spacetime. We briefly outlined some recent explicit embeddings with more complex geometry; provided examples of embeddings for conical geometries; and discussed astrophysical constraints in higher-dimensional models. Finally, we provided an explicit construction of the inherited Killing geometry for SSS embeddings.

Future work could profitably proceed along the following lines: the generation of more explicit embeddings, and, in particular, those associated with astrophysical models; exact solutions for bulks associated with astrophysical models in various phenomenological brane world models, including the imposition of asymptotic requirements; an analysis of astrophysical constraints in Gauss–Bonnet gravity; and the inclusion of bulk topological effects, such as those associated with identifications and singularities. Finally, it would be nice to close the gaps in the existence theorems by resolving the non-isometric nature of the Ricci-equivalent requirement, and by extending them to include Ricci-degenerate metrics.

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