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Statistical mechanics of gravitating systems ... *and some curious history of Chandra's rare misses!*

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Abstract. Chandra's academic life had several phases each culminating in a monograph describing that subject. I shall deal with aspects of his work in the two earliest phases. I shall describe the overall structure of statistical mechanics of gravitating systems, the relevance of isothermal sphere in the mean-field approximation and issues related to collisional relaxation and dynamical friction in self-gravitating system of particles. There are several curious features in the history of these topics which I comment upon.

Keywords. Antonov instability; isothermal sphere; dynamical friction; negative specific heat.

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1. Introduction

Chandra's academic career can be divided into six different phases (which led a colleague of mine to come up once with a horrible pun 'Six phases of Subrahmanya'). The first two of these involved the study of stellar structure and principles of stellar dynamics leading to the first two monographs [1,2] in the years 1939 and 1942. The investigations during this period also led Chandrasekhar to dabble quite a bit with the issues which arise when principles of statistical mechanics are applied to the study of self-gravitating systems. In this talk, I shall concentrate on several aspects of this problem and shall try to present Chandrasekhar's contribution in the proper historical context of the work done by several others. Unlike the usual picture we have of Chandrasekhar as a perfectionist one notices that there were some curious rare misses of fundamental significance in his study of these topics.

2. Statistical mechanics of gravitating systems

2.1 *Basics of statistical mechanics*

In the study of laboratory systems involving short-range interaction between constituent particles, a central quantity which we use is the entropy functional $S(E, V)$ that gives the entropy of the system as a function of its energy and its volume. This, in turn, is related to the density of states of the system $g(E)$ by $S(E) = \ln g(E)$ with

$$g(E) = \frac{\mathrm{d}\Gamma(E)}{\mathrm{d}E}; \qquad \Gamma(E) = \int \mathrm{d}p\ \mathrm{d}q \theta_D[E - H(p, q)]. \tag{1}$$

(We shall suppress exhibiting the explicit dependence on the volume V when it is not relevant.) In this microcanonical description of the system, the temperature and the pressure can be obtained by

$$T(E) = \left(\frac{\partial S}{\partial E}\right)^{-1}; \qquad P = T\left(\frac{\partial S}{\partial V}\right) \tag{2}$$

which shows that the relation between temperature and energy can be determined once we know the Hamiltonian $H(p, q)$ of the system. For example, an ideal gas of N particles with $H \propto \sum p_i^2$ will lead to the familiar relations

$$\Gamma \sim V^N E^{3N/2} \sim g(E); \qquad T(E) = (2E/3N); \qquad P/T = N/V. \tag{3}$$

Very often one uses the equivalent canonical description based on the partition function $Z(T)$ given by the Laplace transform of the density of states

$$Z(T) = \int \mathrm{d}E g(E) \exp[-\beta E] = \int \mathrm{d}p\ \mathrm{d}q \exp[-\beta H(p, q)], \tag{4}$$

where $\beta = 1/T$. In this case, one determines the (mean) energy and pressure by the relations

$$\bar{E} = -(\partial \ln Z/\partial\beta); \qquad \bar{P} = T(\partial \ln Z/\partial V). \tag{5}$$

For systems which obey extensivity of energy (viz., when total energy of the system is the sum of its parts to a high degree of accuracy) the canonical and microcanonical descriptions will lead to the same physical quantities to the accuracy $\mathcal{O}(\ln N/N)$ where N is the number of degrees of freedom of the system.

2.2 *Phases of self-gravitating systems*

This equivalence between canonical and microcanonical descriptions fails for systems with gravitational interaction mainly because energy is not an extensive parameter for such systems (see e.g. [3]). If a large gravitating system is divided into two parts, the total energy cannot be expressed as the sum of the energies of the two parts; the gravitational interaction energy makes a significant contribution to the total energy due to the long-range nature of gravity. Hence the fundamental description of gravitating systems has to be based on microcanonical ensemble and any use of canonical ensemble (in some rare occasions) needs to be justified by specific physical considerations. This inequivalence of the two ensembles should also be obvious from the fact that systems in canonical ensemble cannot exhibit negative specific heat while self-gravitating systems often do.

To provide a microcanonical description of the gravitating system, we need to evaluate the density of states in eq. (1). This integral will diverge in the absence of two relevant cut-offs. First is the cut-off at large distances which is required to confine high-energy particles from moving to large distances. This, of course, is not special to self-gravitating systems; even an ideal gas of particles will have a divergent density of states if it is not confined by a box of volume V. The second cut-off is at short distances to prevent particles from approaching each other arbitrarily closely thereby releasing large amount of gravitational potential energy, $-Gm^2/r$, as $r \to 0$. Once again, such a situation arises even in the case of plasmas in which quantum mechanical considerations will provide an effective short-distance cut-off. For gravitating systems relevant to astrophysics, there is usually some other physical process, arising from the finite size of the self-gravitating objects, which will provide this cut-off.

Given a large-distance cut-off R and short-distance cut-off a one can, in principle, compute the density of states and the thermodynamic behaviour of such a system. The two cut-offs define two natural energy scales $E_1 = -Gm^2/a$ and $E_2 = -Gm^2/R$ with $a \ll R$. On the other hand, the application of virial theorem to such a system will lead to a relation of the form

$$2K + U = 3PV + U_0, \tag{6}$$

where K is the kinetic energy of the particles, U is the gravitational potential energy, P is the pressure exerted by the particles on the confining volume and U_0 is the correction to the virial due to the short-distance cut-off. Broadly speaking, the different phases of the gravitating systems can be related [3] to the different ways in which this condition is satisfied:

(a) When the energy of the system is such that $E \gg E_2$, gravity is irrelevant and the system behaves like a gas confined by a container. In this high-temperature phase with positive specific heat, eq. (6) is satisfied with $2K \approx 3PV$ and the other two terms are sub-dominant.

(b) When $E_1 \ll E \ll E_2$, the system is unaffected either by the confining box or the short-distance cut-off. In this phase with negative specific heat, it is dominated entirely by gravity and eq. (6) is satisfied by $2K + U \approx 0$ with the other two terms being sub-dominant. Since canonical ensemble cannot lead to negative specific heat, the description in canonical and microcanonical ensembles differ drastically in this regime. In canonical ensemble, the negative specific heat region is replaced by a rapid phase transition.

(c) As we approach lower energies ($E \to E_1$) the hard core nature of the particles begins to be felt and the gravity is resisted by the other physical processes. This will lead to a low-temperature hard core condensate in which eq. (6) is satisfied by $U \approx U_0$ with the other two terms being sub-dominant.

The existence of negative specific heat phase is characteristic of the inherent instability of self-gravitating system. As the system evolves, it has a tendency to form a centrally condensed core with $U \approx U_0$ releasing large amount of energy which puts the remaining part of the system into a high temperature phase that will exist as a halo around the core. The particles in this halo will be bouncing off the walls of the container in the form of a high temperature gas with a cold core existing as a centrally condensed body.

3. Isothermal sphere, Antonov instability and Chandrasekhar

The self-gravitating phase of the system in the description given above allows a mean-field description under certain circumstances. In this context, one essentially maximizes the mean-field entropy (given by the integral over the phase space of $-f \ln f$ where $f(\mathbf{x}, \mathbf{v}, t)$ is the distribution function) subject to the constancy of total energy and total mass. This leads to the (intuitively obvious) equations

$$\rho(\mathbf{x}) = A \exp(-\beta\phi(\mathbf{x})); \qquad \phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{y})\mathrm{d}^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \tag{7}$$

where ρ is the mass density and ϕ is the gravitational potential. The constants A and β are to be determined in terms of the total mass M and total energy E of the system, which is assumed to be confined in a spherical box of radius R. Equation (7) can be re-written in the form

$$\nabla^2\phi = 4\pi G\rho_c \mathrm{e}^{-\beta[\phi(\mathbf{x})-\phi(0)]}; \qquad \rho_c = \rho(0) \tag{8}$$

which is usually called the equation for the isothermal sphere. As shown by Antonov [4] the entropy extremization problem, solved formally by this equation, actually turns out to be fairly subtle.

It turns out that the behaviour of the system depends crucially on two parameters, one of which can be taken to be $\lambda \equiv RE/GM^2$ while the other is $\rho(0)/\rho(R)$. The behaviour of the system can be summarized [4,5] as follows:

1. Systems with $(RE/GM^2) < -0.335$ cannot evolve into isothermal spheres. Entropy has no extremum for such systems.
2. Systems with $((RE/GM^2) > -0.335)$ and $(\rho(0) > 709\,\rho(R))$ can exist in a meta-stable (saddle point state) isothermal sphere configuration. The entropy extrema exist but they are not local maxima.
3. Systems with $((RE/GM^2) > -0.335)$ and $(\rho(0) < 709\,\rho(R))$ can form isothermal spheres which are local maximum of entropy.

These results were obtained by Antonov [4] by a careful examination of the second variation of the entropy functional in order to distinguish between maxima, minima and saddle point and a simpler derivation of this result is provided in ref. [5]. However, it is possible to understand at least the first of the results from a more elementary considerations which I shall now describe.

Introducing a length, mass and energy scale through $L_0 \equiv (4\pi G\rho_c\beta)^{1/2}$, $M_0 = 4\pi\rho_c L_0^3$, $\phi_0 \equiv \beta^{-1} = GM_0/L_0$ and using the dimensionless variables

$$x \equiv \frac{r}{L_0}, \qquad n \equiv \frac{\rho}{\rho_c}, \qquad m = \frac{M(r)}{M_0}, \qquad y \equiv \beta\,[\phi - \phi(0)] \tag{9}$$

we can reduce the isothermal sphere equation and its boundary condition to

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \mathrm{e}^{-y}; \qquad y(0) = y'(0) = 0. \tag{10}$$

This equation belongs to a class of equations which arise in the study of stellar structure and polytropes and has been analysed extensively by Emden in his book [6]. Chandrasekhar, in his book on stellar structure [1], models the relevant sections exactly as in Emden's and describes the nature of solutions to this equation in a manner identical to Emden. In particular, by choosing the variables $v \equiv m/x$; $u \equiv nx^3/m = nx^2/v$ one can reduce the second-order eq. (10) (and the boundary condition) to the first-order form:

$$\frac{u}{v}\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{(u-1)}{(u+v-3)}; \qquad v(u=3)=0; \qquad v'(u=3) = -\frac{5}{3}. \tag{11}$$

This first-order equation has a singular point where the numerator and denominator of the right-hand side of the above equation simultaneously vanish. This occurs at $u_s = 1$, $v_s = 2$ corresponding to the solution $n = 2/x^2$, $m = 2x$. This is the asymptotic behaviour of the density and mass of an isothermal sphere and all other solutions approach this one in an oscillatory manner. (The solution to eq. (11) is plotted in figure 1 as a spiraling curve.) Both Emden and Chandrasekhar give the spiralling solution curves in the u–v plane on which all isothermal spheres must lie. Chandrasekhar also describes several other properties of the solution basing his discussion almost entirely on Emden's work.

However, somewhat surprisingly, Chandrasekhar does not consider the nature of total energy for the isothermal solutions. It is fairly straightforward to show that $\lambda \equiv (RE/GM^2)$ can be expressed purely as the function of the values of the variables u and v at the outer boundary $r = R$. It turns out that, for a given value of λ, the isothermal sphere must lie [3] on the curve

$$v = \frac{1}{\lambda}\left(u - \frac{3}{2}\right); \qquad \lambda \equiv \frac{RE}{GM^2} \tag{12}$$

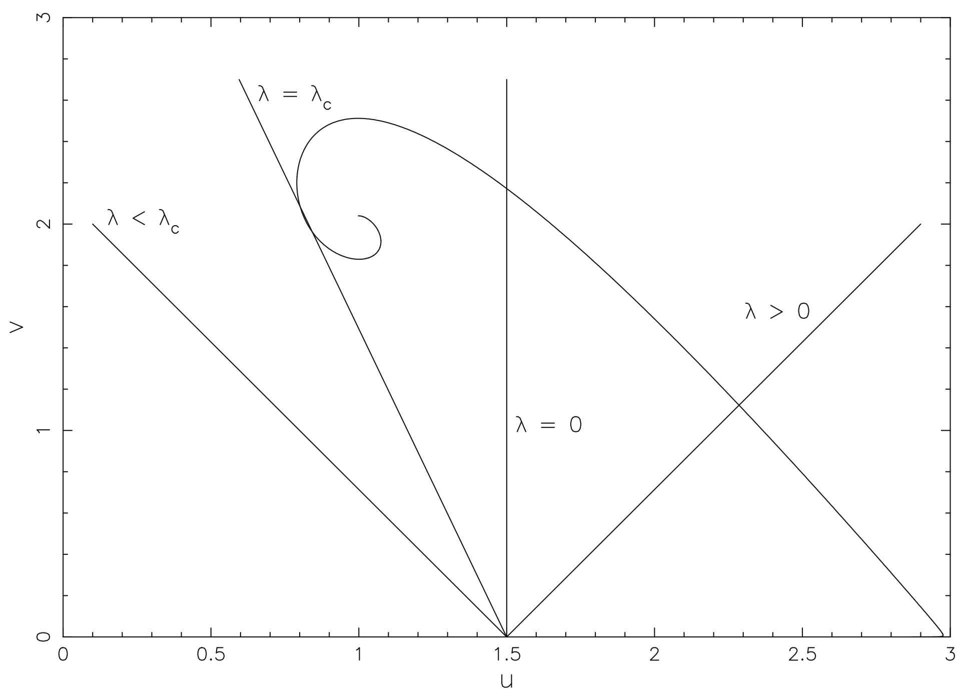

Figure 1. Bound on RE/GM^2 for the isothermal sphere.

which is a straight line through $(1.5, 0)$ with the slope λ^{-1}. It is clear from figure 1 that when λ is less than a critical value $\lambda_c \approx -0.335$, these curves do not intersect and the isothermal sphere cannot exist. The entropy has no maximum when $\lambda < \lambda_c$ which is one of the key results found by Antonov.

It is surprising that Chandrasekhar, who has worked out the isothermal sphere in uv coordinates as early as 1939, missed discovering the energy bound shown in figure 1. Chandrasekhar [1] has the uv curve but does not over-plot lines of constant λ. If he had done that, he might have discovered Antonov instability [4] decades before Antonov did! I suspect this is partially due to Chandrasekhar relying extensively on the earlier work of Emden in his study of isothermal sphere rather than attempting to look at this problem afresh from the point of view of statistical mechanics and entropy maximization.

4. Collisional relaxation and dynamical friction in gravitating systems

The second key topic I want to address is Chandrasekhar's analysis of collisional relaxation and dynamical friction in a system of self-gravitating particles, e.g. stars in a globular cluster. The basic idea behind the collisional relaxation can be understood by considering two kinds of collisions usually called 'hard' and 'soft' collisions. When two particles of mass m scatter off each other gravitationally with an impact parameter b and velocity v, the typical transverse velocity induced by such a collision is about $\Delta v_\perp \simeq \left(Gm/b^2\right)(2b/v) = (2Gm/bv)$. The hard collisions are those with impact parameter $b \lesssim b_c = Gm/v^2$ so that $\Delta v_\perp \approx v$. The relevant relaxation time-scale for hard collisions is given by

$$t_{\text{hard}} \simeq \frac{1}{(n\sigma v)} \simeq \frac{R^3 v^3}{N\left(G^2 m^2\right)} \simeq \frac{N R^3 v^3}{G^2 M^2} \approx N(R/v), \tag{13}$$

where n is the number density of stars and $\sigma \approx \pi b_c^2$ is the cross-section for hard collisions. This is typically N times the dynamical time-scale of the system when approximate virial condition holds.

What is more interesting is the effect of soft collisions which operates at a time-scale shorter by a factor $\ln N$. The soft collisions make a particle diffuse in the velocity space with $(\Delta v_\perp)^2$ of the individual collisions adding up linearly with time. In a time interval Δt, this process will lead to an increase in the mean-square velocity given by

$$\langle(\delta v_\perp)^2\rangle_{\text{total}} \simeq \Delta t \int_{b_1}^{b_2} (2\pi b\ \mathrm{d}b)\,(vn) \left(\frac{G^2 m^2}{b^2 v^2}\right) = \frac{2\pi n G^2 m^2}{v} \Delta t \ln\left(\frac{b_2}{b_1}\right). \tag{14}$$

It seems reasonable to take $b_2 \approx R, b_1 \approx b_c$. Then it is easy to show that, if the system is in approximate virial equilibrium, soft collisions significantly alter the velocity dispersion in a time-scale given by

$$t_{\text{soft}} \simeq \frac{v^3}{2\pi G^2 m^2 n \ln N} \simeq \left(\frac{N}{\ln N}\right)\left(\frac{R}{v}\right) \simeq \left(\frac{t_{\text{hard}}}{\ln N}\right). \tag{15}$$

One of the key results in Chandrasekhar's book [2] is the derivation of this collisional relaxation time. He essentially obtains the result in eq. (15) after devoting about 25 pages (from pages 48 to 73) for the algebraic derivation which includes a 'three-dimensional' picture! For comparison, the same result has been obtained earlier by James Jeans [7] in 1929 using about 3 pages (pages 317 to 320) in his book *Astronomy and Cosmogony*. The result was doubtless known to many others and – in fact – the explicit use of $\ln N$ in the time-scale for soft collisions exists in a 1938 paper of Ambartsumian [8].

Chandrasekhar defends his elaborate calculation of this previously known result by saying: "Though the physical ideas were correctly formulated by Jeans .... a completely rigorous evaluation of the time of relaxation was not available until recently". Chandrasekhar does not seem to have been bothered by the fact that any estimation of time of relaxation will necessarily be uncertain by factors of order unity both because of the variation of density – Chandrasekhar assumes a constant density star cluster – and by the uncertainties in the upper and lower cut-offs inside the logarithm. I believe this is typical of Chandrasekhar's attitude of rigour for rigour's sake!

There is, however, a more interesting twist to this tale which illustrates one of the rare occasions in which Chandrasekhar completely missed a key physical effect.

Some amount of thought will convince one that the diffusion in the velocity space is a run-away process and – left to itself – it will continuously drive larger proportion of stars to higher and higher velocities, which is clearly unphysical. There should exist a frictional, damping, mechanism which will act against this diffusion so that, when both processes act together, the system will be driven to an equilibrium distribution in velocity space. It seems that Chandrasekhar realized this soon after – but only after – the publication of his book in stellar dynamics. He addresses this issue and obtains the expression for dynamical friction as a separate physical phenomenon in his works published shortly afterwards [9].

Given the fact that the dynamical friction and diffusion in velocity space are just two aspects of the same physical phenomena, it is not surprising that an elegant and unified derivation of these effects exists. I shall now briefly summarize this derivation.

We can describe the diffusion in velocity space, that obeys standard conservation laws, through a source term which is a divergence in momentum space. Hence the evolution of distribution function will be governed by an equation of the form

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \nabla\phi\cdot\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial J^{\alpha}}{\partial p^{\alpha}}. \tag{16}$$

The form of the current J_α can be determined by considering the elementary collisional process and one obtains the result

$$J_\alpha(\mathbf{l}) = \frac{B_0}{2}\int \mathrm{d}\mathbf{l}'\left\{f\frac{\partial f'}{\partial l_\beta} - f'\frac{\partial f}{\partial l_\beta}\right\}\cdot\left\{\frac{\delta_{\alpha\beta}}{k} - \frac{k_\alpha k_\beta}{k^3}\right\}; \quad \mathbf{k} = \mathbf{l} - \mathbf{l}', \tag{17}$$

where

$$B_0 = 4\pi G^2 m^5 L; \qquad L = \int_{b_1}^{b_2}\frac{\mathrm{d}b}{b} = \ln\left(\frac{b_2}{b_1}\right). \tag{18}$$

In this current, the term proportional to f leads to dynamical friction while the term proportional to $\partial f/\partial l_\beta$ leads to the increase in the velocity dispersion. The form in eq. (17) is quite elegant and a simple calculation also tells us immediately that the current vanishes for the Maxwellian distribution which should arise as the steady-state configuration. The current can be also written in an alternate form

$$J_\alpha(l) \equiv a_\alpha(\mathbf{l})f(\mathbf{l}) - \frac{1}{2}\frac{\partial}{\partial l_\beta}\left\{\sigma^2_{\alpha\beta}f\right\}, \tag{19}$$

where $a_\alpha = (\partial\eta/\partial l_\alpha)\,, \sigma^2_{\alpha\beta} = \left(\partial^2\psi/\partial l_\alpha \partial l_\beta\right)$ where ψ and η satisfy the equations with Laplacians defined in the velocity space:

$$\nabla^2\psi = \eta; \qquad \nabla^2\eta(\mathbf{l}) = -8\pi f(\mathbf{l}). \tag{20}$$

As an interesting aside (and an illustration of the power of this formalism) note that the dynamical friction term arising from a_α is like a force arising from the potential η which satisfies the Poisson equation in momentum space with f as the source. When f depends only on the magnitude of the velocity (so that f is 'spherically symmetric' in velocity space) we can immediately conclude that a_α at some velocity is only contributed by stars with lower speeds. So the dynamical friction on a star is contributed only by stars with lower speeds whenever $f(\mathbf{l}) = f(l)$.

Some more insights into the two physical processes can be obtained by treating these coefficients a_α and $\sigma^2_{\alpha\beta}$ as constants and studying a simpler differential equation in one-dimensional velocity space given by

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial}{\partial v}\left\{(\alpha v)f + \frac{\sigma^2}{2}\frac{\partial f}{\partial v}\right\} \equiv -\frac{\partial J}{\partial v}. \tag{21}$$

The solution to this equation with the initial condition $f(v,0) = \delta_D(v - v_0)$ is given by

$$f(v,t) = \left[\frac{\alpha}{\pi\sigma^2(1-\mathrm{e}^{-2\alpha t})}\right]^{1/2} \exp\left[-\frac{\alpha(v - v_0\mathrm{e}^{-\alpha t})^2}{\sigma^2(1-\mathrm{e}^{-2\alpha t})}\right]. \tag{22}$$

This solution shows that the mean drift velocity decays to zero as $\langle v\rangle = v_0\mathrm{e}^{-\alpha t}$ due to the nonzero coefficient of dynamical friction, α. On the other hand, the velocity dispersion asymptotically reaches the value

$$\langle v^2\rangle - \langle v\rangle^2 = \frac{\sigma^2}{\alpha}(1-\mathrm{e}^{-2\alpha t}) \to \frac{\sigma^2}{\alpha} \tag{23}$$

with the distribution function becoming a Gaussian. The existence of dynamical friction due to nonzero α is vital for this value to be finite. When $\alpha \to 0$ – which is the situation described in Chandrasekhar's book [2] that has diffusion in velocity space without dynamical friction – the velocity dispersion of all the stars will diverge asymptotically.

Curiously enough, this elegant derivation obtaining both the dynamical friction and diffusion was already known before Chandrasekhar's work! These results were first obtained and published – in 1936, about six years before Chandrasekhar's work was published – by Landau [10]. (He was discussing Coulomb interactions in a plasma but everything could be trivially translated to gravitational interaction.) Strangely enough, the elegance and power of this result were not appreciated, occasionally even by plasma physicists. A detailed discussion of this approach by Rosenbluth *et al* [11] in 1957 cites Chandrasekhar's work but not Landau's though they have cited Cohen *et al* [12] with a comment "A more complete list of references is given in this..." The paper by Cohen *et al* does cite Landau's paper but it is clear they have not understood it because they make a statement "In this reference, the important terms representing dynamical friction which should appear in the diffusion equation are set equal to zero as a result of certain approximations" which is incorrect. Landau, in his usual elegant but terse style, has captured all the essential physics.

(A textbook derivation of this result in the context of plasmas [13] as well as gravitating systems [14] is now available.)

5. Conclusions

I raised several of these issues with Chandrasekhar when I wrote my review [3] on statistical mechanics of gravitating systems and tried to obtain his reaction – especially as regards Anatonov instability of isothermal sphere and Landau's work on dynamical friction – but did not succeed much. In reply to my queries by mail I got a polite reply saying " .... it relates to matters that I was interested in, some forty years ago ... I am afraid my present remembrance is faded." When I met him later in person, at IUCAA, I had a detailed session with him explaining the Landau's approach but I got the impression that its elegance and beauty did not somehow impress him!

It is probably correct to say that Chandrasekhar pioneered the use of statistical physics and stochastic processes in the study of astrophysical systems dominated by gravity. His approach to this subject shows the characteristic rigour, employed as a matter of policy, rather than out of necessity or relevance. The two 'misses' I have described are probably due to this overly mathematical approach. It would be worthwhile for someone – who is more of an expert on history of contemporary astrophysics that I am – to put together the contributions of different individuals to this fascinating subject, in the initial stages of its development.

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