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Integrability of two coupled Kadomtsev–Petviashvili equations

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Abstract. The integrability of two coupled KP equations is studied. The simplified Hereman form of Hirota's bilinear method is used to examine the integrability of each coupled equation. Multiple-soliton solutions and multiple singular soliton solutions are formally derived for each coupled KdV equation.

Keywords. Coupled KP equation; Hereman's method; multiple-soliton solutions; multiple singular soliton solutions.

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1. Introduction

The celebrated Korteweg–de Vries (KdV) equation is considered as one of the most significant equation in the theory of integrable systems. It gives multiple-soliton solutions, an infinite number of conservation laws, bi-Hamiltonian structure, a Lax pair, and many other physical properties [1–27].

The celebrated KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

gives multiple-soliton solutions. The KdV equation models many nonlinear phenomena that appear in many scientific applications. The derivative u_t characterizes the time evolution of the wave propagating in one direction, the nonlinear term uu_x describes the steepening of the wave and the linear term u_{xxx} accounts for the spreading or dispersion of the wave.

The Kadomtsev–Petviashvili (KP) equation [7] extends the KdV equation, and is given by

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0. \tag{2}$$

This equation is also completely integrable and gives multiple-soliton solutions. The KP equation is used to model shallow-water waves with weakly nonlinear restoring forces. The

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KP equation is a model for shallow long waves in the x-direction with some mild dispersion in the y-direction. It is a natural generalization of the KdV equation and is completely integrable by the inverse scattering transform method. Kadomtsev and Petviashvili [7] generalized the KdV equation from (1+1) to (2+1) dimensions. They developed this equation when they relaxed the restriction that the waves be strictly one-dimensional of the KdV equation [7,20].

The coupled KdV and the coupled KP equations attracted considerable attraction because of their significance in many scientific applications [1–6]. Several distinct approaches have been employed to establish coupled KdV and coupled KP equations. For example, singularity analysis combined with the prolongation technique was used in [1] to establish a new coupled KdV equation. In [2], the 4×4 matrix spectral problem with three potentials was used to develop one hierarchy of coupled KdV equation. In [3–6], the algebraic-geometrical and Wronskian methods were used to handle the coupled KdV and the coupled KP equations.

Solitons arise as solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. Solitons result from the delicate balance between the nonlinearity effect of uu_x and the dispersion effect of u_{xxx}. Solitons, after a full interaction with others, re-emerge retaining their identities with the same speed and shape.

Many reliable methods are used in the literature to examine completely integrable coupled KdV equations. The Hirota bilinear method, the Bäcklund transformation method, the inverse scattering method, the Painlevé analysis, and others were effectively used in [1–27] and the references therein to determine multiple soliton solutions for completely integrable equations. Each method has its own significant properties. The Hirota's bilinear method is rather heuristic and possesses significant features that make it ideal for the determination of multiple-soliton solutions [14–24] for a wide class of nonlinear evolution equations.

In this work, we aim to study two new hierarchies of nonlinear evolution equations. An interesting equation in this hierarchy is a new coupled KP equation

$$\begin{aligned} &\left(u_t + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x\right)_x + u_{yy} = 0, \\ &\left(v_t + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x\right)_x + v_{yy} = 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &(u_t + u_{xxx} + 3uu_x + 3ww_x)_x + u_{yy} = 0, \\ &(v_t + v_{xxx} + 3vv_x + 3ww_x)_x + v_{yy} = 0, \\ &\left(w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x\right)_x + w_{yy} = 0. \end{aligned} \tag{4}$$

It is interesting that for $u_{yy} = v_{yy} = 0$, the resulting coupled KdV equations were introduced first in [3–6]. These coupled KdV equations were investigated in [3–6] using the Wronskian and the Lax pair approaches.

The aim of this work is two-fold. First, we aim to apply the simplified form of Hirota's bilinear method, developed by Hereman and Nuseir [14] to show that the two coupled KdV systems (3) and (4) are completely integrable. Our second aim is to determine multiple-soliton solutions and multiple singular soliton solutions for each coupled equation.

2. The simplified Hirota's method

In what follows, we only summarize the main steps of the simplified form of Hirota's method. More details can be found in [14]. We first substitute

$$u(x, y, t) = K\mathrm{e}^{kx+ry-ct}, \tag{5}$$

where K is the amplitude of the wave, into the linear terms of the evolution equation, to determine the dispersion relation between k, r and c. We then substitute the single-soliton solution

$$u(x, y, t) = R(\ln\ f(x, y, t))_{xx}, \tag{6}$$

where R is a constant, into the given equation, where the auxiliary function $f(x, y, t)$ is given by

$$f(x, y, t) = 1 + C_1 f_1(x, y, t) = 1 + C_1\mathrm{e}^{\theta_i}, \tag{7}$$

such that

$$\theta_i = k_i x + r_i y - c_i t, \quad i = 1, 2, \ldots, N, \tag{8}$$

and solve the resulting equation to determine the numerical value for R.

Notice that the N-soliton solutions can be obtained using the following forms for $f(x, y, t)$ into (6):
(i) For single-soliton solution, we use

$$f(x, y, t) = 1 + C_1\mathrm{e}^{\theta_1}. \tag{9}$$

(ii) For two-soliton solutions, we use

$$f(x, y, t) = 1 + C_1\mathrm{e}^{\theta_1} + C_2\mathrm{e}^{\theta_2} + C_1C_2a_{12}\mathrm{e}^{\theta_1+\theta_2}, \tag{10}$$

where a_{12} is the phase-shift that should be determined.
(iii) For three-soliton solutions, we use

$$\begin{aligned} f(x, y, t) = {} & 1 + C_1\mathrm{e}^{\theta_1} + C_2\mathrm{e}^{\theta_2} + C_3\mathrm{e}^{\theta_3} + C_1C_2a_{12}\mathrm{e}^{\theta_1+\theta_2} + C_2C_3a_{23}\mathrm{e}^{\theta_2+\theta_3} \\ & + C_1C_3a_{13}\mathrm{e}^{\theta_1+\theta_3} + C_1C_2C_3b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{11}$$

The determination of three-soliton solutions confirms that N-soliton solutions exist for any finite order $N \geq 1$. Multiple-soliton solutions are obtained for $C_1 = C_2 = C_3 = +1$. However, multiple singular soliton solutions are obtained if $C_1 = C_2 = C_3 = -1$.

It is interesting to point out that the determination of three-soliton solutions leads to the determination of multiple-soliton solutions. This in fact confirms the complete integrability of the model under discussion. This has been confirmed by Hereman and Nuseir [14].

3. The first coupled KP equation

We first consider the coupled KP equation

$$\begin{aligned} &\left(u_t + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x\right)_x + u_{yy} = 0, \\ &\left(v_t + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x\right)_x + v_{yy} = 0. \end{aligned} \tag{12}$$

To determine the regular soliton solutions, we use $C_1 = C_2 = C_3 = 1$. Substituting

$$u(x, y, t) = K\mathrm{e}^{k_i x + r_i y - c_i t},$$
$$v(x, y, t) = A\mathrm{e}^{k_i x + r_i y - c_i t}, \tag{13}$$

where A and K are constants, into the linear terms of (12), gives the dispersion relation by

$$c_i = \frac{k_i^4 + r_i^2}{k_i}. \tag{14}$$

This means that

$$\theta_i = k_i x + k_i y - \frac{k_i^4 + r_i^2}{k_i} t. \tag{15}$$

The multi-soliton solutions of (12) is assumed to be

$$u(x, y, t) = R_1 (\ln f)_{xx} = R_1 \frac{f f_{xx} - f_x^2}{f^2},$$
$$v(x, y, t) = R_2 (\ln f)_{xx} = R_2 \frac{f f_{xx} - f_x^2}{f^2}, \tag{16}$$

where R_1 and R_2 are constants that will be determined. The auxiliary function $f(x, y, t)$ for the single-soliton solution is given by

$$f(x, y, t) = 1 + \mathrm{e}^{\theta_1} = 1 + \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}, \quad C_1 = 1. \tag{17}$$

Substituting (17) in (12), and solving for R_1 and R_2 we find a set of possible values given by

$$R_1 = -16, -48, -64,$$
$$R_2 = -32, -48, -80. \tag{18}$$

Substituting (17) in (16) gives the following set of single-soliton solutions:

$$u(x, y, t) = -\frac{16 k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 + \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2},$$
$$v(x, y, t) = -\frac{32 k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 + \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}, \tag{19}$$

$$u(x, y, t) = -\frac{48 k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 + \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2},$$
$$v(x, y, t) = -\frac{48 k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 + \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}, \tag{20}$$

and

$$u(x,y,t) = -\frac{64k_1^2\,\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}}{\left(1+\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2},$$

$$v(x,y,t) = -\frac{80k_1^2\,\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}}{\left(1+\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2}. \tag{21}$$

For the two-soliton solutions we set

$$\begin{aligned} f(x,y,t) =& 1+\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2}+a_{12}\mathrm{e}^{\theta_1+\theta_2},\\ =& 1+\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}+\mathrm{e}^{k_2x+k_2y-\frac{k_2^4+r_2^2}{k_2}t}\\ &+\ a_{12}\mathrm{e}^{(k_1+k_2)x+(r_1+r_2)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}\right)t}. \end{aligned} \tag{22}$$

Using (22) in (12), we obtain the phase-shift as

$$a_{12} = \frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}. \tag{23}$$

and hence we set

$$a_{ij} = \frac{3k_i^2k_j^2(k_i-k_j)^2-(k_ir_j-k_jr_i)^2}{3k_i^2k_j^2(k_i+k_j)^2-(k_ir_j-k_jr_i)^2}, \quad 1\le i<j\le 3. \tag{24}$$

Notice that the phase-shifts depend on both coefficients of the spatial variables k_i and r_i. The second set of two-soliton solutions is obtained by substituting (23) and (22) into (16), where R_1 and R_2 are defined in (18).

For the three-soliton solutions, we set

$$\begin{aligned} f(x,y,t) = 1 &+ \mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}+\mathrm{e}^{k_2x+k_2y-\frac{k_2^4+r_2^2}{k_2}t}+\mathrm{e}^{k_3x+k_3y-\frac{k_3^4+r_3^2}{k_3}t}\\ &+\frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}\mathrm{e}^{(k_1+k_2)x+(r_1+r_2)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}\right)t}\\ &+\frac{3k_1^2k_3^2(k_1-k_3)^2-(k_1r_3-k_3r_1)^2}{3k_1^2k_3^2(k_1+k_3)^2-(k_1r_3-k_3r_1)^2}\mathrm{e}^{(k_1+k_3)x+(r_1+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_3^4+r_3^2}{k_3}\right)t}\\ &+\frac{3k_2^2k_3^2(k_2-k_3)^2-(k_2r_3-k_3r_2)^2}{3k_2^2k_3^2(k_2+k_3)^2-(k_2r_3-k_3r_2)^2}\mathrm{e}^{(k_2+k_3)x+(r_2+r_3)y-\left(\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t}\\ &+b_{123}\mathrm{e}^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t}. \end{aligned} \tag{25}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{26}$$

This shows that three sets of three-soliton solutions can be obtained. The three-soliton solutions are obtained by substituting (25) into (16). This shows that the coupled KP equation (12) is completely integrable and N-soliton solutions can be determined for $u(x, y, t)$ and $v(x, y, t)$, for finite N, where $N \geq 1$.

3.1 *Multiple singular soliton solutions*

As stated before, the singular soliton solutions can be obtained by setting $C_1 = C_2 = C_3 = -1$. For singular soliton solutions we find the following sets of single singular soliton solutions:

$$\begin{aligned} u(x, y, t) &= \frac{16k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 - \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}, \\ v(x, y, t) &= \frac{32k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 - \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}, \end{aligned} \tag{27}$$

$$\begin{aligned} u(x, y, t) &= \frac{48k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 - \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}, \\ v(x, y, t) &= \frac{48k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 - \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} u(x, y, t) &= \frac{64k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 - \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}, \\ v(x, y, t) &= \frac{80k_1^2\, \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}}{\left(1 - \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t}\right)^2}. \end{aligned} \tag{29}$$

For the two singular soliton solutions we substitute

$$\begin{aligned} f(x, y, t) = {} & 1 - \mathrm{e}^{k_1 x + k_1 y - \frac{k_1^4 + r_1^2}{k_1} t} - \mathrm{e}^{k_2 x + k_2 y - \frac{k_2^4 + r_2^2}{k_2} t} \\ & + a_{12} \mathrm{e}^{(k_1 + k_2) x + (r_1 + r_2) y - \left(\frac{k_1^4 + r_1^2}{k_1} + \frac{k_2^4 + r_2^2}{k_2}\right) t}, \end{aligned} \tag{30}$$

in (16) to find that the phase-shift a_{12} is the same as obtained in (23).

For the three singular soliton solutions, we set

$$\begin{aligned} f(x,y,t) = 1 &- \mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t} - \mathrm{e}^{k_2x+k_2y-\frac{k_2^4+r_2^2}{k_2}t} - \mathrm{e}^{k_3x+k_3y-\frac{k_3^4+r_3^2}{k_3}t} \\ &+ \frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}\mathrm{e}^{(k_1+k_2)x+(r_1+r_2)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}\right)t} \\ &+ \frac{3k_1^2k_3^2(k_1-k_3)^2-(k_1r_3-k_3r_1)^2}{3k_1^2k_3^2(k_1+k_3)^2-(k_1r_3-k_3r_1)^2}\mathrm{e}^{(k_1+k_3)x+(r_1+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_3^4+r_3^2}{k_3}\right)t} \\ &+ \frac{3k_2^2k_3^2(k_2-k_3)^2-(k_2r_3-k_3r_2)^2}{3k_2^2k_3^2(k_2+k_3)^2-(k_2r_3-k_3r_2)^2}\mathrm{e}^{(k_2+k_3)x+(r_2+r_3)y-\left(\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t} \\ &- b_{123}\mathrm{e}^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t}. \end{aligned} \tag{31}$$

The three singular soliton solutions are obtained by substituting the last equation into (16) where $b_{123} = a_{12}a_{13}a_{23}$. This shows that the system of eqs (12) gives N singular soliton solutions for $u(x, y, t)$ and $v(x, y, t)$, for finite N, where $N \geq 1$.

4. The second coupled KP equation

We next consider the coupled KP equation

$$\begin{aligned} &(u_t + u_{xxx} + 3uu_x + 3ww_x)_x + u_{yy} = 0, \\ &(v_t + v_{xxx} + 3vv_x + 3ww_x)_x + v_{yy} = 0, \\ &\left(w_t + w_{xxx} + \frac{3}{2}(uw)_x + \frac{3}{2}(vw)_x\right)_x + w_{yy} = 0. \end{aligned} \tag{32}$$

We proceed as before and set $C_1 = C_2 = C_3 = 1$. Substituting

$$u(x, y, t) = \mathrm{e}^{k_ix+r_iy-c_it}, \tag{33}$$

into the linear terms of (32) gives the dispersion relation as

$$c_i = \frac{k_i^4 + r_i^2}{k_i}, \tag{34}$$

and as a result we obtain

$$\theta_i = k_ix + k_iy - \frac{k_i^4 + r_i^2}{k_i}t. \tag{35}$$

The multi-soliton solutions of (32) is assumed to be

$$\begin{aligned} u(x, y, t) &= R_1(\ln f)_{xx} = R_1\frac{ff_{xx} - f_x^2}{f^2}, \\ v(x, y, t) &= R_2u, \\ w(x, y, t) &= R_3u, \end{aligned} \tag{36}$$

where R_1, R_2 and R_3 are constants that will be determined. The auxiliary function $f(x, y, t)$ for the single-soliton solution is given by

$$f(x, y, t) = 1 + \mathrm{e}^{\theta_1} = 1 + \mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}. \tag{37}$$

Substituting (37) in (32), and solving for R_1, R_2 and R_3 we find

$$\begin{aligned} R_1 &= \frac{4}{1+\alpha^2}, \\ R_2 &= \alpha^2, \\ R_3 &= \alpha, \end{aligned} \tag{38}$$

where α is any real constant.

Substituting (37) in (36) gives the single soliton solutions

$$\begin{aligned} u(x,y,t) &= \frac{4k_1^2\,\mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}}{(1+\alpha^2)\left(1+\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2}, \\ v(x,y,t) &= \frac{4\alpha^2k_1^2\,\mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}}{(1+\alpha^2)\left(1+\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2}, \\ w(x,y,t) &= \frac{4\alpha k_1^2\,\mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}}{(1+\alpha^2)\left(1+\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2}. \end{aligned} \tag{39}$$

For the two-soliton solutions we set

$$f(x,y,t) = 1+\mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}+\mathrm{e}^{k_2x+k_2y-\frac{k_1^4+r_1^2}{k_1}t}+a_{12}\mathrm{e}^{(k_1+k_2)x-(k_1^3+k_2^3)t}. \tag{40}$$

Using (40) in (32), we obtain the phase-shift as

$$a_{12} = \frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}, \tag{41}$$

and hence we set

$$a_{ij} = \frac{3k_i^2k_j^2(k_i-k_j)^2-(k_ir_j-k_jr_i)^2}{3k_i^2k_j^2(k_i+k_j)^2-(k_ir_j-k_jr_i)^2}, \quad 1\le i<j\le 3. \tag{42}$$

The two-soliton solutions are obtained by substituting (40) and (41) into (36), where R_1, R_2, and R_3 are defined in (38).

For the three-soliton solutions, we set

$$\begin{aligned} f(x,y,t) = {} & 1+\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}+\mathrm{e}^{k_2x+k_2y-\frac{k_2^4+r_2^2}{k_2}t}+\mathrm{e}^{k_3x+k_3y-\frac{k_3^4+r_3^2}{k_3}t} \\ & +\frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}\mathrm{e}^{(k_1+k_2)x+(r_1+r_2)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}\right)t} \\ & +\frac{3k_1^2k_3^2(k_1-k_3)^2-(k_1r_3-k_3r_1)^2}{3k_1^2k_3^2(k_1+k_3)^2-(k_1r_3-k_3r_1)^2}\mathrm{e}^{(k_1+k_3)x+(r_1+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_3^4+r_3^2}{k_3}\right)t} \\ & +\frac{3k_2^2k_3^2(k_2-k_3)^2-(k_2r_3-k_3r_2)^2}{3k_2^2k_3^2(k_2+k_3)^2-(k_2r_3-k_3r_2)^2}\mathrm{e}^{(k_2+k_3)x+(r_2+r_3)y-\left(\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t} \\ & +b_{123}\mathrm{e}^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t}. \end{aligned} \tag{43}$$

Proceeding as before, we find

$$b_{123} = a_{12}a_{23}a_{13}. \tag{44}$$

The three-soliton solutions are obtained by substituting (43) into (36). This shows that the system of eqs (32) is completely integrable and N-soliton solutions can be determined for $u(x, y, t)$, $v(x, y, t)$, and $w(x, y, t)$, for finite N, where $N \geq 1$.

4.1 *Multiple singular soliton solutions*

As stated before, the singular soliton solutions can be obtained by setting $C_1 = C_2 = C_3 = -1$. For single singular soliton solutions we find

$$\begin{aligned}
u(x, y, t) &= -\frac{4k_1^2\,\mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}}{(1+\alpha^2)\left(1-\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2},\\
v(x, y, t) &= -\frac{4\alpha^2k_1^2\,\mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}}{(1+\alpha^2)\left(1-\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2},\\
w(x, y, t) &= -\frac{4\alpha k_1^2\,\mathrm{e}^{k_1x+r_1y-\frac{k_1^4+r_1^2}{k_1}t}}{(1+\alpha^2)\left(1-\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}\right)^2}.
\end{aligned} \tag{45}$$

For the two singular soliton solutions we substitute

$$\begin{aligned}
f(x,y,t) &= 1-\mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t}-\mathrm{e}^{k_2x+k_2y-\frac{k_2^4+r_2^2}{k_2}t}\\
&\quad + a_{12}\mathrm{e}^{(k_1+k_2)x+(r_1+r_2)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}t\right)},
\end{aligned} \tag{46}$$

in (36) to find that the phase-shift a_{12} is the same as obtained above.

For the three singular soliton solutions, we set

$$\begin{aligned}
f(x, y, t) = 1 &- \mathrm{e}^{k_1x+k_1y-\frac{k_1^4+r_1^2}{k_1}t} - \mathrm{e}^{k_2x+k_2y-\frac{k_2^4+r_2^2}{k_2}t} - \mathrm{e}^{k_3x+k_3y-\frac{k_3^4+r_3^2}{k_3}t}\\
&+\frac{3k_1^2k_2^2(k_1-k_2)^2-(k_1r_2-k_2r_1)^2}{3k_1^2k_2^2(k_1+k_2)^2-(k_1r_2-k_2r_1)^2}\mathrm{e}^{(k_1+k_2)x+(r_1+r_2)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}\right)t}\\
&+\frac{3k_1^2k_3^2(k_1-k_3)^2-(k_1r_3-k_3r_1)^2}{3k_1^2k_3^2(k_1+k_3)^2-(k_1r_3-k_3r_1)^2}\mathrm{e}^{(k_1+k_3)x+(r_1+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_3^4+r_3^2}{k_3}\right)t}\\
&+\frac{3k_2^2k_3^2(k_2-k_3)^2-(k_2r_3-k_3r_2)^2}{3k_2^2k_3^2(k_2+k_3)^2-(k_2r_3-k_3r_2)^2}\mathrm{e}^{(k_2+k_3)x+(r_2+r_3)y-\left(\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t}\\
&- b_{123}\mathrm{e}^{(k_1+k_2+k_3)x+(r_1+r_2+r_3)y-\left(\frac{k_1^4+r_1^2}{k_1}+\frac{k_2^4+r_2^2}{k_2}+\frac{k_3^4+r_3^2}{k_3}\right)t}.
\end{aligned} \tag{47}$$

The three singular soliton solutions are obtained by substituting the last equation into (36). This shows that the system of eqs (32) gives N singular soliton solutions for $u(x, y, t)$, $v(x, y, t)$, and $w(x, y, t)$, for finite N, where $N \geq 1$.

5. Discussion

The simplified form of the Hirota's bilinear method is used to study the integrability of two coupled KP equations. Multiple-soliton solutions and multiple singular soliton solutions are formally derived for each system. The analysis confirms that certain equations which have N-soliton solutions, have simultaneously, N-singular soliton solutions. The properties of solitons and interaction of solitons can be found in the literature, and more details can be found in [8–14]. The graphs for single-soliton solution, two-soliton solutions and three-soliton solutions can be found in the literature. More details and graphs can be found in [20].

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