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On the Lie point symmetry analysis and solutions of the inviscid Burgers equation

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Abstract. Lie point symmetries of the first-order inviscid Burgers equation in a general setting are studied. Some new and interesting solutions are presented.

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1. Introduction

The first-order nonlinear evolution equation that arises from the inviscid Burgers equation presents some interesting features. As is well known, the Burgers equation can be obtained as a limiting case of the Navier–Stokes equations [1] and serves as an adequate model for some gas dynamics and transport problems [2]. For the inviscid case, the Burgers equation reduces to $u_t + uu_x = 0$ where $u(x, t)$ is the particle velocity. However, a more general inviscid Burgers equation arises if the diffusivity is considered to be a general function of u. Towards this end, Ouhadan and El Kinani [3] considered the general inviscid Burgers equation $u_t + f(u)\, u_x = 0$ and used Lie symmetry method [4] for obtaining exact solutions in some cases. However, the analysis presented therein is not complete. The procedure for calculating the Lie point symmetries leads to an underdetermined system of partial differential equations. Ouhadan and El Kinani presented a few simple cases and obtained exact solutions for only those cases. The study was extended by Nadjafikhah [5] to include other cases of interest. In this paper we consider more general cases.

2. Symmetry generators

Consider the inviscid Burgers equation,

$$u_t(x, t) + f(u)u_x(x, t) = 0, \tag{1}$$

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where $f(u)$ is a smooth function having a non-vanishing first derivative. Requiring that eq. (1) be invariant with respect to the first prolonged symmetry generator,

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t}, \tag{2}$$

with ξ, τ and η functions of t, x and u, gives [6–8]

$$\eta f'(u)\, u_x + \eta^t + f(u)\eta^x = 0, \tag{3}$$

where

$$\eta^x = \eta_x + u_x\eta_u - u_x(\xi_x + u_x\xi_u) - u_t(\tau_x + u_x\tau_u), \tag{4}$$

$$\eta^t = \eta_t + u_t\eta_u - u_x(\xi_t + u_t\xi_u) - u_t(\tau_t + u_t\tau_u). \tag{5}$$

Comparing coefficients of the derivatives of u that occur in the determining equation,

$$X(u_t + f(u)\,u_x)|_{u_t + f(u)\,u_x = 0} = 0,$$

we obtain the following system of differential equations:

$$\eta_t + f(u)\eta_x = 0, \tag{6}$$

$$-\xi_t + f(u)\tau_t + \eta f'(u) - f(u)\xi_x + f(u)^2\tau_x = 0. \tag{7}$$

Note that the above system is an underdetermined system consisting of only two equations to be solved for four unknown functions, ξ, τ, η and $f(u)$. Hence, to solve this system one needs various ansätze regarding ξ, τ and η. In the earlier paper, Nadjafikhah [5] presented solutions of systems (6) and (7) for only one case by assuming $\eta = \eta(x, t, u)$, $\xi = \xi(x, t)$ and $\tau = \tau(x, t)$. In our classification of solutions we assume in the first case,

$$\xi = k(u)a(x, t), \qquad \tau = k(u)b(x, t)$$

and in the second case

$$\xi = f(u)a(x, t) + b(x, t), \qquad \tau = f(u)c(x, t) + d(x, t)$$

with η an arbitrary function of x, t and u. We present a complete reduction of (1) in the first case and briefly state the results in the second case.

2.1 $\xi = k(u)a(x, t)$, $\tau = k(u)b(x, t)$ *and* $\eta = \eta(x, t, u)$

In this case (7) becomes

$$-a_t + f(u)b_t + \frac{\eta f'(u)}{k(u)} - f(u)\,a_x + f(u)^2 b_x = 0. \tag{8}$$

From eq. (8) we obtain

$$\eta = \frac{k(u)}{f'(u)}\left[a_t + f(u)\,(a_x - b_t) - f^2(u)b_x\right]. \tag{9}$$

Substituting for η from (9) in eq. (6) and using the fact that $f(u)$ is arbitrary, we obtain an overdetermined system of partial differential equations for $a(x, t)$ and $b(x, t)$, namely,

$$b_{xx} = 0,$$
$$2b_{xt} + a_{xx} = 0,$$
$$b_{tt} - 2a_{xt} = 0,$$
$$a_{tt} = 0.$$

Solving the system of equations gives the components of the symmetry generator as

$$\begin{aligned} \xi &= k(u)\,[\alpha_1 x^2 + \alpha_2 xt + \alpha_3 t + (\alpha_4 + \alpha_5)\,x + \alpha_6], \\ \tau &= k(u)\,[\alpha_2 t^2 + \alpha_1 xt + \alpha_5 t - \alpha_7 x + \alpha_8], \\ \eta &= \frac{k(u)}{f'(u)}\left\{f^2(u)\,[-\alpha_1 t + \alpha_7] + f(u)\,[\alpha_1 x - \alpha_2 t + \alpha_4] + \alpha_2 x + \alpha_3\right\}. \end{aligned} \tag{10}$$

The symmetry generators corresponding to the above infinitesimals are given below.

$$\begin{aligned} X_1 &= k(u)\Big[x^2\frac{\partial}{\partial x} + xt\frac{\partial}{\partial t} + \frac{xf(u) - tf^2(u)}{f'(u)}\frac{\partial}{\partial u}\Big], \\ X_2 &= k(u)\Big[xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} + \frac{x - tf(u)}{f'(u)}\frac{\partial}{\partial u}\Big], \\ X_3 &= k(u)\Big[t\frac{\partial}{\partial x} + \frac{1}{f'(u)}\frac{\partial}{\partial u}\Big], \\ X_4 &= k(u)\Big[x\frac{\partial}{\partial x} + \frac{f(u)}{f'(u)}\frac{\partial}{\partial u}\Big], \\ X_5 &= k(u)\Big[x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\Big], \\ X_6 &= k(u)\frac{\partial}{\partial x}, \\ X_7 &= k(u)\Big[-x\frac{\partial}{\partial t} + \frac{f^2(u)}{f'(u)}\frac{\partial}{\partial u}\Big], \\ X_8 &= k(u)\frac{\partial}{\partial t}. \end{aligned} \tag{11}$$

When $k(u) = 1$, eqs (11) reduce to those given by Nadjafikhah [5] and they form a closed algebra as shown in table 1.

2.2 $\xi = f(u)a(x, t) + b(x, t)$, $\tau = f(u)c(x, t) + d(x, t)$ *and* $\eta = \eta(x, t, u)$

For this case eq. (7) gives

$$-fa_t - b_t + f^2c_t + fd_t + \eta f' - f^2a_x - fb_x + f^3c_x + f^2d_x = 0. \tag{12}$$

To determine the unknown functions a, b, c and d in the above expression we differentiate eq. (12) thrice with respect to u to obtain

$$(((\eta f')_u/f')_u)/2f')_u)/3f' = -c_x. \tag{13}$$

Table 1. The commutator table for $k(u) = 1$.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	$-X_2$	$-X_1$	$-X_1$	$-X_4 - X_5$	0	X_7
X_2	0	0	0	0	$-X_2$	$-X_3$	X_1	$2X_5 - X_4$
X_3	X_2	0	0	X_3	0	0	$2X_4 - X_5$	$-X_6$
X_4	X_1	0	$-X_3$	0	0	$-X_6$	X_7	0
X_5	X_1	X_2	0	0	0	$-X_6$	0	$-X_8$
X_6	$X_4 + X_5$	X_3	0	X_6	X_6	0	$-X_8$	0
X_7	0	$-X_1$	$-2X_4 + X_5$	$-X_7$	0	X_8	0	0
X_8	$-X_7$	$-2X_5 + X_4$	X_6	0	X_8	0	0	0

It follows immediately from eq. (13) that

$$\eta = \frac{1}{f'(u)}\left[f^3 A(x,t) + f^2 B(x,t) + f D(x,t) + E(x,t)\right], \tag{14}$$

where $A(x,t) = -c_x$. Upon substituting for η given by (14) into the systems (6) and (7) it is necessary that

$$c_t + d_x - a_x = -B(x,t), \tag{15}$$

$$d_t - a_t - b_x = -D(x,t), \tag{16}$$

$$b_t = E(x,t) \tag{17}$$

and

$$f^4 A_x + f^3(A_t + B_x) + f^2(B_t + D_x) + f(D_t + E_x) + E_t = 0. \tag{18}$$

Since $f(u)$ is arbitrary, solving the above equations gives the components of the symmetry generator as

$$\begin{aligned}
\xi &= f(u)\left[d(x,t) - (2\alpha_1 t + \alpha_2)x^2 + (2\alpha_4 x + \alpha_5)t^2\right.\\
&\qquad\left. + (\alpha_7 x + \alpha_9 - \alpha_{11})t - (2\alpha_{10} t + \alpha_{13})x - \alpha_{14}\right]\\
&\quad + (\alpha_1 x^3 - \alpha_4 x^2 t - \alpha_5 x t - \alpha_6 t + \alpha_{10} x^2 + \alpha_{11} x + \alpha_{12}),\\
\tau &= f(u)\left[-\alpha_1 x t^2 - \alpha_2 x t - \alpha_3 x + \alpha_4 t^3 + (\alpha_7 - \alpha_{10})t^2\right.\\
&\quad\left. + (\alpha_8 - \alpha_{13})t + \alpha_{15}\right] + d(x,t),\\
\eta &= \frac{1}{f'(u)}\left[f^3(u)(\alpha_1 t^2 + \alpha_2 t + \alpha_3) - f^2(u)(2\alpha_1 x t + \alpha_2 x + \alpha_4 t^2\right.\\
&\quad + \alpha_7 t + \alpha_8) + f(u)(\alpha_1 x^2 + 2\alpha_4 x t + \alpha_5 t + \alpha_7 x + \alpha_9)\\
&\quad \left. - (\alpha_4 x^2 + \alpha_5 x + \alpha_6)\right]
\end{aligned} \tag{19}$$

where $a(x,t)$ and $d(x,t)$ are two arbitrary functions of x and t that are related via the equation

$$\begin{aligned}
d(x,t) - a(x,t) &= (2\alpha_1 t + \alpha_2)\,x^2 - (2\alpha_4 x + \alpha_5)\,t^2\\
&\quad - (\alpha_7 x + \alpha_9 - \alpha_{11})\,t + (2\alpha_{10} t + \alpha_{13})\,x + \alpha_{14}.
\end{aligned} \tag{20}$$

The corresponding generators are given below.

$$
\begin{aligned}
Z_1 &= \left(-2x^2tf(u) + x^3\right)\frac{\partial}{\partial x} - xt^2 f(u)\frac{\partial}{\partial t}\\
&\quad + \frac{t^2 f^3(u) - 2xtf^2(u) + x^2 f(u)}{f'(u)}\frac{\partial}{\partial u},\\
Z_2 &= -x^2 f(u)\frac{\partial}{\partial x} - xtf(u)\frac{\partial}{\partial t} + \frac{tf^3(u) - xf^2(u)}{f'(u)}\frac{\partial}{\partial u},\\
Z_3 &= -xf(u)\frac{\partial}{\partial t} + \frac{f^3(u)}{f'(u)}\frac{\partial}{\partial u},\\
Z_4 &= \left(2xt^2 f(u) - x^2 t\right)\frac{\partial}{\partial x} + t^3 f(u)\frac{\partial}{\partial t}\\
&\quad - \frac{t^2 f^2(u) - 2xtf(u) + x^2}{f'(u)}\frac{\partial}{\partial u},\\
Z_5 &= \left(t^2 f(u) - xt\right)\frac{\partial}{\partial x} + \frac{tf(u) - x}{f'(u)}\frac{\partial}{\partial u},\\
Z_6 &= -t\frac{\partial}{\partial x} - \frac{1}{f'(u)}\frac{\partial}{\partial u},\\
Z_7 &= xtf(u)\frac{\partial}{\partial x} + t^2 f(u)\frac{\partial}{\partial t} + \frac{xf(u) - tf^2(u)}{f'(u)}\frac{\partial}{\partial u},\\
Z_8 &= tf(u)\frac{\partial}{\partial t} - \frac{f^2(u)}{f'(u)}\frac{\partial}{\partial u},\\
Z_9 &= tf(u)\frac{\partial}{\partial x} - \frac{f(u)}{f'(u)}\frac{\partial}{\partial u},\\
Z_{10} &= \left(2xtf(u) + x^2\right)\frac{\partial}{\partial x} - t^2 f(u)\frac{\partial}{\partial t},\\
Z_{11} &= (x - tf(u))\frac{\partial}{\partial x},\\
Z_{12} &= \frac{\partial}{\partial x},\\
Z_{13} &= xf(u)\frac{\partial}{\partial x} - tf(u)\frac{\partial}{\partial t},\\
Z_{14} &= f(u)\frac{\partial}{\partial x},\\
Z_{15} &= f(u)\frac{\partial}{\partial t}.
\end{aligned}
\tag{21}
$$

Note that a number of symmetries exist for the inviscid Burgers equation (1) for which ξ and τ depend on u which were not found or mentioned in earlier works. *Inter alia*, we have

$$
\begin{aligned}
V_1 &= L(u)\frac{\partial}{\partial x}, \qquad V_2 = M(u)\frac{\partial}{\partial t},\\
V_3 &= Q(u)a(x,t)\frac{\partial}{\partial x}, \qquad V_4 = R(u)b(x,t)\frac{\partial}{\partial t},
\end{aligned}
$$

where $L(u)$, $M(u)$, $Q(u)$ and $R(u)$ are arbitrary smooth functions of u and

$$a_t + f(u)a_x = b_t + f(u)b_x = 0.$$

3. Reductions for the case $\xi = k(u)a(x,t),\ \tau = k(u)b(x,t),\ \eta = \eta(x,t,u)$

As a first example, we perform a reduction of (1) using the generator

$$X_1 = x^2\frac{\partial}{\partial x} + xt\frac{\partial}{\partial t} + \frac{xf - tf^2}{f'}\frac{\partial}{\partial u}$$

by choosing $k(u) = 1$. From this generator we find the invariants to be $y = x/t$ and $w = f/(x - tf)$ so that

$$f'u_t = \frac{x}{(1+tw)^2}\left(-\frac{x}{t^2}w' - w^2\right)$$

and

$$f'u_x = \frac{1}{(1+tw)^2}\left(w + tw^2 + w'\frac{x}{t}\right)$$

which, when substituted into (1), leads to the reduced equation $w' = 0$. The invariant solution is therefore

$$u(x,t) = f^{-1}\left(\frac{kx}{1+kt}\right)$$

for some constant k. Now if $f(u) = u^2$ or $f(u) = u/(1+u^2)$, then we have the solutions

$$u(x,t) = \sqrt{\frac{kx}{1+kt}}$$

and

$$u(x,t) = \frac{(1+kt) \pm \sqrt{(1+kt)^2 - 4k^2x^2}}{2kx}$$

respectively. Similarly, the boost-type symmetry given by

$$X_3 = t\frac{\partial}{\partial x} + \frac{1}{f'}\frac{\partial}{\partial u}$$

with $f(u) = u$ has invariants $y = t$ and $w = f(u) - x/t$ that reduces (1) to the ordinary differential equation $w/w' = -1/y$. The invariant solution is therefore

$$u(x,t) = f^{-1}\left(\frac{x+k}{t}\right).$$

Following the same procedure, the generators X_2, X_4, X_5, X_7 and $X_1 + X_2$ yield the respective solutions,

$$u(x,t) = f^{-1}\left(\frac{x-k}{t}\right),$$

$$u(x,t) = f^{-1}\left(\frac{x}{t-k}\right),$$

$$u(x,t) = f^{-1}\left(\frac{x}{t}\right),$$

$$u(x,t) = f^{-1}\left(\frac{x}{t+k}\right),$$

$$u(x,t) = f^{-1}\left(\frac{x+k}{t-k}\right).$$

From all the above solutions, we conclude that

$$u(x,t) = f^{-1}\left(\frac{\alpha x + \beta}{\gamma t + \delta}\right)$$

is an invariant solution of the generalized inviscid Burgers equation (1) resulting from any of the generators X_i, $i = 1, 2, ..., 8$, or any of their linear combinations if and only if $\alpha = \gamma$ where α, β, γ and δ are arbitrary constants.

The linear combination,

$$X_4 + X_5 = 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{f(u)}{f'(u)}\frac{\partial}{\partial u},$$

with invariants $y = x/t^2$ and $w = g(u)/t$ reduce (1) to the ordinary differential equation

$$w' + \frac{w}{w - 2y}.$$

Using the substitution $w = 2yz(y)/(z(y) - 1)$, we obtain the invariant solution

$$u(x,t) = f^{-1}\left(\frac{2x}{t + \sqrt{t^2 - kx}}\right).$$

We now perform a class of reductions leading to invariant solutions using transformations of x and t that depend on u. The combination of V_1, X_6, and the boost lead to the symmetry generator

$$(t + 1 + L(u))\frac{\partial}{\partial x} + \frac{1}{f'}\frac{\partial}{\partial u},$$

which has invariants

$$y = t, \quad w = -x + (t+1)f + \int Lf'\,\mathrm{d}u.$$

Then,

$$u_t = \frac{w' - f}{(t + 1 + L)\,f'} \qquad \text{and} \qquad u_x = \frac{f' + 1}{(t + 1 + L)\,f'},$$

which, when substituted into (1), leads to the invariant solution

$$k + x = (t+1)f + \int Lf'\mathrm{d}u.$$

This corresponds to an infinite number of solutions ranging from the simple to the complex depending on the choices for f and L. For instance, if $L = f$, then the invariant solution is of the form

$$u(x,t) = f^{-1}\left[-(t+1) \pm \sqrt{(t+1)^2 + 2x + 2k}\right].$$

If $f(u) = u/(u+1)$, $u \neq -1$, then the solution of (1) is

$$u(x,t) = \frac{-(t+1) \pm \sqrt{(t+1)^2 + 2x + 2k}}{(t+2) \mp \sqrt{(t+1)^2 + 2x + 2k}}.$$

Similarly, the generator,

$$V_2 + X_7 = (M(u) - x)\frac{\partial}{\partial t} + \frac{f^2(u)}{f'(u)}\frac{\partial}{\partial u},$$

with invariants,

$$y = x, \quad w = -t + \frac{x}{f} + \int \frac{M}{f^2} f' \mathrm{d}u,$$

leads to the solution

$$k + t = \frac{x}{f} + \int \frac{M}{f^2} f' \mathrm{d}u.$$

As an example, suppose $\sqrt{M} = f = \cosh u$. Then

$$u(x,t) = \cosh^{-1}\left(\frac{k + t \pm \sqrt{(k+t)^2 - 4x}}{2}\right)$$

is an invariant solution of eq. (1).

4. Conclusions

We find some interesting exact solutions for the inviscid Burgers equation corresponding to the situation when the space and time transformation are velocity dependent. These more general cases have not been considered in the earlier works cited above.

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