PRAMANA — journal of physics | © Indian Academy of Sciences | Vol. 77, No. 2 August 2011 pp. 263–275

Exact travelling solutions for some nonlinear physical models by (G'/G)-expansion method

B SALIM BAHRAMI[1], H ABDOLLAHZADEH[2], I M BERIJANI[1], D D GANJI[1] and M ABDOLLAHZADEH[1,*]

[1]Department of Mechanical Engineering, Babol University of Technology, P.O. Box 484, Babol, Iran
[2]Department of Industrial Engineering, University of Kurdistan, Sanandaj, Iran
*Corresponding author. E-mail: mm.abdollahzadeh@yahoo.com

MS received 17 December 2010; revised 8 March 2011; accepted 21 March 2011

Abstract. In this paper, we establish exact solutions for some special nonlinear partial differential equations. The (G'/G)-expansion method is used to construct travelling wave solutions of the two-dimensional sine-Gordon equation, Dodd–Bullough–Mikhailov and Schrödinger–KdV equations, which appear in many fields such as, solid-state physics, nonlinear optics, fluid dynamics, fluid flow, quantum field theory, electromagnetic waves and so on. In this method we take the advantage of general solutions of second-order linear ordinary differential equation (LODE) to solve many nonlinear evolution equations effectively. The (G'/G)-expansion method is direct, concise and elementary and can be used with a wider applicability for handling many nonlinear wave equations.

Keywords. (G'/G)-expansion method; travelling wave solutions; two-dimensional sine-Gordon equation; Dodd–Bullough–Mikhailov equation; Schrödinger–KdV equation.

PACS Nos 05.45.Yv; 02.30.Hq; 02.30.Jr

1. Introduction

The nonlinear partial differential equations are frequently used for modelling natural and social phenomena and systems. Nonlinear wave phenomena appear in various scientific and engineering fields, such as solid-state physics, fluid mechanics, chemical kinetics, plasma physics, population models, nonlinear optics etc. Analytical exact solutions to nonlinear partial differential equation play an important role in nonlinear science, especially they may provide us much physical information and more insight into the physical aspects of the problem and may lead to further applications.

A variety of methods, such as tanh–sech method [1,2], exp-function method [3–7], the sine–cosine method [8,9], the simplest equation method [10], the Jacobi elliptic function method [11,13], simplest equation method [12], F-expansion method [14,15], Li group analysis [16], He's variational iteration method [17] and homogeneous balance method [18,19] are used for finding exact solutions of nonlinear evolution equations.

DOI: 10.1007/s12043-011-0100-9; *e*Publication: 9 July 2011

The (G'/G)-expansion method was proposed to look for exact solutions of nonlinear wave equations in [20]. The most important advantage of this method is that the special functions that one takes to expand the exact solution are the general solutions of the second-order linear equation and these solutions are well-known. This simplicity in searching the general solutions of many differential equations is an attractive characteristic of this method. Moreover, this method allows us to obtain a rich variety of exact solutions that contain more arbitrary constants compared to the solutions presented by other authors [20–24].

Physics and engineering often provide us with complicated nonlinear equations, such as sine-Gordon equation, Schrödinger equation, coupled Schrödinger–KdV equations etc. In this paper, we shall apply the (G'/G)-expansion method to obtain the exact travelling wave solution of the two-dimensional sine-Gordon equation, Dodd–Bullough–Mikhailov equation and the Schrödinger–KdV equation

$$u_{tt} - u_{xx} - u_{yy} + m^2 \sin u = 0, \tag{1.1}$$

$$u_{xt} + p\mathrm{e}^{u} + q\mathrm{e}^{-2u} = 0, \tag{1.2}$$

$$\begin{cases} iu_t = u_{xx} + uv \\ v_t + 6vv_x + v_{xxx} = \left(|u|^2\right)_x \end{cases}. \tag{1.3}$$

The rest of the paper is organized as follows. In §2, we present a methodology of the generalized (G'/G)-expansion method. In §3, we apply our method to the mentioned equations. In §4, some conclusions are given.

2. The (G'/G)-expansion method

Wang has summarized the main steps for using (G'/G)-expansion method, as follows:

(1) Introducing the wave variable into the PDE, we get

$$\phi\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, u_{xxx}, \ldots\right) = 0, \tag{2.1}$$

where $u(x,t)$ is the travelling wave solution. This enables us to use the following changes:

$$u\left(x,t\right) = U\left(\xi\right), \tag{2.2}$$

$$\frac{\partial}{\partial t} = -c\frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \frac{\partial^2}{\partial t^2} = c^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2},$$
$$\frac{\partial}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \frac{\partial^2}{\partial x^2} = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}, \ldots \tag{2.3}$$

and the other derivatives are calculated in the same way. Using (2.3) and (2.2), the nonlinear partial differential equation (PDE) (2.1) changes to a nonlinear ordinary differential equation (ODE):

$$\psi\left(U, -cU', U', c^2U'', U'', -cU', U''', \ldots\right) = 0. \tag{2.4}$$

If all the terms of the resulting ODE contain derivatives in ξ, then by integrating this equation and considering the constant of integration to be zero, we obtain a simplified ODE.

(2) Suppose that the solution of ODE (2.2) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i = a_m \left(\frac{G'}{G}\right)^m + \cdots, \tag{2.5}$$

where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation (LODE) in the form:

$$G'' + \lambda G' + \mu G = 0, \tag{2.6}$$

where

$$G' = \mathrm{d}G(\xi)/\mathrm{d}\xi, \quad G'' = \mathrm{d}^2 G(\xi)/\mathrm{d}\xi^2.$$

$a_i, \ldots, a_m$, λ and μ are constants to be determined later, $a_m \neq 0$, the unwritten part in (2.5) is a polynomial in (G'/G), but the degree of which is generally equal to or less than $m-1$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and higher order nonlinear terms appearing in ODE (2.4).

(3) By substituting eq. (2.5) along with the second-order LODE (2.6) into eq. (2.4) and collecting all terms with the same order of (G'/G) together, the left-hand side of eq. (2.4) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero, yield a set of algebraic equations involving $a_i, \ldots, a_m$, λ, c and μ.

(4) By assuming that the constants $a_i, \ldots, a_m$, λ, c and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second-order LODE (2.6) have been well known for us, then substituting $a_i, \ldots, a_m$, c and general solutions of eq. (2.6) into eq. (2.5) we have more travelling wave solutions of the nonlinear evolution equation (2.1).

It should be noted that (G'/G)-expansion method is similar or we can say, it is a special case of the so-called simplest equation method. The basic idea of the simplest equation method is the assumption of the solution as finite polynomial series expansion of the standard anzats ψ'/ψ, where ψ are functions that satisfy some ordinary differential equations. These ordinary differential equations are referred to as the simplest equations. These simplest special equations are, for example, the functions tanh $(k\xi)$ for the tanh-method, Weierstrass elliptic function, the Ricatti equation and the Bernoulli equation.

3. Applications to special nonlinear equations

3.1 *Two-dimensional sine-Gordon equation*

Let us consider a two-dimensional sine-Gordon equation:

$$u_{tt} - u_{xx} - u_{yy} + m^2 \sin u = 0, \tag{3.1}$$

which appear in many fields such as the propagation of fluxons in Josephson junctions [25,26] between two superconductors, dislocation dynamics in crystal lattices, vortex states in spin systems with an anisotropy created by an external magnetic field, solid-state physics, nonlinear optics, fluid dynamics and so on [27,28]. This is why the sine-Gordon equation was studied successively by many authors [29,30].

To look for the travelling wave solutions of eq. (3.1), we make transformation $v = e^{in}$, $v(x,t) = V(\xi)$, $\xi = x + \alpha y + \beta t$, and generate the reduced nonlinear ODE in the form

$$2\left(\beta^2 - \alpha^2 - 1\right)\left(VV'' - V'^2\right) + m^2\left(V^3 - V\right) = 0. \tag{3.2}$$

Suppose that the solution of ODE (3.2) can be expressed as

$$V(\xi) = a_m \left(\frac{G'}{G}\right)^m + \cdots, \tag{3.3}$$

where $G = G(\xi)$ satisfies the second-order LODE of the form

$$G'' + \lambda G' + \mu G = 0. \tag{3.4}$$

By using eqs (3.3) and (3.4) the following equations can be easily derived:

$$\begin{aligned}
V^3(\xi) &= a_m^3 \left(\frac{G'}{G}\right)^{3m} + \cdots, \\
V'^2(\xi) &= m^2 a_m^2 \left(\frac{G'}{G}\right)^{2m+2} + \cdots, \\
V(\xi)V''(\xi) &= m(m+1)\, a_m^2 \left(\frac{G'}{G}\right)^{2m+2} + \cdots.
\end{aligned} \tag{3.5}$$

Considering the homogeneous balance between V^3 and VV'' in eq. (3.2), based on eq. (3.5), we required that $3m = 2m + 2 \Rightarrow m = 2$. So we can write eq. (3.3) as

$$V(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \tag{3.6}$$

By using eqs (3.4) and (3.6), the following equations can be derived:

$$\begin{aligned}
V^3(\xi) &= a_0^3 + 3a_0^2 a_1 \left(\frac{G'}{G}\right) + \left(3a_0^2 a_2 + 3a_0 a_1^2\right)\left(\frac{G'}{G}\right)^2 \\
&\quad + \left(a_1^3 + 6a_0 a_1 a_2\right)\left(\frac{G'}{G}\right)^3 + \left(3a_1^2 a_2 + 3a_0 a_2^2\right)\left(\frac{G'}{G}\right)^4 \\
&\quad + 3a_1 a_2^2 \left(\frac{G'}{G}\right)^5 + a_2^3 \left(\frac{G'}{G}\right)^6,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
V'^2 &= a_1^2\mu^2 + \left(4a_2\mu^2 a_1 + 2a_1^2\lambda\mu\right)\left(\frac{G'}{G}\right) \\
&\quad + \left(a_1^2\lambda^2 + 8a_1\lambda a_2\mu + 4a_2^2\mu^2 + 2a_1^2\mu\right)\left(\frac{G'}{G}\right)^2 \\
&\quad + \left(2a_2^2\lambda + 4a_1\lambda^2 a_2 + 8a_1 a_2\mu + 8a_2^2\lambda\mu\right)\left(\frac{G'}{G}\right)^3 \\
&\quad + \left(8a_1 a_2\lambda + 4a_2^2\lambda^2 + 8a_2^2\mu + a_1^2\right)\left(\frac{G'}{G}\right)^4 \\
&\quad + \left(8a_2^2\lambda + 4a_1 a_2\right)\left(\frac{G'}{G}\right)^5 + 4a_2^2\left(\frac{G'}{G}\right)^6
\end{aligned} \tag{3.8}$$

and

$$
\begin{aligned}
VV'' &= 2a_0a_2\mu^2 + a_0a_1\lambda\mu \\
&\quad + \left(2a_1a_2\mu^2 + 2a_0a_1\mu + a_0a_1\lambda^2 + 6a_0a_2\lambda\mu + a_1^2\lambda\mu\right)\left(\frac{G'}{G}\right) \\
&\quad + \left(2a_1^2\mu + 4a_0a_2\lambda^2 + 8a_0a_2\mu + 2a_2^2\mu^2 + a_1^2\lambda^2 + 3a_0a_1\lambda + 7a_1a_2\lambda\mu\right)\left(\frac{G'}{G}\right)^2 \\
&\quad + \left(2a_0a_1 + 3a_1^2\lambda + 10a_0a_2\lambda + 6a_2^2\lambda\mu + 5a_1a_2\lambda^2 + 10a_1a_2\mu\right)\left(\frac{G'}{G}\right)^3 \\
&\quad + \left(4a_2^2\lambda^2 + 13a_1a_2\lambda + 6a_0a_2 + 8a_2^2\mu + 2a_1^2\right)\left(\frac{G'}{G}\right)^4 \\
&\quad + \left(10a_2^2\lambda + 8a_1a_2\right)\left(\frac{G'}{G}\right)^5 + 6a_2^2\left(\frac{G'}{G}\right)^6. \qquad (3.9)
\end{aligned}
$$

By substituting eqs (3.6)–(3.9) into eq. (3.2) and collecting all terms with the same power of (G'/G) together, the left-hand side of eq. (3.2) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for a_0, a_1, a_2, λ, μ, α and β as follows:

$$
\begin{aligned}
\left(\frac{G'}{G}\right)^0 &: m^2a_0^3 - 2\alpha^2a_0a_1\lambda\mu - m^2a_0 + 2a_1^2\mu^2 - 2a_0a_1\lambda\mu \\
&\quad + 2\beta^2a_0a_1\lambda\mu + 2\alpha^2a_1^2\mu^2 - 4a_0a_2\mu^2 + 4\beta^2a_0a_2\mu^2 \\
&\quad - 2\beta^2a_1^2\mu^2 - 4\alpha^2a_0a_2\mu^2 = 0, \qquad (3.10a)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{G'}{G}\right)^1 &: -12a_0a_2\lambda\mu - 4\alpha^2a_0a_1\mu - m^2a_1 - 4a_0a_1\mu + 12\beta^2a_0a_2\lambda\mu \\
&\quad - 2\beta^2a_1^2\lambda\mu + 4\beta^2a_0a_1\mu + 2\alpha^2a_1^2\lambda\mu + 2a_1^2\lambda\mu + 3m^2a_1a_0^2 \\
&\quad - 2a_0a_1\lambda^2 + 4a_1a_2\mu^2 - 4\beta^2a_1a_2\mu^2 - 2\alpha^2a_0a_1\lambda^2 \\
&\quad - 12\alpha^2a_0a_2\lambda\mu + 2\beta^2a_0a_1\lambda^2 + 4\alpha^2a_1a_2\mu^2 = 0, \qquad (3.10b)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{G'}{G}\right)^2 &: 2a_1a_2\lambda\mu + 6\beta^2a_0a_1\lambda - 6a_0a_1\lambda - 8a_0a_2\lambda^2 - 16\beta^2a_0a_2\mu \\
&\quad + 4a_2^2\mu^2 + 4\alpha^2a_2^2\mu^2 - m^2a_2 - 8\alpha^2a_0a_2\lambda^2 + 3m^2a_2a_0^2 \\
&\quad - 6\alpha^2a_0a_1\lambda + 3m^2a_1^2a_0 + 8\beta^2a_0a_2\lambda^2 - 16\alpha^2a_0a_2\mu \\
&\quad + 2\alpha^2a_1a_2\lambda\mu - 4\beta^2a_2^2\mu^2 - 2\beta^2a_1a_2\lambda\mu = 0, \qquad (3.10c)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{G'}{G}\right)^3 &: -4\alpha^2a_0a_1 - 2a_1a_2\lambda^2 - 4a_0a_1 + 4a_2^2\lambda\mu - 20\alpha^2a_0a_2\lambda \\
&\quad + 6m^2a_1a_2a_0 - 4\alpha^2a_1a_2\mu + 20\beta^2a_0a_2\lambda - 2a_1^2\lambda - 4\beta^2a_2^2\lambda\mu \\
&\quad + m^2a_1^3 - 20a_0a_2\lambda - 4a_1a_2\mu + 2\beta^2a_0a_1 + 2\beta^2a_1a_2\lambda^2 \\
&\quad - 2\alpha^2a_1a_2\lambda^2 + 4\beta^2a_1a_2\mu - 2\alpha^2a_1^2\lambda = 0, \qquad (3.10d)
\end{aligned}
$$

$$\left(\frac{G'}{G}\right)^4 : 10\beta^2 a_1 a_2 \lambda - 12\alpha^2 a_0 a_2 + 3m^2 a_1^2 a_2 - 10\alpha^2 a_1 a_2 \lambda + 12\beta^2 a_0 a_2 - 2a_1^2 + 2\beta^2 a_1^2 - 2\alpha^2 a_1^2 - 10 a_1 a_2 \lambda + 3m^2 a_2^2 a_0 - 12 a_0 a_2 = 0, \tag{3.10e}$$

$$\left(\frac{G'}{G}\right)^5 : -4a_2^2 + m^2 a_2^3 - 4\alpha^2 a_2^2 + 4\beta^2 a_2^2 = 0, \tag{3.10f}$$

$$\left(\frac{G'}{G}\right)^6 : -4\alpha^2 a_2^2 \lambda - 4a_2^2 \lambda + 4\beta^2 a_2^2 \lambda - 8\alpha^2 a_1 a_2 - 8a_1 a_2 + 8\beta^2 a_1 a_2 + 3m^2 a_1 a_2^2 = 0. \tag{3.10g}$$

Solving these algebraic equations yields

$$a_0 = \mp\frac{\lambda^2}{4\mu - \lambda^2}, \quad a_1 = \mp\frac{4\lambda}{4\mu - \lambda^2}, \quad a_2 = \mp\frac{4}{4\mu - \lambda^2},$$

$$\alpha = \sqrt{-\frac{\beta^2\lambda^2 \mp m^2 - 4\beta^2\mu + 4\mu - \lambda^2}{4\mu - \lambda^2}}, \tag{3.11}$$

where λ and μ are arbitrary constants.

By using eqs (3.14), expression (3.6) can be written as

$$V(\xi) = \mp\frac{1}{4\mu - \lambda^2}\left(\lambda^2 + 4\lambda\left(\frac{G'}{G}\right) + 4\left(\frac{G'}{G}\right)^2\right), \tag{3.12}$$

where

$$\xi = x + \sqrt{-\frac{\beta^2\lambda^2 \mp m^2 - 4\beta^2\mu + 4\mu - \lambda^2}{4\mu - \lambda^2}}\, y + \beta t.$$

By using eqs (3.4), (3.12), we can understand that the solutions to eq. (3.2) turn out to involve the expressions (3.13a) or (3.13b) defined, respectively, as

$$\phi_1 = \frac{C_1 \sinh(k\xi) + C_2 \cosh(k\xi)}{C_1 \cosh(k\xi) + C_2 \sinh(k\xi)}, \quad k = \frac{\sqrt{\lambda^2 - 4\mu}}{2} \tag{3.13a}$$

$$\phi_2 = \frac{-C_1 \sin(k\xi) + C_2 \cos(k\xi)}{C_1 \cos(k\xi) + C_2 \sin(k\xi)}, \quad k = \frac{\sqrt{4\mu - \lambda^2}}{2}. \tag{3.13b}$$

Substituting the general solution of eq. (2.6) into eq. (3.12) and considering (3.13), we have the solutions of eq. (3.2) as follows:

When $\lambda^2 - 4\mu > 0$,

$$V_1(\xi) = \pm(\phi_1)^2, \tag{3.14}$$

where

$$\xi = x + \sqrt{-\frac{\beta^2 + \lambda^2 \mp m^2 - 4\beta^2\mu + 4\mu - \lambda^2}{4\mu - \lambda^2}}\,y + \beta t.$$

When $\lambda^2 - 4\mu < 0$,

$$V_2(\xi) = \pm (\phi_2)^2\,, \tag{3.15}$$

where

$$\xi = x + \sqrt{-\frac{\beta^2 + \lambda^2 \mp m^2 - 4\beta^2\mu + 4\mu - \lambda^2}{4\mu - \lambda^2}}\,y + \beta t.$$

Recalling that $v = \mathrm{e}^{iu}$, and using the travelling wave reduction $v(x,t) = V(\xi)$, we have:

When $\lambda^2 - 4\mu > 0$,

$$u_1(x, y, t) = \arccos\left(\frac{1 + \phi_1^4}{\pm 2\phi_1^2}\right), \tag{3.16}$$

where

$$\xi = x + \sqrt{-\frac{\beta^2 + \lambda^2 \mp m^2 - 4\beta^2\mu + 4\mu - \lambda^2}{4\mu - \lambda^2}}\,y + \beta t.$$

When $\lambda^2 - 4\mu < 0$,

$$u_2(x, y, t) = \arccos\left(\frac{1 + \phi_2^4}{\pm 2\phi_2^2}\right), \tag{3.17}$$

where

$$\xi = x + \sqrt{-\frac{\beta^2 + \lambda^2 \mp m^2 - 4\beta^2\mu + 4\mu - \lambda^2}{4\mu - \lambda^2}}\,y + \beta t$$

C_1, C_2 are arbitrary constants. If C_1 and C_2 are taken as special values, the various results can be rediscovered. For instance, if $C_2^2 < C_1^2$ in (3.13a), we put $C_2/C_1 = \tanh(k\xi_0)$ and then $\phi_1 = \tanh(k\,(\xi + \xi_0))$ and if $C_2^2 > C_1^2$ in (3.13a), we put $C_1/C_2 = \tanh(k\xi_0)$ and then $\phi_1 = \tanh(k\,(\xi + \xi_0))$. So $u_1(x,y,t)$ can be written as

$$u_1(x, y, t) = \arccos\left(\frac{1 + \tanh^4(k\,(\xi + \xi_0))}{\pm 2\tanh^2(k\,(\xi + \xi_0))}\right), \tag{3.18}$$

$$u_1(x, y, t) = \arccos\left(\frac{1 + \tan^4(k\,(\xi + \xi_0))}{\pm 2\tan^2(k\,(\xi + \xi_0))}\right), \tag{3.19}$$

where

$$\xi = x + \sqrt{-\frac{\beta^2 k^2 \mp m^2 - 4k^2}{4k^2}}\,y + \beta t.$$

The above results are equal to the exact solution of eq. (3.1) obtained by Fan and Hon [29]. Note that ξ_0 may be removed trivially by translating the x-coordinate origin.

3.2 *Dodd–Bullough–Mikhailov equation*

In this section we apply the (G'/G)-expansion method to the Dodd–Bullough–Mikhailov equation:

$$u_{xt} + pe^{u} + qe^{-2u} = 0, \tag{3.20}$$

which becomes Liouville equation at $q = 0$. The Dodd–Bullough–Mikhailov equation appears in problems varying from fluid flow to quantum field theory and has been investigated by many authors [31,32].

We make transformation $v = e^{u}$, $v(x,t) = V(\xi)$ and $\xi = x - ct$, then eq. (3.20) changes to a nonlinear ODE:

$$cVV'' - c\left(V'\right)^2 + pV^3 + q = 0. \tag{3.21}$$

Considering the homogeneous balance between V^3 and VV'' in eq. (3.21), we required that $3m = 2m + 2 \Rightarrow m = 2$. So we suppose that the solution of eq. (3.21) is of the form

$$V(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \tag{3.22}$$

By using eqs (3.22), (3.21) and (2.6), and after simple manipulations we reach a system of algebraic equation which yields

$$a_0 = \frac{(-q/p)^{1/3}\left(2\mu + \lambda^2\right)}{-4\mu + \lambda^2}, \quad a_1 = \frac{6\lambda\,(-q/p)^{1/3}}{-4\mu + \lambda^2},$$

$$a_2 = \frac{6\,(-q/p)^{1/3}}{-4\mu + \lambda^2}, \quad c = \frac{3p\,(-q/p)^{1/3}}{-4\mu + \lambda^2}. \tag{3.23}$$

By using eqs (3.4), (3.23), (3.22) and recalling that $v = e^{u}$ and using the travelling wave reduction $v(x,t) = V(\xi)$ we have:

When $\lambda^2 - 4\mu > 0$,

$$u_1(x,t) = \ln\left(-\frac{1}{2}\left(-\frac{q}{p}\right)^{1/3} + \frac{3}{2}\left(-\frac{q}{p}\right)^{1/3}(\phi_1)^2\right), \tag{3.24}$$

where

$$\xi = x - \left(\frac{3p\,(-q/p)^{1/3}}{-4\mu + \lambda^2}\right)t.$$

When $\lambda^2 - 4\mu < 0$,

$$u_2(x,t) = \ln\left(-\frac{1}{2}\left(-\frac{q}{p}\right)^{1/3} - \frac{3}{2}\left(-\frac{q}{p}\right)^{1/3}(\phi_2)^2\right), \tag{3.25}$$

where

$$\xi = x - \left(\frac{3p\,(-q/p)^{1/3}}{-4\mu + \lambda^2}\right)t,$$

C_1, C_2 are arbitrary constants. If C_1 and C_2 are taken as special values, the various results can be rediscovered. For instance, if $C_2^2 < C_1^2$ in (3.13a), we put $C_2/C_1 = \tanh(k\xi_0)$ and then $\phi_1 = \tanh(k(\xi + \xi_0))$ and if $C_2^2 > C_1^2$ in (3.13a), we put $C_1/C_2 = \tanh(k\xi_0)$ and then $\phi_1 = \tanh(k(\xi + \xi_0))$. So $u_1(x, t)$ can be written as

$$u_1(x,t) = \ln\left(\frac{1}{2}\left(\frac{q}{p}\right)^{1/3}\left(1 - 3\tanh^2(k(\xi + \xi_0))\right)\right), \tag{3.26a}$$

$$u_1(x,t) = \ln\left(\frac{1}{2}\left(\frac{q}{p}\right)^{1/3}\left(1 - 3\tan^2(k(\xi + \xi_0))\right)\right), \tag{3.26b}$$

where

$$\xi = x - \left(\frac{3p\,(-q/p)^{1/3}}{4k^2}\right)t.$$

Equation (3.26) is equal to the exact solution of eq. (3.20) obtained by Fan and Hon [29]. (There is a typographical error in the solutions in [29]: the k in the expressions for ξ should be k^2.)

3.3 *Schrödinger–KdV equation*

The coupled Schrödinger–KdV equation

$$\begin{aligned} &iu_t - u_{xx} - uv = 0,\\ &v_t + 6vv_x + v_{xxx} - \left(|u|^2\right)_x = 0, \end{aligned} \tag{3.27}$$

is known to describe various processes in dusty plasma, such as Langmuir, dust-acoustic wave and electromagnetic waves [31–34]. Exact solution of eq. (3.27) was studied by many authors in refs [9,35,36]. Here the (G'/G)-expansion method is applied to system (3.27) and gives some new solutions. Let

$$u = e^{i\theta}U(\xi), \quad v = V(\xi), \quad \theta = \alpha x + \beta t, \quad \xi = x + ct, \tag{3.28}$$

where α, β and c are constants.

Substituting eq. (3.28) into eq. (3.27), we find that $c = 2\alpha$, and V, U satisfy the following coupled nonlinear ordinary differential system:

$$U'' + \left(\beta - \alpha^2\right)U + UV = 0, \tag{3.29a}$$

$$2\alpha V' + 6VV' + V''' - \left(U^2\right)' = 0. \tag{3.29b}$$

Considering the homogeneous balance between UV and U'' in eq. (3.29a) and that between $(U^2)'$ and V''' in eq. (3.29b) we required that $m + n = n + 2$, $2n + 1 = m + 2 \Rightarrow n = 2, m = 2$. So we suppose that the solution of eqs (3.29a) and (3.29b) is of the form

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0, \tag{3.30}$$

$$V(\xi) = b_0 + b_1 \left(\frac{G'}{G}\right) + b_2 \left(\frac{G'}{G}\right)^2, \quad b_2 \neq 0, \tag{3.31}$$

where $G = G(\xi)$ satisfies the second-order LODE (2.6) and a_0, a_1, a_2, b_0, b_1, b_2, λ and μ are constants to be determined later.

By using eqs (3.30) and (3.31) and eq. (2.6), and solving the derived system of algebraic equations above, we have

$$a_0 = \sqrt{2}\left(\lambda^2 + 2\mu\right), \quad a_1 = 6\sqrt{2}\lambda, \quad a_2 = 6\sqrt{2}$$

$$b_0 = -\frac{5}{6}\lambda^2 - \frac{1}{3}\alpha - \frac{8}{3}\mu, \quad b_1 = -6\lambda, \quad b_2 = -6,$$

$$\beta = -\frac{10}{3}\mu + \frac{5}{6}\lambda^2 + \frac{1}{3}\alpha + \alpha^2 \tag{3.32}$$

or

$$a_0 = 6\sqrt{2}\mu, \quad a_1 = 6\sqrt{2}\lambda, \quad a_2 = 6\sqrt{2}$$

$$b_0 = -\frac{1}{6}\lambda^2 - \frac{1}{3}\alpha - \frac{16}{3}\mu, \quad b_1 = -6\lambda, \quad b_2 = -6,$$

$$\beta = \frac{10}{3}\mu - \frac{5}{6}\lambda^2 + \frac{1}{3}\alpha + \alpha^2, \tag{3.33}$$

where λ, α and μ are arbitrary constants.

By using eqs (3.30)–(3.33), and the general solution of eq. (2.6) and the travelling wave reductions $u = e^{i\theta}U(\xi)$ and $v = V(\xi)$, we have the solutions of eq. (3.27) as follows:

When $\lambda^2 - 4\mu > 0$,

$$u_1(x,t) = -\frac{\sqrt{2}\left(\lambda^2 - 4\mu\right)}{2} e^{i\theta} + \frac{3\sqrt{2}\left(\lambda^2 - 4\mu\right)}{2} (\phi_1)^2 e^{i\theta}, \tag{3.34}$$

$$v_1(x,t) = \frac{2\left(\lambda^2 - 4\mu\right)}{3} - \frac{1}{3}\alpha - \frac{3\left(\lambda^2 - 4\mu\right)}{2} (\phi_1)^2, \tag{3.35}$$

where

$$\xi = x + 2\alpha t, \quad \theta = \alpha x + \left(-\frac{10}{3}\mu + \frac{5}{6}\lambda^2 + \frac{1}{3}\alpha + \alpha^2\right) t,$$

or

$$u_2(x,t) = -\frac{3\sqrt{2}\left(\lambda^2 - 4\mu\right)}{2} e^{i\theta} + \frac{3\sqrt{2}\left(\lambda^2 - 4\mu\right)}{2} (\phi_1)^2 e^{i\theta}, \tag{3.36}$$

$$v_2(x,t) = \frac{4\left(\lambda^2 - 4\mu\right)}{3} - \frac{1}{3}\alpha - \frac{3\left(\lambda^2 - 4\mu\right)}{2} (\phi_1)^2, \tag{3.37}$$

where

$$\xi = x + 2\alpha t, \quad \theta = \alpha x + \left(\frac{10}{3}\mu - \frac{5}{6}\lambda^2 + \frac{1}{3}\alpha + \alpha^2\right) t.$$

When $\lambda^2 - 4\mu < 0$,

$$u_3(x,t) = \frac{\sqrt{2}\left(4\mu - \lambda^2\right)}{2} e^{i\theta} + \frac{3\sqrt{2}\left(4\mu - \lambda^2\right)}{2} (\phi_2)^2 e^{i\theta}, \tag{3.38}$$

$$v_3(x,t) = -\frac{2\left(4\mu - \lambda^2\right)}{3} - \frac{1}{3}\alpha - \frac{3\left(4\mu - \lambda^2\right)}{2} (\phi_2)^2, \tag{3.39}$$

where

$$\xi = x + 2\alpha t, \quad \theta = \alpha x + \left(-\frac{10}{3}\mu + \frac{5}{6}\lambda^2 + \frac{1}{3}\alpha + \alpha^2\right) t,$$

or

$$u_4(x,t) = \frac{3\sqrt{2}\left(4\mu - \lambda^2\right)}{2} e^{i\theta} + \frac{3\sqrt{2}\left(4\mu - \lambda^2\right)}{2} (\phi_2)^2 e^{i\theta}, \tag{3.40}$$

$$v_4(x,t) = -\frac{4\left(4\mu - \lambda^2\right)}{3} - \frac{1}{3}\alpha - \frac{3\left(4\mu - \lambda^2\right)}{2} (\phi_2)^2, \tag{3.41}$$

where

$$\xi = x + 2\alpha t, \quad \theta = \alpha x + \left(\frac{10}{3}\mu - \frac{5}{6}\lambda^2 + \frac{1}{3}\alpha + \alpha^2\right) t.$$

C_1, C_2 are arbitrary constants. If C_1 and C_2 are taken as special values, the various results can be rediscovered. For instance, if $C_2^2 < C_1^2$ in (3.13a), we put $C_1/C_2 = \tanh(k\xi_0)$ and then $\phi_1 = \tanh(k(\xi + \xi_0))$ and if $C_2^2 > C_1^2$ in (3.13a), we put $C_1/C_2 = \tanh(k\xi_0)$ and then $\phi_1 = \tanh(k(\xi + \xi_0))$. Then $u_2(x,t)$ and $v_2(x,t)$ can be written as

$$u_2(x,t) = -6\sqrt{2}k^2\left(1 - \tanh^2(k(\xi + \xi_0))\right) e^{i\theta}, \tag{3.42}$$

$$v_2(x,t) = \frac{1}{3}\left(16k^2 - \alpha\right) - 6k^2 \tanh^2(k(\xi + \xi_0)), \tag{3.43}$$

where

$$\xi = x + 2\alpha t, \quad \theta = \alpha x + \left(-\frac{10}{3}k^2 + \frac{1}{3}\alpha + \alpha^2\right) t.$$

which is equal to the exact solution of eq. (3.27) obtained by Fan and Hon [29].

4. Conclusions

We have successfully implemented the (G'/G)-expansion method to establish travelling wave solution of the two-dimensional sine-Gordon equation, Dodd–Bullough–Mikhailov equation and the Schrödinger–KdV equation.

- It is shown that nonlinear wave equations can be handled by the (G'/G)-expansion method and that the performance of this method is reliable and effective and gives more solutions.
- It is shown that the (G'/G)-expansion method is a special case of the simplest equation method and that the (G'/G)-expansion method can be equivalent to the extended tan expansion method.

It should be noted that the availability of computer systems like Mathematica or Maple facilitates the tedious algebraic calculations. The method we have used in this paper is also standard, direct and computerizable, which allows us to solve complicated and tedious algebraic calculation. The exact solution obtained by this method can be used to check computer codes or as initial condition for computer programs that simulate processes in the corresponding system.

References

[1] W Malfliet, *Am. J. Phys.* **60**, 650 (1992)
[2] W Malfliet and W Hereman, *Phys. Scr.* **54**, 563 (1996)
[3] D D Ganji and M Abdollahzadeh, *Appl. Math. Comput.* **206**, 438 (2008)
[4] J H He and X H Wu, *Chaos, Solitons and Fractals* **30**, 700 (2006)
[5] J H He and M A Abdou, *Chaos, Solitons and Fractals* **34**, 1421 (2007)
[6] X H Wu and J H He, *Chaos, Solitons and Fractals* **38**, 903 (2008)
[7] M Yaghobi Moghaddam, A Asgari and H Yazdani, *Appl. Math. Comput.* **210**, 422 (2009)
[8] A M Wazwaz, *Appl. Math. Comput.* **159**, 559 (2004)
[9] S Liu, Z Fu, S Liu and Q Zhao, *Phys. Lett.* **A289**, 69 (2001)
[10] E Fan and J Zhang, *Phys. Lett.* **A305**, 383 (2002)
[11] M Hosseini, H Abdollahzadeh and M Abdollahzadeh, *J. Math. Comp. Sci.* **2**, 376 (2011)
[12] N Roshan, N Roshan, H Abdollahzadeh, D D Ganji and M Abdollahzadeh, *Int. J. Nonlin. Dyn. Engng. Sci.* **2**, 247 (2010)
[13] E J Parkes, B R Duffy and P C Abbott, *Phys. Lett.* **A295**, 280 (2002)
[14] Y Zhou, M Wang and Y Wang, *Phys. Lett.* **A308**, 31 (2003)
[15] G Cai, Q Wang and J Huang, *Int. J. Nonlin. Sci.* **2(2)**, 122 (2006)
[16] P J Olver, *Application of Lie group to differential equation* (Springer, New York, 1986)
[17] A Sadighi and D D Ganji, *Comput. Math. Appl.* **54**, 1112 (2007)
[18] E Fan and H Zhang, *Phys. Lett.* **A246**, 403 (1998)
[19] L Wang, J Zhou and L Ren, *Int. J. Nonlin. Sci.* **1(1)**, 58 (2006)
[20] M Wang, X Li and J Zhang, *Phys. Lett.* **A372**, 417 (2008)
[21] D D Ganji and M Abdollahzadeh, *J. Math. Phys.* **50**, 1 (2009)
[22] S Zhang, J L Tong and W Wang, *Phys. Lett.* **A372**, 2254 (2008)
[23] J Zhang, X Wei and Y Lu, *Phys. Lett.* **A372**, 3653 (2008)
[24] R Abazari, *Comput. Fluids* **39**, 1957 (2010)
[25] W Hauck, *Math. Methods Appl. Sci.* **24**, 1189 (2001)
[26] J K Perring and T H R Skyrme, *Nucl. Phys.* **31**, 550 (1962)
[27] E Infeld and G Rowlands, *Nonlinear waves, solitons and chaos* (Cambridge Univ. Press, Cambridge, 2000)
[28] A Polyanin and V F Zaitsev, *Handbook of nonlinear partial differential equations* (CRC Press, Boca Raton, FL, 2004)

[29] E Fan and Y C Hon, *Appl. Math. Comput.* **141**, 351 (2003)
[30] D Kaya, *Appl. Math. Comput.* **159**, 1 (2004)
[31] A M Wazwaz, *Commun. Nonlin. Sci. Numer. Simulat.* **13**, 584 (2008)
[32] T Yoshinaga, M Wakamiya and T Kakutani, *Phys. Fluids* **A3**, 83 (1991)
[33] N N Rao, *Pramana – J. Phys.* **49**, 109 (1997)
[34] S V Singh, N N Rao and P K Shukla, *J. Plasma Phys.* **60**, 551 (1998)
[35] D Kaya and S M El-Sayed, *Phys. Lett.* **A313**, 82 (2003)
[36] C L Bai and H Zhao, *Phys. Lett.* **A355**, 32 (2006)