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Anharmonic solution of Schrödinger time-independent equation

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Abstract. We present here a mathematical explanation of how the Schrödinger equation for a class of harmonic oscillators possesses exact solutions. Some of the extended potentials used here are not present in the literature.

Keywords. Anharmonic solutions; Yukawa potential.

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1. Introduction

The problem of finding exact (classical and quantum mechanical) solutions for anharmonic oscillators is of methodological and practical interest in some model field theories (ϕ^4, ϕ^6 etc.), in the context of various fields of applied physics; for instance, in the mean field theory of first-order structural phase transitions [1], in many other cases in atomic and molecular physics or in charmonium systems which provide us with mathematical formulations of physical problems. Moreover, in other cases non-relativistic exact results can be used for extrapolation into the relativistic regime, for example in the context of the theory of S-matrix of strong interaction [2]. Keen interest was shown in the use of multi-term potential in various articles [3–8]. Most of them use doubly harmonic oscillators. Also, if the potential can be treated by perturbation methods [9], then the exact solutions provide an excellent check of the perturbation expansion for those states for which they exist. If the potential cannot be treated by a perturbation method, then the exact solution do at least give some information about the energy levels. Also, Yan [10] discussed solution for non-linear Schrödinger equation.

For these reasons, various types of time-independent solutions and eigenvalues for the Schrödinger equation are important. So we present a class of new exact solutions and eigenvalues for the Schrödinger equation.

2. One-dimensional oscillations

The Hamiltonian for one-dimensional oscillation is well known [4] to be

$$H = \frac{1}{2}p^2 + V(q)\,, \tag{2.1}$$

where p is the momentum and q is the function of generalized coordinates.

Let us now consider a more general potential

$$V(q) = \frac{1}{2}\left\{-\sum_{n=2}^{N} 2n\,(2n-1)\,a_n q^{2n-2} + \left(\sum_{n=1}^{N} 2na_n q^{2n-1}\right)^2\right\}, \tag{2.2}$$

where a_n are constants.

The time-independent Schrödinger equation is

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + V(q)\right)\psi(q) = E\psi(q)\,. \tag{2.3}$$

For the potential (2.2) the solution of eq. (2.3) is of the form

$$\psi(q) = \exp\left(-\sum_{n=1}^{N} a_n q^{2n}\right), \tag{2.4}$$

where $E = a_1$.

This is a ground state solution of one-dimensional time-independent Schrödinger equation and the ground state energy level is positive or negative depending on whether a_1 is positive or negative. For $N = 2$, in potential (2.2), the solution (2.4) can be reduced to the potential of Leach ([6], expression (1.3)), i.e.

$$V(q) = aq^2 + bq^4 + cq^6, \tag{2.5}$$

and his solution (expression (1.5)), i.e.

$$\psi(q) = \exp\left(-\frac{1}{2}\alpha q^2 - \frac{1}{4}\beta q^4\right), \tag{2.6}$$

provided

$$a = 2E^2 - 6a_2, \quad b = 8Ea_2, \quad c = 8a_2^2$$

and

$$\alpha = 2a_1 = 2E, \quad \beta = 4a_2.$$

Also for $N = 2$ the potential (2.2) reduces to that of Flessas's ([3–5], expression (1)) and of Khare ([7], expression (1)), i.e.

$$V(q) = \frac{1}{2}\omega^2 q^2 + \frac{1}{4}\lambda q^4 + \frac{1}{6}\eta q^6, \quad \eta > 0, \tag{2.7}$$

provided

$$E = a_1 = \frac{1}{8}\lambda\left(\frac{3}{\eta}\right)^{1/2}, \quad a_2 = \frac{1}{4}\left(\frac{\eta}{3}\right)^{1/2}, \quad \omega^2 = \frac{3\lambda^2}{16\eta} - (3\eta)^{1/2}\,,$$

and the corresponding solution

$$\psi(q) = \exp\left\{-\frac{\lambda}{8}\left(\frac{3}{\eta}\right)^{1/2} q^2 - \frac{1}{4}\left(\frac{\eta}{3}\right)^2 q^4\right\}. \tag{2.8}$$

Again for $N = 3$ the potential (2.2) reduces to the potential of Flessas ([4], expression (3)), i.e.

$$V(q) = \frac{1}{2}\omega^2 q^2 + \frac{1}{4}b_2 q^4 + \frac{1}{6}b_3 q^6 + \frac{1}{8}b_4 q^8 + \frac{1}{10}b_5 q^{10}, \quad b_5 > 0, \tag{2.9}$$

provided

$$E = a_1, \quad \omega^2 = 4\left(E^2 - 3a_2\right), \quad b_2 = 4\left(8Ea_2 - 15a_3\right),$$

$$b_3 = 6\left(8a_2^2 + 12Ea_2\right), \quad b_4 = 192a_2a_3, \quad b_5 = 180a_3^2,$$

and the corresponding solution

$$\psi(q) = \exp\left(-\sum_{n=1}^{3} a_n q^{2n}\right). \tag{2.10}$$

Now, consider the potential

$$V = E - \frac{1}{2}\left\{Q'(q)\mathrm{e}^{-2\beta q^2} + Q'(q)\mathrm{e}^{-\beta q^2} - 2\beta q Q(q)\mathrm{e}^{-\beta q^2}\right\}, \tag{2.11}$$

where $Q(q)$ is a function of q and $Q'(q) = \mathrm{d}Q/\mathrm{d}q$, $\beta = \text{const.}$

For this potential, Schrödinger equation (2.3) gives the solution

$$\psi(q) = A\exp\left\{\int^q Q\left(q'\right)\mathrm{e}^{-\beta q'^2}\mathrm{d}q'\right\}. \tag{2.12}$$

For bounded support, one may consider the wave function [11,12]

$$\begin{aligned} \psi(q) &= A\exp\left\{\frac{-\alpha^2}{q_0^2 - q^2}\right\}, \quad && \text{if } |q| < q_0, \\ &= 0, && \text{if } |q| \geq q_0. \end{aligned} \tag{2.13}$$

(Note that $\psi(q)$ and all its derivatives vanish strictly for $|q| \geq q_0$, $q_0 > 0$.)

Thus the Schrödinger equation (2.3) gives the potential

$$V = E - \frac{\alpha^2 q_0^2 + 2\alpha^2\left(q_0^2 - 1\right)q^2 - 3\alpha^2 q^4}{\left(q_0^2 - q^2\right)^4}. \tag{2.14}$$

The time-independent one-dimensional Schrödinger equation (2.3) may also be written as [6]

$$\left(D^2 + \lambda - X\right)\psi = 0, \tag{2.15}$$

where $D \equiv \mathrm{d}/\mathrm{d}q$. λ is the eigenvalue and $X(q)$ is the potential (to within a constant multiplier). This differential equation may be written in the operator form [6] as

$$D^2 + \lambda - X = (D - \alpha)(D - \beta) \tag{2.16}$$

from which it follows that

$$\beta = -\alpha, \quad \alpha^2 - \alpha' = X - \lambda. \tag{2.17}$$

Equation (2.11) now reads [6] as

$$(D - \alpha)\,(D + \alpha)\,\psi = 0, \tag{2.18}$$

the general solution of (2.18) as follows:

$$\psi(q) \;=\; A\exp\!\left(-\int^{q} \alpha(u)\,\mathrm{d}u\right) + B\,\exp\!\left(-\int^{q} \alpha(u)\,\mathrm{d}u\right) \times \int^{q} \exp\!\left(2\int^{u} \alpha(v)\,\mathrm{d}v\right)\mathrm{d}u. \tag{2.19}$$

In the context of an oscillator potential, $\alpha(q)$ is a polynomial and either A or B will be zero according to the behaviour of $\alpha(q)$ as $|q| \to \infty$. The solution (2.19) is a ground-state solution. To obtain higher states factorization of the form [6]

$$(D - \alpha)\,(fD - \beta) = f\left(D^2 + \lambda - X\right), \tag{2.20}$$

is assumed, which gives $\beta = f' - \alpha f$, and

$$X - \lambda = \alpha^2 - \alpha' - \left(2\alpha f' - f''\right)/f. \tag{2.21}$$

Then the solution changes to

$$\psi(q) \;=\; Af(q)\exp\!\left(-\int^{q} \alpha(u)\,\mathrm{d}u\right) + Bf(q)\exp\!\left(-\int^{q} \alpha(u)\,\mathrm{d}u\right) \times \int^{q} f^{-2}(u)\exp\!\left(2\int^{u} \alpha(v)\,\mathrm{d}v\right)\mathrm{d}u, \tag{2.22}$$

having the same analysis as for (2.19).

In (2.17) the choice of a polynomial for $\alpha(q)$ will give a polynomial $X(q)$. For the wave function to be square integrable the degree of $\alpha(q)$ must be odd, i.e., $2N + 1$, whence the degree of $X(q)$ is $4N + 2$. It is possible to obtain more than one explicit solution (corresponding to different energy levels) for a particular potential by assuming that $\psi(q)$ has the form

$$\psi(q) = f(q)\exp\!\left(-\sum_{n=1}^{N}\int \alpha_n q^{2n-1}\mathrm{d}q\right), \tag{2.23}$$

where α_n are constants.

For odd states, we need to consider

$$f(q) = \sum_{n=1}^{p} f_{2n-1}q^{2n-1}, \quad \alpha(q) = \sum_{n=1}^{N}\alpha_{2n-1}q^{2n-1}, \quad A = \sum_{n=0}^{N-1} a_{2n}q^{2n}. \tag{2.24}$$

Then (2.21) implies the choice

$$2\alpha f' - f'' = Af, \tag{2.25}$$

which gives N sets of relations for $k = 1,2,\ldots$, where p is a positive integer, equating coefficients of powers of q:

$$q^{2k-1} : 2\sum_{n=1}^{p} (2n-1)\alpha_{2k-2n+1} f_{2n-1} - 2k(2k+1) f_{2k+1} = \sum_{n=1}^{P} a_{2k-2n} f_{2n-1}, \tag{2.26}$$

from which we can obtain all unknowns a_{2i}, $i = 1,2,3,\ldots$.

For even states, we have to take f as an even function, i.e.

$$f(q) = \sum_{n=0}^{p} f_{2n} q^{2n}.$$

Then for this f, eq. (2.25) gives N number of relations

$$q^{2k} : 2\sum_{n=1}^{p} 2n\alpha_{2k-2n+1} f_{2n} - 2(k+1)(2k+1) f_{2k+2}$$
$$= \sum_{n=0}^{p} a_{2k-2n} f_{2n}, \tag{2.27}$$

where

$$a_{-n} = f_{-n} = 0, \quad \alpha_{2N+i} = f_{2P+j} = a_{2N+k} = 0.$$

Using the above process we can obtain different states of solution of the form (2.23). To clarify this process we cite some examples.

Case 1. Consider $\alpha(q) = \alpha_1 q + \alpha_3 q^3 + \alpha_5 q^5$, $f = f_1 q$. Then eq. (2.25), i.e., $2\alpha f' - f'' = (a_0 + a_2 q^2 + a_4 q^4) f$, gives

$$a_0 = 2\alpha_1, \quad a_2 = 2\alpha_3, \quad a_4 = 2\alpha_5.$$

Then for $\lambda = 3\alpha_1$, the potential (2.21) becomes

$$X(q) = \left(\alpha_1^2 - 5\alpha_3\right) q^2 + (\alpha_1\alpha_3 - 7\alpha_5) q^4 + \left(\alpha_3^2 + 2\alpha_1\alpha_5\right) q^6$$
$$+ 2\alpha_3\alpha_5 q^8 + \alpha_5^2 q^{10}, \tag{2.28}$$

and the solution is

$$\psi(q) = (f_1 q) \exp\left\{-\frac{\alpha_1}{2} q^2 - \frac{\alpha_3}{4} q^4 - \frac{\alpha_5}{6} q^6\right\}. \tag{2.29}$$

Case 2. Consider $f = f_0 + f_2 q^2$. Then (2.27) gives for $(f_2/f_0) = g$, $a_0 = 2g$, $a_2 = 4\alpha_1 g + 2g^2$, $a_4 = 4\alpha_3 g - 4\alpha_1 g^2 - 2g^3$, $\alpha_5 = \alpha_3 g - \alpha_1 g^2 - \frac{1}{2} g^3$, which indicates that g has three values.

The potential (2.21) for $\lambda = \alpha_1 + 2g$ gives

$$X(q) = \left(\alpha_1^2 - 3\alpha_3 - 4\alpha_1 g - 2g^2\right) q^2 + \left(2\alpha_1\alpha_3 - 5\alpha_5 - 4\alpha_3 g + 4\alpha_1 g^2 + 2g^3\right) q^4 + \left(\alpha_3^2 + 2\alpha_1\alpha_5\right) q^6 + 2\alpha_3\alpha_5 q^8 + \alpha_5 q^{10}, \tag{2.30}$$

and the solution is

$$\psi(q) = f_0\left(1 + gq^2\right) \exp\left\{-\frac{\alpha_1}{2}q^2 - \frac{\alpha_3}{4}q^4 - \frac{\alpha_5}{6}q^6\right\}. \tag{2.31}$$

Now, if we take $f = 1 + f_2 q^2$ and $\alpha_5 = a_4 = 0$, then using (2.27) the above result after calculations, gives

$$f_2 = -\alpha_1 \pm \sqrt{\left(\alpha_1^2 + 2\alpha_3\right)},$$

and for $\lambda = \alpha_1 - 2f_2$, the potential (2.21) becomes

$$X(q) = \left(\alpha_1^2 - 7\alpha_3\right) q^2 + 2\alpha_1\alpha_3 q^4 + \alpha_3^2 q^6, \tag{2.30a}$$

and the corresponding solution is

$$\psi = \left[1 + \left\{-\alpha_1 \pm \sqrt{\left(\alpha_1^2 + 2\alpha_3\right)}\right\} q^2\right] \exp\left(-\frac{\alpha_1^2}{2}q^2 - \frac{\alpha_3}{4}q^4\right). \tag{2.31a}$$

Now for $\alpha_1 = \beta/2\sqrt{\gamma}$, $\alpha_3 = \sqrt{\gamma}$, eq. (2.30a) gives

$$X(q) = \left(\frac{\beta^2}{4\gamma} - 7\sqrt{\gamma}\right) q^2 + \beta q^4 + \gamma q^6, \tag{2.30b}$$

$$\psi = \left[1 + \frac{1}{2}\left\{-\frac{\beta}{\gamma} \pm \sqrt{\left(\frac{\beta^2}{\gamma} + 8\sqrt{\gamma}\right)}\right\} q^2\right] \exp\left(-\frac{\beta}{4\sqrt{\gamma}}q^2 - \frac{\sqrt{\gamma}}{4}q^4\right), \tag{2.31b}$$

$$\begin{aligned} E = \lambda &= \frac{\beta}{2\sqrt{\gamma}} + 2\left\{\frac{\beta}{2\sqrt{\gamma}} \pm \sqrt{\left(\frac{\beta^2}{4\gamma} + 2\sqrt{\gamma}\right)}\right\} \\ &= \frac{3\beta}{2\sqrt{\gamma}} \pm \sqrt{\left(\frac{\beta^2}{\gamma} + 8\sqrt{\gamma}\right)} \end{aligned} \tag{2.30c}$$

and

$$\alpha = \frac{\beta^2}{4\gamma} - 7\sqrt{\gamma}, \tag{2.30d}$$

which shows that our results (eqs (2.30b–d)) and (eq. (2.31b)) are exactly matching with the results of Atre *et al* ([13], expressions (13), (14)) and their potential in Hamiltonian (in expression (6)) for $n = 2$ except their polynomial $P_{2\pm}$ (in expression (15)) but our method gives $\left[1 + \frac{1}{2}\left\{-\frac{\beta}{\gamma} \pm \sqrt{\left(\frac{\beta^2}{\gamma} + 8\sqrt{\gamma}\right)}\right\} q^2\right]$. So eqs (2.30) and (2.31) are more general than that of [13].

Again for $\beta = \frac{\alpha_1}{q} + \alpha_3 q^3$, $f = f_0 + f_2 q^2$. Then using (2.25), i.e. $2\beta f' - f'' = Af$, gives $f_2 = 4\alpha_3 f_0/(1 - 2\alpha_1)$ and for $\lambda = E = -(4\alpha_3 f_0/f_2)$ gives the potential

$$X(q) = \frac{(\alpha_1^2 + \alpha_1)}{q^2} + (2\alpha_1\alpha_3 - 7\alpha_3)\, q^2 + \alpha_3^2 q^6, \tag{2.30e}$$

and the solution is

$$\psi = \left(f_0 \pm \sqrt{\frac{4\alpha_3 f_0}{1 - 2\alpha_1} q^2}\right) q^{-\alpha_1} \exp\!\left(-\frac{\alpha_3}{4} q^4\right). \tag{2.31c}$$

Then the potential (2.30e) can be matched with the potential of the Hamiltonian (in expression (16)) of Atre *et al* [13] iff

$$\sigma = \alpha_1(\alpha_1 + 1), \quad \alpha = 2\alpha_1\alpha_3 - 7\alpha_3, \quad \gamma = \alpha_3^2.$$

If we choose $f_2 = \sqrt{\gamma} = \alpha_3 = a \Rightarrow f_0 = -\frac{\lambda}{4} = -\frac{E}{4}$ that gives

$$\alpha = (2\alpha_1 - 7)\sqrt{\gamma} = -4\,(s - 2)\, a, \quad \text{for } \alpha_1 = -2l = -2\left(s - \frac{1}{4}\right),$$

and

$$\sigma = -2l\,(-2l + 1) = 4\left(s - \frac{1}{4}\right)\left(s - \frac{3}{4}\right),$$

this agrees with that in ref. [13] for $\mu = -5/2$.

Then the solution (2.31c) becomes

$$\psi = \left(-\frac{E}{4} \pm a q^2\right) q^{2l} \exp\!\left(-\frac{a}{4} q^4\right),$$

which agrees with the solution of Atre *et al* [13] (p. 49).

Case 3. Consider $f = f_1 q + f_3 q^3$. Then eq. (2.25) gives for $f_3/f_1 = g$, $a_0 = 2\alpha_1 - 6g$, $a_2 = 2\alpha_3 + 4\alpha_1 g + 6g^2$, $a_4 = 6(\alpha_3 g - \alpha_1 g^2 - 6g^3)$, $\alpha_5 = \alpha_3 g - \alpha_1 g^2 - 6g^3$, which indicates that g has three values.

For $\lambda = 6g - \alpha_1$, eq. (2.21) reduces to

$$\begin{aligned} X(q) = {} & \left(\alpha_1^2 - 5\alpha_3 - 4\alpha_1 g - 6g^2\right) q^2 + \left(2\alpha_1\alpha_3 - 6\alpha_3 g + 6\alpha_1 g^2 + 36 g^3\right) q^4 \\ & + \left(\alpha_3^2 + 2\alpha_1\alpha_5\right) q^6 + 2\alpha_3\alpha_5 q^8 + \alpha_5^2 q^{10}, \end{aligned} \tag{2.32}$$

and the solution is

$$\psi(q) = f_1\left(q + g q^3\right) \exp\left(-\frac{\alpha_1}{2} q^2 - \frac{\alpha_3}{4} q^4 - \frac{\alpha_5}{6} q^6\right). \tag{2.33}$$

Case 4. Consider $f = 1 + f_2 q^2 + f_4 q^4$, for $\alpha_5 = a_4 = 0$. Then eq. (2.25) yields $a_0 = -2f_2$, $a_2 = 8\alpha_3$, $f_4 = \frac{\alpha_1}{3} f_2 + \frac{f_2^2}{6} - \frac{2\alpha_3}{3}$, $f_2^3 + 4\alpha_1 f_2^2 + 4\left(\alpha_1^2 - 4\alpha_3\right) f_2 - 8\alpha_1\alpha_3 = 0$, which indicates that f_2 has three values.

For $\lambda = E = \alpha_1 + 2f_2$, eq. (2.21) reduces to

$$X(q) = \left(\alpha_1^2 - 11\alpha_3\right) q^2 + 2\alpha_1\alpha_3 q^4 + \alpha_3^2 q^6,$$

and the solution for $f = P_4(q)$ is

$$\psi(q) = P_4(q)\exp\Big(-\frac{\alpha_1}{2}q^2 - \frac{\alpha_3}{4}q^4\Big),$$

for the three values of f_2, $P_4(q)$ gives three polynomials.

Now consider the special case to match with the results of Atre *et al* [13] (p. 50). For $\alpha_3 = 1$, choose $\alpha_1 = 0$ yielding $f_2 = 0, \ \pm 4$. Hence

$$f_4 = \begin{cases} -\dfrac{2}{3}, & \text{for } f_2 = 0, \\ 2, & \text{for } f_2 = \pm 4. \end{cases}$$

The potential now becomes

$$X(q) = -11q^2 + q^6,$$
$$\psi(q) = P_4\,(q)\exp\Big(-\frac{q^4}{4}\Big),$$

where

$$P_4(q) = \begin{cases} 1 - \dfrac{2}{3}q^4, & \text{for } f_2 = 0 \\ 1 \pm 4q^2 + 2q^4, & \text{for } f_2 = \pm 4 \end{cases}$$

and

$$E = \lambda = 0, \ \pm 8$$

which agree with the results of Atre *et al* [13] (p. 50).

Similarly for higher states we can extend the result for $X(q)$ which is a polynomial in q^2. However, it is not necessary that $X(q)$ be a polynomial in q^2 but the polynomial of $\alpha(q)$ must be an odd degree. The solution $\psi(q)$ admits solution of Magyari's ([8], expression (3)).

3. Three-dimensional oscillators

The Schrödinger equation in cylindrical polar coordinate system when ψ is independent of θ and z (by replacing ρ by q) reduces to

$$-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}q}\right)\psi + (V - E)\psi = 0. \tag{3.1}$$

The potential,

$$V = \frac{1}{2}\left\{\left(\sum_{n=1}^{N} 2na_n q^{2n-1}\right)^2 - \sum_{n=2}^{N} 4n^2 a_n q^{2n-2}\right\}, \tag{3.2}$$

gives the solution of (3.1) for $E = 3a_1$ as

$$\psi(q) = A\exp\left\{-\sum_{n=1}^{N} a_n q^{2n}\right\}. \tag{3.3}$$

Again, the Schrödinger equation in spherical polar coordinate system when ψ is independent of θ and φ (by replacing r by q) reduces to

$$-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}q^2}+\frac{2}{q}\frac{\mathrm{d}}{\mathrm{d}q}\right)\psi+(V-E)\,\psi=0. \tag{3.4}$$

The potential

$$V=\frac{1}{2}\left\{\left(\sum_{n=1}^{N}2na_nq^{2n-1}\right)^2-\sum_{n=2}^{N}2n(2n+1)a_nq^{2n-2}\right\}, \tag{3.5}$$

yields the solution of (3.4) for $E=3a_1$ as

$$\psi(q)=A\exp\left\{-\sum_{n=1}^{N}a_nq^{2n}\right\}. \tag{3.6}$$

The time-independent Schrödinger equation for cylindrical polar coordinate system is

$$\left(-\frac{1}{2}\nabla^2+V\right)\psi=E\psi, \tag{3.7}$$

where

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}.$$

Consider ψ is independent of θ.

Case (a). Consider the potential

$$V(r)=\frac{1}{2}\left\{-\sum_{i=2}^{N}4\alpha_i^2m_ir^{2\alpha_i-2}+\left(\sum_{i=1}^{N}2\alpha_im_ir^{2\alpha_i-1}\right)^2\right\}. \tag{3.8}$$

For $E=\frac{1}{2}k^2=2m_1$ and $\alpha_1=1$, the solution of (3.7) for the above potential is

$$\psi(rz)=\left(a\mathrm{e}^{kz}+b\mathrm{e}^{-kz}\right)\exp\left(\sum_{i=1}^{N}m_ir^{2\alpha_i}\right). \tag{3.9}$$

Case (b). Let us assume the potential

$$V(rz)=\frac{1}{2}\left\{\frac{m^2}{r^2}+\left(\sum_{i=1}^{N}\beta_i\alpha_iz^{\beta_i-1}\right)^2-\sum_{i=1}^{N}\beta_i(\beta_i-1)\alpha_iz^{\beta_i-2}\right\}. \tag{3.10}$$

For $E=\frac{1}{2}k^2$, the required solution of (3.7) for the above potential is

$$\psi=\{C_1J_m(kr)+C_2J_{-m}(kr)\}\exp\left(-\sum_{i=1}^{N}\alpha_iz^{\beta_i}\right),\quad \text{for } m\text{-fraction}, \tag{3.11}$$

$$\psi=\{C_1J_m(kr)+C_2Y_m(kr)\}\exp\left(-\sum_{i=1}^{N}\alpha_iz^{\beta_i}\right),\quad \text{for } m\text{-integer}, \tag{3.12}$$

where J_m, Y_m are respectively Bessel's functions of the first kind and Neumann's Bessel's function of the second kind, of order n.

Case (c). Again, consider the potential

$$\begin{aligned} V(r,z) &= V(r) + V(z) \\ &= 2\left(\sum_{i=1}^{n} \beta_i q_i r^{2q_i-1}\right) - 2\sum_{i=2}^{n} \beta_i q_i^2 r^{2q_i-2} + 2\left(\sum_{i=1}^{n} m_i \alpha_i z^{2m_i-1}\right)^2 \\ &\quad + \sum_{i=1}^{n} m_i(2m_i-1) z^{2m-2}. \end{aligned} \tag{3.13}$$

For $q_1 = 1$, $E = 2\beta_1$, the corresponding solution of (3.7) for the above potential is

$$\psi = \exp\left\{\sum_{i=1}^{n} \left(\alpha_i z^{2m_i} - \beta_i r^{2q_i}\right)\right\}. \tag{3.14}$$

The time-independent Schrödinger equation for the spherical coordinate system is

$$\left(-\frac{1}{2}\nabla^2 + V\right)\psi = E\psi, \tag{3.15}$$

where

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Let $V = V(r)$ is the function of r only. Then for $\psi = R\Theta\Phi$, where $R = R(r)$, $\Theta = \Theta(\theta)$, $\Phi = \Phi(\varphi)$, the above equation reduces to Morse and Freshbach [9],

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left\{2(E - V(r))r^2 - l(l+1)\right\}R = 0, \tag{3.16a}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2} + k^2\Phi = 0, \tag{3.16b}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\varphi}\right) + l(l+1)\Theta - \frac{k^2\Theta}{\sin^2\theta} = 0. \tag{3.16c}$$

The solution of eqs (3.16b) and (3.16c) are respectively

$$\Phi = a_1\cos k\varphi + b_1\sin k\varphi, \tag{3.17a}$$

$$\Theta(\cos\theta) = aP_l^k(\cos\theta) + bQ_l^k(\cos\theta), \tag{3.17b}$$

where $P_l^k(x)$ and $Q_l^k(x)$ are the associated Legendre polynomials or functions of the first and second kind respectively, and

$$\psi = (a_1 \cos k\varphi + b_1 \sin k\varphi)\left\{a P_l^k(\cos\theta) + b Q_l^k(\cos\theta)\right\} \exp\left(-\sum_{i=1}^{n} a_i \gamma^{2m_i}\right)$$

$$Q_l^k(\cos\theta) = (-1)^l \sin^k\theta \frac{d^k Q_l(\cos\theta)}{d(\cos\theta)^k}.$$

Case (i). Consider

$$V(r) = \frac{n(n+1) - l(l+1)}{2r^2},$$

then eq. (3.16a) gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left\{2Er^2 - n(n+1)\right\} R = 0. \tag{3.18}$$

Choosing $2E = k^2$ and then using the transformation $x = kr$, eq. (3.18) reduces to

$$x^2 \frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}R}{\mathrm{d}x} + \left\{x^2 - n\,(n+1)\right\} R = 0.$$

Again, by choosing $Z = \sqrt{x} R$, the above equation becomes

$$x^2 \frac{\mathrm{d}^2 Z}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}Z}{\mathrm{d}x} + \left\{x^2 - \left(n + \tfrac{1}{2}\right)^2\right\} Z = 0,$$

which is a Bessel's equation of order $n + \frac{1}{2}$ and so the solution of eq. (3.16a) will be

$$R = \frac{1}{kr}\sqrt{\frac{2}{\pi}}\left\{A J_{n+\frac{1}{2}}(kr) + (-1)^{n+1} B J_{-n-\frac{1}{2}}(kr)\right\}. \tag{3.19}$$

The required solution of (3.15) is

$$\psi = \frac{1}{kr}\sqrt{\frac{2}{\pi}}\left\{A J_{n+\frac{1}{2}}(kr) + (-1)^{n+1} B J_{-n-\frac{1}{2}}(kr)\right\} \times \left\{a P_l^k(\cos\theta) + b Q_l^k(\cos\theta)\right\}\left\{a_1 \cos k\varphi + b_1 \sin k\varphi\right\}. \tag{3.20}$$

Case (ii). Now, consider the potential

$$V(r) = 2\left(\sum_{i=1}^{n} m_i a_i r^{2m_i - 1}\right)^2 - \sum_{i=2}^{n} m_i (2m_i + 1) a_i r^{2m_i - 2} - l(l+1)/2r^2. \tag{3.21}$$

For $E = 3a_1$ and $m_1 = 1$, the solution of eq. (3.16a) for the potential (eq. (3.21)) is

$$R = \exp\left(-\sum_{i=1}^{n} a_i r^{2m_i}\right). \tag{3.22}$$

So the required solution of (3.15) is

$$\psi = (a_1 \cos k\varphi + b_1 \sin k\varphi)\left\{a P_l^k(\cos\theta) + b Q_l^k(\cos\theta)\right\} \times \exp\left(-\sum_{i=1}^{n} a_i r^{2m_i}\right). \tag{3.23}$$

Case (iii). For the extended Yukawa potential, which may be considered as a hadron potential,

$$V(r) = E - \left(a + br + cr^2\right) \mathrm{e}^{-\mu r}. \tag{3.24}$$

For the above potential, eq. (3.16a) gives

$$R = r^l \exp\left(-\sum_{m=1}^{\propto} a_m r^m\right), \tag{3.25}$$

provided that they satisfy the following recurrence relations:

$$-4lma_{2m} + m^2 a_m^2 + 2\sum_{p=1}^{\propto} p(2m-p)a_p a_{(2m-p)} - 2m(2m+1)a_{2m}$$

$$+\frac{2a\mu^{2m-2}}{(2m-2)!} - \frac{2b\mu^{2m-3}}{(2m-3)!} + \frac{2c\mu^{2m-4}}{(2m-4)!} = 0, \quad m \neq p \tag{3.26a}$$

$$-2l(2m-1)a_{2m-1} + 2\sum_{p=1}^{\propto} p(2m-p-1)a_p a_{(2m-p-1)} - 2m(2m-1)a_{2m-1}$$

$$-\frac{2a\mu^{2m-3}}{(2m-3)!} + \frac{2b\mu^{2m-4}}{(2m-4)!} - \frac{2c\mu^{2m-5}}{(2m-5)!} = 0, \quad 2m \neq 2p+1 \tag{3.26b}$$

which gives

$$a_2 = \frac{a}{(2l+3)}, \quad a_3 = \frac{(b - a\mu)}{(3l+6)},$$

$$a_4 = \frac{1}{4(2l+5)}\left\{a\mu^2 - 2b\mu + 2c - \frac{4a^2}{(2l+3)^2}\right\},$$

$$a_5 = \frac{1}{(10l+3)}\left\{b\mu^2 - 2c\mu - \frac{a\mu^3}{3} + \frac{6a(b-a\mu)}{(2l+3)(3l+6)}\right\},$$

and

$$a_r = f_r(a, b, c, l, \mu), \qquad r \geq 4.$$

4. Higher states

To obtain wave functions and eigenvalues for higher energy states we make a modification of the factorization as was done in (2.16). We assume that the energy states occur for the radial functions only.

Case (α). For the cylindrical coordinate system, we write the radial equation as

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}+\left(\lambda-X(r)-k^2\right)R=0.$$

So, for higher states we assume

$$(D-\alpha)(fD-\beta)\equiv f\left(D^2+\frac{1}{r}D+\lambda-X(r)-k^2\right), \tag{4.1}$$

which yields

$$\beta=f'-\alpha f-\frac{f}{r},$$
$$X-\lambda+k^2=\alpha^2+\frac{\alpha}{r}-\alpha'+\frac{1}{r^2}+\frac{1}{f}\left(f''-2\alpha f'-\frac{f'}{r}\right). \tag{4.2}$$

The solution of $R(r)$ is given by

$$R=\frac{A}{r}f(r)\exp\left(-\int^r\alpha(u)\mathrm{d}u\right)+\frac{B}{r}f(r)\exp\left(-\int^r\alpha(u)\mathrm{d}u\right)$$
$$\times\int^r\left[\frac{u}{f^2(u)}\exp\left(\int^u 2\alpha(v)\mathrm{d}v\right)\right]\mathrm{d}u. \tag{4.3}$$

For X to represent an oscillator potential, the constant A will be zero if $\alpha(r)\to-\infty$ as $r\to-\infty$ and B will be zero if $\alpha(r)\to\infty$ as $r\to\infty$. For higher state solutions, we choose

$$\alpha=\sum_{i=1}^{n+1}\alpha_{2i-1}r^{2i-1}+\frac{m}{r},\quad A=\sum_{i=1}^{n}a_{2i}r^{2i}-\frac{b}{r^2}. \tag{4.4}$$

Case (i). For odd states: Consider $f=\sum_{i=1}^{Q}f_{2i-1}r^{2i-1}$. Then using (4.4) $f''-2\alpha f'-\frac{f'}{r}=Af$ gives the following recurrence relations, for $m=0$, $b=1$:

$$\left\{(2i+1)^2+1\right\}f_{2i+1}-2\sum_{j=0}^{n-1}(2i-2j-1)\alpha_{2i+1}f_{2(i-j)-1}$$
$$=\sum_{j=0}^{n}a_{2j}f_{2(i-j)-1}, \tag{4.5}$$

where $i=1,2,3,\ldots,n-1$ and Q is a positive integer, which determines the unknown values of a_{2i} uniquely.

The potential is, to within a constant multiplier, for $\lambda = k^2 - a_0$;

$$X(r) = \left(\sum_{i=1}^{n+1} \alpha_{2i-1} r^{2i-1}\right)^2 - 2\sum_{i=2}^{n+1} (i-1)\alpha_{2i-1} r^{2i-2} + \sum_{i=1}^{n} a_{2i} r^{2i},$$

and the solution is

$$R = \left(\frac{1}{r}\sum_{i=1}^{Q} f_{2i-1} r^{2i-1}\right) \exp\left\{-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right\}$$
$$\times \left[A + B\int^{r} \frac{u}{\left(\sum_{i=1}^{Q} f_{2i-1} r^{2i-1}\right)^2} \exp\left(2\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} u^{2i}\right) \mathrm{d}u\right].$$

The solution of eq. (3.7) is

$$\psi = \left(\frac{1}{r}\sum_{i=1}^{Q} f_{2i-1} r^{2i-1}\right) \exp\left\{-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right\}$$
$$\times \left[A + B\int^{r} \frac{u}{\left(\sum_{i=2i-1}^{Q} f_{2i-1} r^{2i-1}\right)^2} \exp\left(2\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} u^{2i}\right) \mathrm{d}u\right]$$
$$\times \left\{a\mathrm{e}^{kz} + b\mathrm{e}^{ikz}\right\}.$$

Case (ii). For even states: Consider $f = \sum_{i=0}^{G} f_{2i} r^{2i}$, where G is a positive integer. Then using (4.4) $f'' - 2\alpha f' - \frac{f'}{r} = Af$ gives the following recurrence relation, for $m = -1$, $b = 0$, $a_0 = 0$:

$$4\,(i+1)^2 f_{2i+2} - 2\sum_{j=0}^{i} 2(i-j)\alpha_{2j+1} f_{2(i-j)} = \sum_{j=1}^{n} a_{2j} f_{2(i-j)},$$

where $i = 1, 2, 3, \ldots, n-1$, which determines the unknown values of a_{2i} uniquely.

The potential is, to within a constant multiplier, for $\lambda = k^2 - a_0 + 2\alpha_1$,

$$X(r) = \left(\sum_{i=1}^{n+1} \alpha_{2i-1} r^{2i-1}\right)^2 - \sum_{i=2}^{n+1} 2i\alpha_{2i-1} r^{2i-2} + \sum_{i=1}^{n} a_{2i} r^{2i},$$

and the solution of $R(r)$ is

$$R = \left(\sum_{i=1}^{G} f_{2i} r^{2i}\right) \exp\left\{-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right\}$$
$$\times \left[A + B\int^{r} \frac{1}{u\left(\sum_{i=0}^{G} f_{2i} r^{2i}\right)^2} \exp\left(2\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} u^{2i}\right) \mathrm{d}u\right].$$

So the solution of eq. (3.7) is

$$\psi = \left(\sum_{i=1}^{G} f_{2i} r^{2i}\right) \exp\left\{-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right\}$$
$$\times \left[A + B\int^{r} \frac{2}{\left(\sum_{i=0}^{G} f_{2i} r^{2i}\right)^2} \exp\left(2\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} u^{2i}\right) \mathrm{d}u\right] \left\{a\mathrm{e}^{kz} + b\mathrm{e}^{ikz}\right\}.$$

We now cite two examples.

Case (i). Let $f = f_1 r$, $p = n - 1$ and $b = 1$. Then $f'' - 2\alpha f' - \frac{f'}{r} = Af$ gives the recurrence relation

$$2i(2i+1)\alpha_{2i+1} - 2f_1\alpha_{2i-1} = f_1 a_{2i-2}, \tag{4.5a}$$

for $i = 1, \ldots, n+1$, the above relation determines a_{2i} uniquely.

For $\lambda = k^2 - a_0$, the potential (4.2) becomes

$$X(r) = \left(\sum_{i=1}^{n+1} \alpha_{2i-1} r^{2i-1}\right)^2 - 2\sum_{i=2}^{n+1} (i-1)\alpha_{2i-1} r^{2i-2} + \sum_{i=1}^{n-1} a_{2i} r^{2i}, \tag{4.6}$$

which gives

$$R = A\left(f_1 r^2\right) \exp\left\{-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right\} + B\left(f_1 r^2\right) \exp\left(-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right)$$
$$\times \int \frac{1}{f_1 r^2} \exp\left(2\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right) \mathrm{d}r.$$

Hence the solution of (3.7) is

$$\psi = \left[A\left(f_1 r^2\right) \exp\left\{-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right\} + B(f_1 r^2) \exp\left(-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right)\right.$$
$$\left.\times \int \frac{1}{(f_1 r)^2 r} \exp\left(2\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right) \mathrm{d}r\right] \left\{a\mathrm{e}^{kz} + b\mathrm{e}^{-kz}\right\} \tag{4.7}$$

which is the second state solution.

Case (ii). Let $f = f_0 + f_2 r^2$, $p = n-2$, $b = 0$. Then using (4.4) $f'' - 2\alpha f' - (f'/r) = Af$ gives the recurrence relation

$$4f_2\alpha_{2i-1} + f_0 a_{2i} + f_2 a_{2i-2} = 0, \quad \text{for } i = 1, 2, \ldots, n-1. \tag{4.8}$$

Relation (4.8) determines a_{2i} uniquely, which gives $a_0 = 0$. For $\lambda = k^2$, we get the potential

$$X(r) = \left(\sum_{i=1}^{n+1} \alpha_{2i-1} r^{2i-1}\right)^2 + 2\sum_{i=2}^{n+1} (i-1)\alpha_{2i-1} r^{2i-2} + \sum_{i=1}^{n-2} a_{2i} r^{2i} + \frac{1}{r^2} \tag{4.9}$$

and

$$R = \left(f_0 + f_2 r^2\right)\left[r \exp\left(-\sum_{i=1}^{n+1}\frac{1}{2i}\alpha_{2i-1}r^{2i}\right)\right.$$
$$\left.\times\left\{A + B\int \frac{1}{\left(f_0 + f_1 r^2\right)^2 r}\exp\left(2\sum_{i=1}^{n+1}\frac{1}{2i}\alpha_{2i-1}r^{2i}\right)\mathrm{d}r\right\}\right]. \tag{4.10}$$

Hence the solution of (3.7) is

$$\psi = \left[A\left(f_0 + f_2 r^2\right) r \exp\left\{-\sum_{i=1}^{n+1}\frac{1}{2i}\alpha_{2i-1}r^{2i}\right\} + B\left(f_0 + f_2 r^2\right) r\right.$$
$$\left.\times \exp\left(-\sum_{i=1}^{n+1}\frac{1}{2i}\alpha_{2i-1}r^{2i}\right)\int \frac{1}{\left(f_0 + f_1 r^2\right)^2 r}\exp\left(2\sum_{i=1}^{n+1}\frac{1}{2i}\alpha_{2i-1}r^{2i}\right)\mathrm{d}r\right]$$
$$\times\left\{a\mathrm{e}^{kz} + b\mathrm{e}^{-kz}\right\}$$

which is a third state solution. Similarly we can extend for higher states for different values of G.

Case (β). For the spherical polar coordinate systems the radial part of the equation for higher state is assumed to be

$$(D-\alpha)(fD-\beta) \equiv f\left(D^2 + \frac{2}{r}D + \lambda - X - l(l+1)/r\right) \tag{4.11}$$

which gives

$$\beta = f' - \alpha f - \frac{2f}{r} \tag{4.12}$$

$$X - \lambda + l(l+1)/r^2 = \alpha^2 - \alpha' + 2\frac{\alpha}{r} + \frac{2}{r} + \frac{1}{f}\left(f'' - 2\alpha f' - 2\frac{f'}{r}\right). \tag{4.13}$$

Then the function R is given by

$$R = \frac{A}{r^2}f(r)\exp\left\{-\int^r \alpha(u)\mathrm{d}u\right\} + \frac{B}{r^2}f(r)\exp\left\{-\int^r \alpha(u)\mathrm{d}u\right\}$$
$$\times\int^r\left\{u^2 f^{-2}(u)\exp\int^u 2\alpha(v)\mathrm{d}v\right\}\mathrm{d}u. \tag{4.14}$$

Let us consider

$$\alpha = \sum_{i=1}^{n+1}\alpha_{2i-1}r^{2i-1} + \frac{m}{r}, \quad A = \sum_{i=0}^{p}a_{2i}r^{2i} + \frac{p}{r^2}.$$

Case (1). $f = \sum_{i=0}^{Q} f_{2i-1} r^{2i-1}$, for odd states. Then $f'' - 2\alpha f' - 2\frac{f'}{r} = Af$ gives the recurrence relation by equating the coefficients of r^{2i-1}, which are

$$2i(2i+1) f_{2i+1} - 2\sum_{j=0}^{j\leq i} (2i-2j-1)\alpha_{2j+1} f_{2i-2j-1}$$

$$= \sum_{j=0}^{j\leq i} a_{2j} f_{2i-2j-1} + p f_{2i+1}$$

which give the values of unknown a_{2i}, $i = 0, 1, 2, \dots$.

The potential, to within a constant multiplier, for $m = l$, $p = -2(l+1)$ and $\lambda = -a_0 - (2l+1)\alpha_1$, is

$$X = \left(\sum_{i=1}^{n+1} \alpha_{2i-1} r^{2i-1}\right)^2 + \sum_{i=2}^{n+1} (2l+3-2i)\,\alpha_{2i-1} r^{2i-2} + \sum_{i=1}^{n} a_{2i} r^{2i}$$

and the solution of $R(r)$ is

$$R = \left(\sum_{i=1}^{Q} f_{2i-1} r^{2i-1}\right) r^{l-2} \exp\left(-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right)$$

$$\times \left\{ A + B\int^{r} \left(\frac{u^{l+2}}{\left(\sum_{i=1}^{Q} f_{2i-1} u^{2i-1}\right)^2} \exp\left(2\sum_{j=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} u^{2i}\right)\right) \mathrm{d}u \right\}.$$

Case 2. Consider $f = \sum_{i=0}^{Q} f_{2i} r^{2i}$, for even stages. Then for $p = 0$, $f'' - 2\alpha f' - 2\frac{f'}{r} = Af$. This gives the recurrence relation by equating the coefficient of r^{2i} as

$$2(i+1)(2i+1) f_{2i+2} - 4\sum_{j=0}^{j\leq i} (i-j)\alpha_{2j+1} f_{2(i-j)} - 2(i+1) f_{2i+2}$$

$$= \sum_{j=1}^{j\leq i} a_{2j} f_{2(i-j)}$$

which gives the values of unknown a_{2i} uniquely for $i = 0, 1, 2, 3, \dots$.

The potential, to within a constant multiplier for $m = l-1$, and $\lambda = -a_0 - (2l-1)\,\alpha_1$ is

$$X = \left(\sum_{i=1}^{n+1} \alpha_{2i-1} r^{2i-1}\right)^2 + \sum_{i=2}^{n+1} (2l+1-2i)\,\alpha_{2i-1} r^{2i-2} + \sum_{i=1}^{n} a_{2i} r^{2i}$$

and the solution for $R(r)$ is

$$R = \left(\sum_{i=1}^{Q} f_{2i-1} r^{2i-1}\right) r^{l-3} \exp\left(-\sum_{i=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} r^{2i}\right)$$
$$\times \left\{A + B\int^{r} \left(\frac{u^{l+1}}{\left(\sum_{i=1}^{Q} f_{2i-1} u^{2i-1}\right)^2} \exp\left(2\sum_{j=1}^{n+1} \frac{1}{2i}\alpha_{2i-1} u^{2i}\right)\right) \mathrm{d}u\right\}.$$

Similarly, we can extend for higher states for different values of m.

5. Conclusion

In this article, we have discussed various types of new exact solutions of Schrödinger time-independent equations. To obtain exact solutions of different states we used factorization method for isotropic anharmonic oscillator. For different values of Q or G we get different states of exact solutions. Various theories have been proposed for the interaction of hadrons such as quantum chromodynamics, and certain modifications and approximations, e.g. the non-linear sigma model (Feynman [14], Islam[15–18]). When one considers non-relativistic limits of these and related theories, one arrives at different forms of non-linear Schrödinger equations. Some of the solutions considered in this article may be useful in these contexts. In general, this article (with possible extensions) can be considered as a 'Laboratory' for discussing solutions of various and diverse forms of the Schrödinger and other nonlinear equations in different classical and quantum mechanical contexts of physical interest. The present paper may also be useful in connection with the fairly extensive and interesting on quasi-exactly solvable models in quantum mechanics (Refs. 4–7 of Atre and Panigrahi [13]). There is a scope for examining precisely the extent to which quasi-exact solutions (as opposed to exact solution) are relevant and meaningful in various physical contexts. We hope to examine these and related questions (almost philosophical: see e.g. David Bohm, *Wholeness and the implicate order,* ARK Edition, 1983, London WCIE 7DD, England), in a future paper.

References

[1] H Thomas, *Structural phase transitions and soft modes* edited by E J Samuelsen, E Andersen and J Feder (Universitetforlaget, Oslo, 1971), p-15.
[2] V de Alfaro and T Regge, *Potential scattering* (North-Holland, Amsterdam, 1965)
[3] G P Flessas and K Das, *Phys. Lett*. **A78**, 19 (1980)
[4] G P Flessas, *J. Phys. A: Math. Gen.* **14**, L209 (1981)
[5] G P Flessas, *Phys. Lett.* **A81**, 17 (1981)
[6] P G L Leach, *J. Math. Phys.* **25**, 2974 (1984)
[7] A Khare, *Phys. Lett.* **A83**, 237 (1981)
[8] E Magyari, *Phys. Lett.* **A81**, 116 (1981)
[9] P M Morse and H Freshbach, *Methods of theoretical physics* (McGraw Hill Book Company, Inc., 1953) Vols I and II
[10] Z Yan, *J. Phys. A: Math. Gen.* **39**, L401 (2006)

[11] F Riesz and B Sz-Nagy, *Functional analysis* (Frederick Ungar Publishing Co., New York, 1978)
[12] Z Rahman, *Atmospheric and oceanic energetics and ocean-atmosphere geostrophic balance*, Ph.D. Thesis (University of Chittagong, Nov. 1993)
[13] Rajneesh Atre and Prasanta K Panigrahi, *Phys. Lett*. **A317**, 46 (2003)
[14] R P Feynman, *Nucl. Phys*. **B188**, 479 (1981)
[15] J N Islam, *Proc. R. Soc.* **A389**, 291 (1983)
[16] J N Islam, *Progr. Theor. Phys.* **89**, 161 (1993)
[17] J N Islam, *Found. Phys. (US)* **24**, 593 (1994)
[18] J N Islam, *Confinement and the Schrödinger equation for gauge theories*, Book to appear in 2011.