Chance and Chandra

JEREMY GOODMAN* and ZACHARY SLEPIAN

Princeton University Observatory, Peyton Hall, Princeton, NJ 08544, USA
*Corresponding author. E-mail: jeremy@astro.princeton.edu

Abstract.   A few examples are given of Chandra's work on statistical and stochastic problems that relate to open questions in astrophysics, in particular his theory of dynamical relaxation in systems with inverse-square interparticle forces. The roles of chaos and integrability in this theory require clarification, especially for systems having a dominant central mass. After this prelude, a hypothetical form of bosonic dark matter with a simple but nontrivial statistical mechanics is discussed. This makes for a number of eminently falsifiable predictions, including some exotic consequences for dynamical friction.

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1. Introduction

Probabilistic and statistical considerations dominated several of Chandrasekhar's contributions to astronomy and astrophysics. Among his less well-remembered contributions to astronomy proper, as opposed to astrophysics, are a series of papers with Münch on brightness fluctuations in the star fields of the Milky Way [1] and one on the inference of the distribution of stellar spins from observed values of $v \sin i$ [2]. In a paper that deserves to be better remembered [3], Chandra introduced the use of structure functions to describe the refractive-index fluctuations that degrade ground-based astronomical images ('seeing'). These methods have been used ever since by students of atmospheric and interstellar scintillation, although usually without recognizing the debt owed to Chandrasekhar. Better known is Chandra's magisterial review article on random walks and other stochastic processes [4].

The problems of this type for which Chandra is most famous are those of dynamical relaxation and dynamical friction. He was not solely responsible for these developments – the problem goes back at least as far as Jeans [5], and a fairly complete solution for the velocity diffusion coefficients of a plasma subject to Coulomb encounters was given in the 1930s by Landau [6], but Chandra's subsequent emphasis on the probabilistic aspects was distinctive. He considered the probability distribution of the gravitational force experienced at a given point in space due to an infinite field of point masses whose locations form a Poisson process (the Holtzmark distribution), for which $\mathrm{Prob}(> f) \propto f^{-5/2}$ at large

values of the magnitude f of the force, and he then considered the effect of the motion of these gravitating masses on a test mass [7]. Rather than describing these as a series of impulsive two-body encounters, as done in most textbooks (e.g. [8]) and indeed in his own monograph [9], Chandra treated the summed gravitational forces of the field particles as a time-dependent stochastic process. In Chandra's hands, this approach described the diffusion of the test particle's velocity, but not (at least in the first instance) the back-reaction on the field particles; therefore it predicted indefinite increase in the test particle's kinetic energy. Chandra recognized the need for a drag term – dynamical friction – to preserve a Maxwellian equilibrium distribution [10].

It is well known that the usual treatments encounter a logarithmic divergence, the usual cure for which is to estimate lower and upper bounds for the impact parameters of two-body encounters. The divergence at small impact parameters is straightforward to remove by replacing the impulse approximation with exact formulae for hyperbolic close encounters, as Chandra demonstrated, so that it suffices in most cases to set $b_{\min} \sim G(m_t + m_f)/\langle v_{rel}^2 \rangle$, these symbols having the usual meanings. But the divergence at large impact parameters is more problematic. It is conventional to remark that the treatment of the field particles as statistically uniform breaks down at distances comparable to the size or density scale height of a self-gravitating system (R) and therefore to cut off the impact parameter at $b_{\max} \sim R$, so that (since $\langle v_{rel}^2 \rangle \sim Gm_f N/R$) the argument of the logarithm is of order N, the total number of particles. Chandra, however, argued for a cut-off at the mean interparticle distance, $D \sim R/N^{1/3}$. This choice seems to be in some tension with his view of the gravitational field as a collective process.

Although it is now accepted that $b_{\max}/b_{\min} \propto N$ in self-gravitating systems, much ink has been shed over the proper choice of the constant of proportionality. As a practical matter, it makes little difference for large N. Moreover, having worked in stellar dynamics for his Ph.D. and several years thereafter, the speaker and first author of this paper (JG) takes the view that the whole discussion is futile because there can be no such thing as a universally correct choice. The logarithm must depend at some level upon the details of the density profile of the system and upon where the test particle orbits within it. Furthermore, the root of the difficulty with large impact parameters lies in the insistence upon Markovian approximation, that is, the assumption that the probability distributions or moments of the test particle's velocity and position can be described by differential equations. This may be a very good approximation for gas particles interacting via short-range collisions, or even for Coulomb encounters in a plasma, since they are naturally cut off at the Debye length. But gravitational encounters have a duration that increases, on average, in proportion to impact parameter, b. Even in an infinite system of field particles, the evolution of the probability distribution $P[\mathbf{v}, t|\mathbf{v}_0, t_0]$ for the test particle's velocity, should be insensitive to encounters more distant than $b(t, t_0) \equiv |t - t_0|\langle v_{rel}^2 \rangle^{1/2}$; stars more distant than this exert a net acceleration that is effectively constant during the interval $t - t_0$. The divergence of the net acceleration as the size of the system tends to infinity is removable (by shifting to a freely-falling frame) because it is constant for all finite time intervals. To reformulate dynamical relaxation along these lines would require integral rather than differential equations in time, however. The slight theoretical ambiguity represented by the usual logarithm is a price well worth paying for convenience in practical applications.

There is another bit of unfinished business regarding dynamical friction that is probably more urgent, especially because of the discovery of extrasolar planetary systems and the

consequent revival of interest in celestial mechanics. (Chandra did not work in celestial mechanics. Perhaps it seemed to him to be a closed subject. But in view of the mathematical elegance of the subject and the new questions brought to light by computer calculations as well as planetary discoveries, it is likely that he would take interest if he were alive today.) This unfinished business concerns the relationship between dynamical relaxation and chaos. Chandra's analysis presumes that two given particles encounter one another just once since he treats their unperturbed trajectories as rectilinear. This eliminates the possibility of resonances, since resonances require repeated encounters. Celestial mechanics, however, is rife with resonance; and chaos begins where resonances overlap [11]. It seems that one uses the language of celestial mechanics when the number of gravitating bodies is small, $N \lesssim 10$, but that of dynamical relaxation when $N \gtrsim 10^3$. Ambiguous intermediate cases exist: For example, the minimum number of particles in the core of a cluster of equal point masses undergoing core collapse (and rebound) is thought to be $\lesssim 30$ [12]; and the number of significant bodies in a young planetary system might well lie in the range 10–100 before this number is reduced by consolidation or ejection. It would be interesting, and perhaps useful for application to such intermediate cases, to understand these two domains of relaxation and resonant interaction as different limits of a unified statistical theory of N-body systems. Some steps in this direction have been taken [13,14].

After this prelude, the remainder of this paper is devoted to a rather different topic: a (probably counterfactual) species of interacting dark matter. The connection to Chandra's work is admittedly tenuous, and consists mainly of the unusual properties of dynamical relaxation for such a dark matter.

2. Dark matter and its possible collisionality

Nonbaryonic dark matter (hereafter DM) makes up some 22% of the critical density $3H_0^2/8\pi G$ [15]. Notwithstanding experimental searches and suggested evidence for annihilation signals in cosmic backgrounds, DM is so far known only through its gravitational effects, including galactic rotation curves, virial temperatures of X-ray emitting gas in clusters of galaxies, and the power spectrum of temperature fluctuations of the cosmic microwave background. In compliance with Occam's Razor, DM is therefore usually presumed to have a minimum of nongravitational interactions. However, were he alive today, William of Occam would presumably allow DM to have a minimum of additional properties if those would resolve important problems in particle physics as well as astrophysics and cosmology. In this spirit, the axion and supersymmetric weakly interacting massive particles (WIMPs) have been put forward as dark-matter candidates. The existence of axions might explain CP conservation in the strong interactions [16], while supersymmetry would unify fermions and bosons, and perhaps also the strong, weak, and electromagnetic coupling constants at high energies [17]. WIMPs are favoured by the consideration that if they were produced thermally in the early Universe, then the present-day density of DM is consistent with WIMP masses in the range 1–100 GeV and annihilation cross-sections comparable to what would be expected from the weak interactions [18].

Effectively noninteracting DM would be consistent with most of what is known or believed about the present Universe, and so it is conventional to treat DM as collisionless. This assumption lends itself to predicting the dynamics of DM via N-body simulations.

Occasionally, however, attention is drawn to apparent discrepancies between collisionless, dynamically 'cold' DM and certain features of galactic structure on small scales. It is suggested on this basis that the DM may have additional properties such as an appreciable initial velocity dispersion ('warm' DM), or even some collisionality. One such discrepancy is the dearth of small satellite galaxies compared to the abundance of DM subhalos predicted to lie within the DM halos of luminous ($\gtrsim L_*$) galaxies like the Milky Way [19]. This, however, might be explained by a low efficiency of star formation in the subhaloes [20]. Another discrepancy is the prevalence and persistence of apparently rapidly rotating galactic bars despite the dynamical friction exerted on these bars by the DM halo [21]. But perhaps the dynamical friction is reduced by resonant effects too delicate for standard N-body simulation methods [22]. A decade ago, the most talked-about discrepancy was an apparent minimum core radius for dark halos inferred from rotation curves of the dwarf galaxies [23–26]. To solve these apparent problems, Spergel and Steinhardt [27] suggested that the (WIMPy) DM might have a collisional mean-free path in the range $1\,\mathrm{kpc} \lesssim \lambda_{\mathrm{mfp}} \lesssim 1\,\mathrm{Mpc}$. This would efficiently erode DM subhalos and cause cuspy DM cores with profiles $\rho_{\mathrm{DM}} \propto r^{-\alpha}$ in the range $0 < \alpha < 2$ to be smoothed out, at least initially, because the dynamical temperature associated with such profiles increases with radius. However, as the authors acknowledged, the same process of thermal conduction in a self-gravitating system leads eventually to core collapse, i.e. an indefinite increase in central density. They tried to argue that the time-scale for collapse exceeds a Hubble time. But the time-scale for core collapse cannot greatly exceed that for cusp smoothing except in carefully engineered profiles that would not prevail generally.

Galaxy clusters in collision provide perhaps the most decisive limits on self-interacting DM. Comparison of the distributions of gas (traced by X-rays) and dark matter (traced by weak gravitational lensing) in 1E 0657-56, the 'Bullet Cluster,' has been used to place a limit on the DM scattering cross-section per unit mass: $\sigma/m < 1.25\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ at 68% confidence [28]. This translates to $\lambda_{\mathrm{mfp}} \gtrsim 0.5\,\mathrm{Mpc}$ for the DM in the Solar Neighbourhood.

3. Repulsive dark matter: Overview

A different sort of self-interacting DM was proposed at about the same time as [27]: bosonic, like the axion, but with a repulsive interaction [29,30]. Also like the axion, these particles were supposed to be born in a Bose–Einstein condensate in the early Universe. The interaction would naturally be short-range, in fact point-like, if it corresponded to the nonrelativistic limit of a massive scalar field with a momentum-independent self-interaction term $V(\phi) > 0$. In the condensate, pressure would vary only with density. In fact if $V(\phi) = \kappa\phi^4$, with κ a dimensionless coupling, then it was argued in [30] that in the nonrelativistic limit $P \ll \rho c^2$, the pressure and density would be related as in an $n = 1$ Emden polytrope, $P = K\rho^2$, with $K = 3\kappa/2m^4$ in units $\hbar = c = 1$, where m is the particle mass. As Chandra himself discussed in his monograph on stellar structure [31], the cases $n \in \{0, 1, 5\}$ are the only polytropes for which a (nonsingular) self-gravitating spherical equilibrium can be found analytically; for $n = 1$, the density profile is $\rho(r)/\rho(0) = \sin(r/a)/(r/a)$, $0 \le r \le \pi a$, with scale length $a = \sqrt{K/2\pi G}$. If all of the DM were still in the condensate, then all nonrotating dark halos in virial equilibrium

would have this size and density profile independent of their masses. Clearly this would not be acceptable, especially if the scale length $a \sim 1$ kpc as required to match the DM cores of dwarf galaxies. More complex and extended profiles can be obtained if the DM has a finite temperature, as discussed below, or where it has not yet reached thermal or even virial equilibrium.

It was argued by Peebles [29] and Goodman [30] that if present-day repulsive dark matter (hereafter RDM) derives from a relativistic scalar field, then it has acceptable behaviour in the early Universe, i.e. it does not suppress large-scale structure or unduly affect primordial nucleosynthesis. There is some doubt whether even a massive scalar field has a sensible nonrelativistic limit, but we shall not worry about that here. We treat the RDM as an assemblage of nonrelativistic point particles of mass m having a two-body interaction potential whose range is small compared to the particles' Compton wavelength, so that $U(\mathbf{x}_1 - \mathbf{x}_2) \rightarrow U_0\delta(\mathbf{x}_1 - \mathbf{x}_2)$. The constant $U_0 = \tilde{U}(0)$ is the Fourier transform of $U(\Delta\mathbf{x})$ at zero momentum. If the relativistic correspondence were valid and $V(\phi) = \kappa\phi^4$, then $U_0 = 3\kappa\hbar^3/m^2c$ at tree level. We further assume that the particles are conserved. In the relativistic correspondence, this would require either that κ be very small, or that the particles be protected from annihilation by some symmetry, e.g. by making ϕ a charged scalar field and its action invariant to global phase changes, since otherwise they would have an annihilation cross-section at $O(\kappa^4/m^2)$ (annihilation of two particles into one is kinematically forbidden, so the lowest-order graph has two interaction vertices).

In terms of the two basic parameters of the nonrelativistic model, the core scale length $a = \sqrt{U_0/4\pi Gm^2}$, whereas the scattering cross-section per unit mass is, to lowest order, $\sigma/m = mU_0^2/\pi\hbar^4$. Since these involve independent combinations of m and U_0, the degree of collisionality of RDM is independent of its minimum core size. So it seems possible to evade the constraint on $\sigma_{\mathrm{scatt}}/m$ set by the Bullet Cluster [28]. Also, since $\sigma_{\mathrm{ann}}/\sigma_{\mathrm{scatt}} \propto \kappa^2 \propto (m^2U_0)^2$, reducing the collisionality reduces the annihilation rate faster. As will be shown, however, a long mean-free path leads in this species of DM to unacceptable halo density profiles, at least if collisions are frequent enough for thermal equilibration over a Hubble time.

4. Superfluidity, dynamical friction, and galactic bars

The repulsion makes the condensate, a large number of bosons in the same single-particle state, a superfluid. Superfluidity involves a penalty for excitations out of the degenerate state, which arises from the repulsive interactions. As a consequence of their indistinguishability, the repulsion between a pair of RDM bosons in distinct momentum states $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$ $[\langle\mathbf{p}|\mathbf{p}'\rangle = 0]$ is greater than it would be if both were in $|\mathbf{p}\rangle$ by [32]

$$\Delta E = \nu\tilde{U}(\mathbf{p} - \mathbf{p}') + \frac{\mathbf{p}^2 - \mathbf{p}'^2}{2m} \approx \nu U_0 + \frac{\mathbf{p}^2 - \mathbf{p}'^2}{2m}, \tag{1}$$

since the exchange energy reduces to $\nu\tilde{U}(0)$ for a short-range potential. Consequently, a rigid obstacle ('spoon') passing through the condensate at relative velocity $\mathbf{v}$ causes no dissipation if $|\mathbf{v}| < v_{\mathrm{crit}} \equiv \sqrt{2\nu_c/m}$ because, viewing the problem in the rest frame of the spoon, the energy liberated by reducing the kinetic energy of a particle to zero is less than the exchange energy. Similarly, two streams of RDM condensate pass freely through

each other if their relative velocity is less than $\sqrt{2}v_{\mathrm{crit}}$ even when the mean-free path would otherwise be short. Here v_{c} is the total number density of the condensate summed over both streams.

At the time of [30], one of us (JG) thought that the superfluidity of RDM implies that there should be no dynamical friction on a moving potential, such as a galactic bar, insofar as the core of the halo is dominated by the condensate and $v_{\mathrm{bar}} < v_{\mathrm{crit}}$. Our current view is somewhat different. In the limit that the mean-free path is large compared to the dimensions of the bar or the core, the original argument is perhaps correct. In the opposite limit, while it remains true that the dynamical friction vanishes below a critical speed $\sim v_{\mathrm{crit}}$, the reason is not because the RDM is a superfluid, but rather because it is an ideal fluid in the classical sense, i.e. one in which the viscosity based on the mean-free path and thermal velocity is negligible. The important difference between a galactic bar interacting with DM gravitationally and a spoon interacting with a laboratory fluid is that the gravitational potential is smooth, whereas the spoon has a surface at which a no-slip boundary condition applies to the normal (non-superfluid) component. Thus, even in laminar flow, the spoon exerts a viscous force on the normal component that vanishes more slowly than linearly with the viscosity, probably as $\mathrm{Re}^{-1/2}$, because there is a laminar boundary layer. At moderately high Reynolds number, Re, the boundary layer on the rear side of the spoon becomes unstable and triggers turbulence. For a smooth large-scale potential like that of a rotating bar, however, there is no such surface and no such boundary layer, and therefore perhaps no turbulence. This absence of drag in the limit of small λ_{mfp} would seem to hold even if the RDM had a substantial normal component, except insofar as turbulence may arise even without boundary layers in a smooth velocity field at large Re.

On the other hand, the bar or other moving potential may experience a wave drag if it couples to a wave whose phase velocity, whether linear or angular, matches that of the potential itself. In a pure condensate, below v_{crit}, the only significant waves are sound waves at speed $c_{\mathrm{s}} = \sqrt{2K\rho} = v_{\mathrm{crit}}/\sqrt{2}$, so that a linearly moving potential experiences no wave drag if its velocity is subsonic, though a rotating bar could excite sound waves at large radii were $|\mathbf{r} \times \mathbf{\Omega}_{\mathrm{bar}}| > c_{\mathrm{s}}$. One can see this explicitly from the quantum-mechanical equation of motion for the condensate wave function (Gross–Pitaevskii or nonlinear Schrödinger equation):

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U_0|\Psi|^2\Psi + (\Phi_{\mathrm{self}} + \Phi_{\mathrm{ext}})\Psi, \tag{2}$$

in which Φ_{ext} and Φ_{self} are the gravitational potentials of the bar or other external perturber and of the condensate itself; Φ_{self} satisfies Poisson's equation with the mass density $\rho_{\mathrm{c}} = m|\Psi|^2$ of the condensate as source. Adopting the usual Jeans swindle, consider perturbations to a background state of uniform density and vanishing Φ. The unperturbed wave function Ψ_0 has a constant modulus $|\Psi_0| = \sqrt{v_0}$ but a time-dependent phase, because the term involving U_0 in eq. (2) does not vanish in the background state. One sets $\Psi \to \Psi_0(t)[1 + \varepsilon(\mathbf{r}, t)]$ and expands eq. (2) to first order in the real and imaginary parts of ε, treating Φ_{ext} and Φ_{self} as well as ε as first-order quantities. In the unforced case $\Phi_{\mathrm{ext}} = 0$, the dispersion relation for Fourier modes $\varepsilon(\mathbf{r}, t) \propto \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$ becomes

$$\omega_{\mathbf{k}}^2 = (2U_0k^2 - 4\pi Gm)v_0 + \left(\frac{\hbar k^2}{2m}\right)^2. \tag{3}$$

As usual with the GP equation, the last term on the right represents single-particle exitations; it is small for long-wavelength modes $k \ll \sqrt{2mc_s}/\hbar$, with sound speed $c_s \equiv \sqrt{2U_0\nu_0}$. If one neglects this term, then eq. (3) matches the results of a Jeans analysis for a classical ideal fluid, $\omega^2 = c_s^2 k^2 - 4\pi G\rho_0$ [8]. If one now introduces a rigid perturbing potential that moves at constant velocity $\mathbf{V}$ through the condensate, $\Phi_{\mathrm{ext}}(\mathbf{r}-\mathbf{V}t)$, and writes $\tilde{\Phi}_{\mathrm{ext}}(\mathbf{k})$ for the spatial Fourier transform of this potential at any time, then after transients have decayed, the component of the wave drag along $\mathbf{V}$ is

$$F_{\mathrm{drag}} = -\frac{\rho_0}{V} \int \frac{d\mathbf{k}}{(2\pi)^3} \, \delta(\omega_{\mathbf{k}} - \mathbf{k} \cdot \mathbf{V}) |\mathbf{k}\tilde{\Phi}_{\mathrm{ext}}(\mathbf{k})|^2. \tag{4}$$

Since the Fourier components $\tilde{\Phi}_{\mathrm{ext}}(\mathbf{k})$ are negligible at $k \gtrsim \sqrt{2mc_s}/\hbar$, this is effectively the same drag as for an ideal fluid with equation of state $P = K\rho^2$. The quantum-mechanical nature of the condensate plays no direct role, the drag vanishes unless V is supersonic.

The formal result (4) for the wave drag clearly holds more generally for ideal fluids in which the dispersion relation may differ from (3). Thus it should hold even when the RDM has a normal (nondegenerate) component, as it must at finite temperature. In a realistic case where the background state is not uniform, however, the RDM gas would be stratified, i.e. it would have entropy gradient parallel to the background gravitational field, so that waves restored by buoyancy (internal waves/g modes) might be excited at subsonic velocities. As will be shown in §6, an isothermal RDM halo at nonzero temperature consists of a core that is almost pure condensate, and has nearly the $n = 1$ Emden profile, surrounded by an extended nondegenerate 'atmosphere', with a sharp shelf in the density profile at the edge of the core. Although we have not calculated it, the coupling of a subsonically rotating bar potential to the g modes would probably occur mainly at this shelf, with a drag proportional to the density in the nondegenerate component there; because that density is much less than the central density of the degenerate core, the drag on the bar would be expected to be considerably reduced compared to the estimates of [21] for collisionless DM at the same overall mass-to-light ratio.

5. Thermodynamics

Apart from the above clarification of its implications for dynamical friction, our main progress concerning RDM since [30] has been to construct isothermal spherical halo models, which has led us to a lower bound on the collisionality of the RDM that is, while somewhat imprecise, nevertheless orders of magnitude less than what has been inferred from the Bullet Cluster – hence ruling out RDM, we think.

For this purpose we needed to derive the equation of state of RDM at finite temperature, which we did by calculating the partition function of RDM considered as an almost ideal gas. 'Almost' means that the scattering length $\lambda_{\mathrm{scatt}} \equiv mU_0/\hbar^2 \ (\approx \sqrt{\pi\sigma_{\mathrm{scatt}}})$ is much less than the de Broglie wavelength $\lambda_{\mathrm{dB}} \equiv h/\sqrt{2mkT}$. Then, just as in a classical dilute gas, the (noncondensate) particles can be regarded as belonging to well-defined momentum states between collisions. On the other hand, λ_{dB} should be small compared to galactic scales so that we may consider the partition function of a homogeneous box of RDM.

The almost-ideal boson gas is a well-studied physical model for a superfluid. The standard treatment in textbooks [33] emphasizes the dynamics of quasiparticle excitations when

most of the particles are in the condensate, and is couched in terms of annihilation and creation operators for these excitations. Our approach is cruder but, we hope, sufficient to determine the pressure in local thermodynamic equilibrium (LTE). Insofar as possible, we follow elementary treatments of a noninteracting ideal boson gas, describing microstates by the numbers of quanta $(n_0, n_1, n_2, \ldots, n_i, \ldots) \equiv \vec{n}$ in each single-particle momentum state $|\mathbf{p}_i\rangle$, these states being discrete if the gas is imagined to be confined to a finite volume V, and ordered by their kinetic energies so that $p_i^2 \leq p_j^2$ if $i \leq j$. The total number of particles $N = \sum_i n_i$, and the number density $\nu = N/V$. We presume that there is a unique state $|\mathbf{p}_0\rangle$ of zero kinetic energy (the ground state). The condensate, if present, will be characterized by the occupation number n_0 of the ground single-particle state being macroscopic, i.e. comparable to N. The kinetic energy $|\mathbf{p}_i|^2/2m$ of all other states is higher than that of the ground state by at least $O(\hbar^2/2mV^{2/3})$; while this is small for macroscopic volume V, it is larger than $k_B T/N$ by a factor that is $O(N^{1/3})$ at fixed $\nu = N/V$. Therefore, a macroscopic occupation number in any state other than $|\mathbf{p}_0\rangle$ is always exponentially suppressed.

The energy of a general microstate $\vec{n}$ is

$$E(\vec{n}, V) = \sum_{i \geq 0} \frac{p_i^2}{2m} n_i + U_0 \frac{N^2}{2V} + \frac{U_0}{2V} \sum_i \sum_{i \neq j} n_i n_j$$

$$= \sum_{i \geq 0} \frac{p_i^2}{2m} n_i + \frac{U_0}{V} \left[N^2 - \tfrac{1}{2} \sum_i n_i^2 \right]$$

$$\approx \sum_{i \geq 0} \frac{p_i^2}{2m} n_i + \frac{U_0}{2V} \left[2N^2 - n_0^2 \right]. \tag{5}$$

The double sum in the first line represents the exchange energy for pairs in distinct states, and is rewritten as $(\sum_i n_i)^2 - \sum_i n_i^2$ in the second line. Now

$$\frac{U_0}{2V} \sum_{i>0} n_i^2 \leq \frac{U_0}{2V} \max_{j>0} n_j \sum_{i>0} n_i \leq \tfrac{1}{2}\nu U_0 \max_{j>0} n_j.$$

Following the remarks in the last paragraph, n_j is microscopic for all $j > 0$. So the error in the last line of (5) grows more slowly with N than $E(\vec{n}, V)$ as $N \to \infty$ at constant N/V and therefore can be neglected in the thermodynamic limit. This step is essential, because the negative sign in front of $\sum_i n_i^2$ would otherwise prevent us from summing independently over all the n_i with a chemical potential μ to enforce $\bar{N} = N$, as one does to evaluate the grand canonical partition function of a noninteracting gas, because the sums would diverge. But with the approximation (5), one can eliminate N and n_0 in favour of the intensive variables $\nu_0(\mu, t) \equiv \bar{n}_0/V$ and $\nu(\mu, T) \equiv \bar{N}/V$, leaving V as the sole extensive variable, thereby obtain the grand canonical partition function of the interacting system, and hence the pressure. Here $\nu_0 \leq \nu$ is the number density of condensate particles only, not to be confused with the number density of the background state in the wave-drag calculation of the previous section.

One finds that the condensate appears when the number density is greater than $\nu_{\mathrm{crit}}(T) = \zeta(3/2)(mk_B T/h\hbar)^{3/2}$, which is the same as the critical density for a noninteracting gas. In

terms of scaled number densities $\hat{\nu} \equiv \nu/\nu_{\text{crit}}$ and $\hat{\nu}_0 \equiv \nu_0/\nu_{\text{crit}}$, a scaled pressure $\hat{P} \equiv P/\nu_{\text{crit}}k_{\text{B}}T \propto P/T^{5/2}$, and a dimensionless measure of the repulsion term

$$\theta \equiv \frac{U_0}{k_{\text{B}}T}\nu_{\text{crit}} = \tfrac{1}{2}\zeta(3/2)\frac{\sigma_{\text{scatt}}^{1/2}}{\lambda_{\text{dB}}}, \tag{6}$$

the equation of state is

$$\hat{P} = \left(\hat{\nu}^2 - \tfrac{1}{2}\hat{\nu}_0^2\right)\theta + \text{Li}_{5/2}(z)/\zeta(3/2), \tag{7a}$$

$$\hat{\nu} = \text{Li}_{3/2}(z)/\zeta(3/2) + \max(\hat{\nu}_0, 0), \tag{7b}$$

$$z = \exp(-\theta\hat{\nu}_0) \quad \text{if and only if} \quad \hat{\nu}_0 > 0. \tag{7c}$$

Here $\text{Li}_s(z)$ is the polylogarithm [$\text{Li}_s(z) = \sum_{k=0}^{\infty} z^k/k^s$ where this converges], which relates to the Bose–Einstein integrals when s is an integer or half-integer. The parameter z is a sort of fugacity. When $\hat{\nu}_0 = 0$, meaning that the condensate is absent, then z is an implicit function of $\hat{\nu}$ defined by eq. (7a); otherwise, eq. (7c) gives z explicitly. The noninteracting gas is recovered by setting $\theta = 0$: then pressure is constant once the condensate appears, if temperature is fixed.

A plot of the equation of state (7) is shown in figure 1 for $\theta = 1$, corresponding to strong scattering, although the steps leading to eqs (7) probably cannot be justified unless $\theta \ll 1$. The condensate is absent along the red section, and present along the blue. The part of the curve beneath the dashed line segment is unphysical, and the endpoints of this segment define two phases in contact; it can be shown that $\oint \hat{P}\, d\hat{\nu} = 0$ around the loop defined by the dashed segment and the unphysical lobe below it. The density jump at the phase transition scales $\propto \theta$ when $\theta \ll 1$. In that limit, however, because of the factors of θ in eqs (7b) and (7c), the pressure rises only slowly with increasing $\nu > \nu_{\text{crit}}$ until $\hat{\nu} \gtrsim \theta^{-1}$, at which point $\nu \approx \nu_0$ because z and the polylogarithms become small.
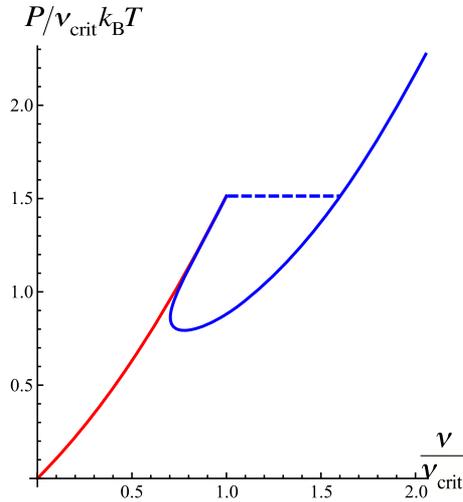
Figure 1. Scaled pressure $\hat{P}$ vs. scaled number density $\hat{\nu}$ for interaction strength $\theta = 1$ (eq. (6)).

In other words, while the actual phase transition is rather weak for weak collisionality ($\sigma_{\text{scatt}} \ll \lambda_{\text{dB}}^2$), in the same limit an isothermal system goes from nondegenerate to almost entirely degenerate over a small increase in pressure but a large increase in density. It is this feature of the equation of state that leads to constraints on the collisionality of RDM from galactic rotation curves.

6. RDM halos

The simplest model for dark halo in virial equilibrium that resembles inferences from observations of galaxies is an isothermal sphere. For a classical ideal gas or system of collisionless WIMPs, the isothermal sphere has a unique density profile, apart from rescalings of the central density and core radius. It is the limit of the sequence of Emden spheres, as is also discussed in Chandrasekhar's first book [31], and is characterized by constant velocity dispersion and constant P/ρ at all radii. Although the density profile cannot be obtained in closed form, $\rho(r) \sim r^{-2}$ at $r \gg r_{\text{c}}$; this implies a mass contained with radius r that is asymptotically linear, $M(r) \sim r$, and therefore a rotation velocity for circular orbits, $v_{\text{c}}(r) = GM(r)/r$, that is asymptotically constant, in general agreement with rotation curves of spiral galaxies out to the largest observable radii. The total mass profile (DM+baryons) appears to be remarkably close to $\rho \propto r^{-2}$ within at least the luminous regions of elliptical galaxies [34].

Using the equation of state (7), one can construct self-gravitating isothermal spheres of RDM. The condensate dominates within the core radius $r_{\text{c}} \equiv \pi a$, and the density profile there is approximately the same as for zero temperature, namely an $n = 1$ Emden model. At much larger radii, $\rho \propto r^{-2}$ as for the classical isothermal sphere. The behaviour at intermediate radii is quite different from the classical case and depends strongly upon the collisionality parameter (6). This is shown in figure 2. All of these models are scaled to the same central density, core radius, and asymptotic rotation velocity, which turns out to require that the scaled central number density $\hat{\nu}(0) \propto \theta^{-1}$; so apart from dimensional scalings, they form a one-parameter sequence. Paradoxically, the strongly collisional model, $\theta = 1$, looks most like the classical case, but for $\theta \ll 1$, the density drops sharply at the edge of the core and is approximately constant out to $r \approx \theta^{-1/2}r_{\text{c}}$. The explanation for this behaviour is that the doubling of the repulsion due to exchange forces acting on the nondegenerate component within the core (the nondegenerate component is never entirely absent) renders this component almost unbound, so that its density becomes small and declines slowly until such a radius that its own interior mass begins to be larger than that of the core. This structure is analogous to that of a red giant, in which the central parts are supported by degeneracy pressure and there is a large increase in entropy across the hydrogen burning shell, leading to a distended convective envelope.

The drop in density at the edge of the core is associated with a drop in the rotation curve, which eventually recovers and is approximately constant beyond $\theta^{-1/2}r_{\text{c}}$. Figure 2 shows that v_{c} drops by a factor ~ 2 for $\theta = 10^{-4}$. A drop this large or larger would seem to be in contradiction with what is inferred for the contribution of the halo to v_{c} after the contribution of the baryonic matter is subtracted. Thus, insofar as RDM halos are (or would be) isothermal, we can safely conclude that $\theta \geq 10^{-4}$.
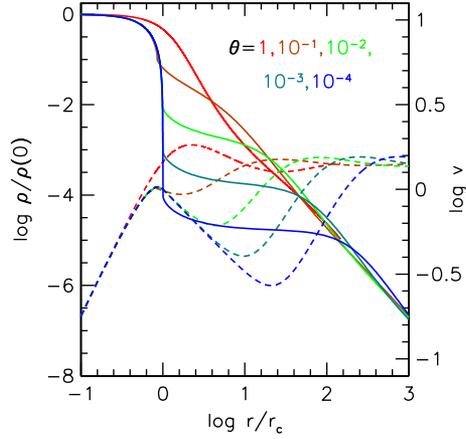
Figure 2. Self-gravitating isothermal spheres of RDM for various collisionalities θ (eq. (6)). Solid curves and left axis: mass density. Dashed curves and right axis: rotation curves.

7. Limits on the RDM boson mass

Eliminating U_0 between $\sigma_{\text{scatt}} = (mU_0)^2/\pi\hbar^4$ and $r_c = \pi a = \sqrt{\pi U_0/4Gm^2}$ leads to $\sigma = (4/\pi^2)G^2\hbar^{-4}m^6r_c^4$. Therefore, one can derive an upper bound on m in terms of the constraint inferred by [28] for σ/m from the Bullet Cluster:

$$m < 7 \times 10^{-4} \left(\frac{\sigma/m}{1.25\,\text{cm}^2\,\text{g}^{-1}}\right)^{1/5} \left(\frac{r_c}{1\,\text{kpc}}\right)^{-4/5} \text{eV}/c^2. \tag{8a}$$

On the other hand, taking $\lambda_{\text{dB}} \approx h/mv_c$ in eq. (6), where v_c is the asymptotic circular velocity of the halo, one has a lower bound on m in terms of the constraint $\theta > 10^{-4}$ obtained in the last section:

$$m > 9 \cdot \left(10^4\theta\right)^{1/4} \left(\frac{v_c}{100\,\text{km s}^{-1}}\right)^{-1/4} \left(\frac{r_c}{1\,\text{kpc}}\right)^{-1/2} \text{eV}/c^2. \tag{8b}$$

This corresponds to $\lambda_{\text{dB}} \lesssim 0.05$ cm for $v_c = 100\,\text{km s}^{-1}$.

The two inequalities (8a) and (8b) are clearly incompatible for any astronomically acceptable values of core radius and circular velocity. Therefore, it seems that repulsive dark matter, at least of the sort we have been considering here, can be ruled out.

The lower bound (8b) derives from our assumption that the halo is isothermal. However, one should ask whether that assumption is reasonable. Local thermodynamic equilibrium should be established on a collision time $t_{\text{coll}} \sim \lambda_{\text{mfp}}/v_c$, but global isothermality is achieved on a conduction time $t_{\text{cond}} \sim r^2/(v_c\lambda_{\text{mfp}})$. If the mean-free path were small compared to galactic scales, one could have $t_{\text{cond}} \gg t_{\text{coll}}$. As noted in our discussion earlier, however, $\sigma/m \approx 1\,\text{g cm}^{-1}$ corresponds to $\lambda_{\text{mfp}} \sim r$ in the solar neighbourhood. So, if eq. (8a) were an approximate equality, then these time-scales would be comparable to one another, and rather less than a Hubble time. On the other hand, if the mass were very much less than (8a), then the RDM would suffer little scattering, so that even t_{coll} could be longer

than the age of the Galaxy. Thermodynamics considerations would not apply. And yet, despite the lack of scattering, the repulsion could still provide a minimal $r_c \sim 1$ kpc since (as we have already noted) the core radius and the scattering cross-section involve different combinations of U_0 and m.

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