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Chandrasekhar's book *An Introduction to the Study of Stellar Structure*

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Abstract. For me, and for many astrophysicists of my generation, Chandrasekhar's book *An Introduction to the Study of Stellar Structure* was very important. I could not have done my PhD (1962–1965) without it. Much more recently (1998) I realized that I could not have written my lecture course on thermodynamics and statistical mechanics without much of it, particularly the first chapter. I shall present anecdotal evidence that the influence of his discussion on the second law of thermodynamics has been important not just for astrophysics but for a much wider range of physics.

Chandrasekhar's discussion of polytropes was masterly. Even today polytropes play an important role as an aid for understanding stellar structure. I believe that to the list of analytic solutions of the polytrope only one more has to be added: a curious $n = 5$ model of Srivastava (1962).

Stellar structure is nowadays a very computationally intensive subject. I shall illustrate this with a couple of topics from my experience with `Djehuty`, a supercomputer code for modelling stars in 3D. Nevertheless it remains true, I believe, that analytical mathematical entities like polytropes are fundamental as aids for understanding what the computers churn out.

How close are we to seeing a book with the title 'The Last Word on the Study of Stellar Structure'? Not very, although much has been learned in 70 years. I shall discuss a few of the aspects of stellar evolution that are problematic today.

I shall discuss a couple of aspects where I believe analysis of 'piecewise polytropic' structures sheds light on the question 'Why do stars become red giants?'

Keywords. S Chandrasekhar; stellar structure; thermodynamics.

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1. Chandrasekhar's book *An Introduction to the Study of Stellar Structure* [1]

This book was the most important source for people of my generation, and it remains today an absolutely fundamental source for students. It was written in the pre-computer age, and stellar structure and evolution is nowadays a very computer-intensive subject; nevertheless no amount of computation can replace the kind of insight that emerges in every chapter of this book.

I did not realize until about twelve years ago how important a source it is in some quite different areas. Twelve years ago I was asked – well, ordered – to give a course on thermodynamics and statistical mechanics. I did not feel uneasy, but nevertheless there was one vital step in thermodynamics that I had often felt, in my experience as a student, was rather badly explained in all of the standard undergraduate textbooks of the time. I hoped to discover some more recent books that would explain this more clearly.

The point that troubled me was the transition from the second law of thermodynamics, as commonly stated, to the mathematical expression of it in the form of a physical quantity, the entropy, and its relation to temperature and density. The second law is a very odd law, when one compares it with other laws of nature such as Newton's law of gravity, or Faraday's law of induction. Usually, such a law can be expressed in words, but is then immediately translatable into mathematics. The force of gravity can be written as 'directly proportional to the product of the masses and inversely proportional to the square of the distance', which immediately translates into

$$F = G\frac{m_1 m_2}{d^2}. \tag{1}$$

But the second law as commonly put in words (there are several variants) is an oddly negative statement, e.g. *Heat cannot flow from a cooler to a hotter body, without external work being done;* and it is not at all clear how that translates into an equation for anything, let alone the entropy. Had Newton said that 'gravity cannot be proportional to the sum of the masses nor inversely to the cube of the distance', it would have been a true but not very helpful statement. Generally, it does not seem reasonable that one can draw sharp mathematical conclusions from a statement that something or other can *not* be done.

Of course many books do manage to make the transition from a negative to a positive statement, but none of them, as it seemed to me in sifting through the library textbooks, made it very clear how this is done. And then I remembered that the first chapter in Chandrasekhar's book did make this transition rather clear. I reread it, and it was a revelation. Chandrasekhar based his exposition on a theorem by Carathéodory [2], which (I believe) no one at that time had taken the trouble to translate into English.

I suppose Chandrasekhar put this discussion of Carathéodory's theorem into a book on stellar structure because he also felt, as I did, that the usual introductions to thermodynamics were not satisfactory. But unlike myself, he actually did something about it.

Only a week or two later, the value of his discussion was brought home to me in a slightly surprising manner. I was sitting at dinner in my college, next to a gentleman who was a guest of one of my colleagues. He told me that he was a physical chemist, with a particular interest in thermodynamics. I said, 'Good; I hope you can help me write my lecture course on this'. I went on to explain my problem more-or-less as I have told you, but then added, 'However, I did finally find a very good treatment, in a somewhat unexpected place'. Right away, he said 'Ah, I expect you mean Chandrasekhar's book on stellar structure'.

So evidently it is not just in the tiny world of stellar structure that Chandrasekhar's treatment of thermodynamics is considered masterly. And I am glad to think that it is not just myself who finds the transition of the second law from words to mathematics rather difficult.

2. A new(ish) analytic $n = 5$ polytrope

Chandrasekhar's discussion of polytropes was not particularly novel, but he brought together in a masterly way the mathematics of polytropes as developed over the previous fifty years. A polytrope is a spherical self-gravitating body in which the pressure relates to the density in a particularly simple way:

$$p \propto \rho^{(n+1)/n}, \text{ or equivalently } p \propto \theta^{n+1}, \ \rho \propto \theta^n. \tag{2}$$

The constant n does not need to be an integer, but the values 0, 1, 3 and 5 are particularly interesting for astrophysics.

The fact that the body is self-gravitating means that it must satisfy (in some units)

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}\theta}{\mathrm{d}r} = -\theta^n. \tag{3}$$

Chandrasekhar's discussion notes the analytic solutions:

(a) $n = 0$ and $n = 1$; they can be solved for in complete generality,
(b) a singular solution for all $n > 3$, which is a power law relation

$$\theta = \left(\frac{2(n-3)}{(n-1)^2 r^2}\right)^{1/(n-1)}, \tag{4}$$

and

(c) another analytic solution, which is for $n = 5$ only. It is the special case of the Emden boundary condition $\mathrm{d}\theta/\mathrm{d}r = 0$ at the centre $r = 0$:

$$\theta = \frac{1}{\sqrt{a + \frac{1}{3}\frac{r^2}{a}}}. \tag{5}$$

I thought this was the last word on analytical solutions of the polytropic equation, until I received in about 1975 a paper to referee that showed that there is a second analytic solution for $n = 5$, one which does *not* satisfy the Emden boundary condition. It turned out, somewhat disappointingly for the authors involved, that this solution had already been discovered and published by Srivastava [3], in a paper which I felt embarassed to have overlooked thus far. Srivastava [3] showed that the following is also a solution of the $n = 5$ polytropic equation:

$$r\theta^2 = \frac{1}{3\cot^2(\frac{1}{2}\ln\frac{r}{a}) + 1}, \tag{6}$$

where a is an arbitrary constant. I shall happily challenge anyone to verify, on one side only of a sheet of paper, that (6) is indeed a solution of (3) for $n = 5$. This solution is plotted in figure 1.

When I first saw the solution (6), I thought that perhaps a use can be found for it as an illustration of some kind of actual star. However, I have been totally unable to think of any application. The Emden solutions ($\mathrm{d}\theta/\mathrm{d}r = 0$ when $r = 0$) separate all solutions into two kinds: those with a finite mass of zero radius at the origin from those with a finite radius containing zero mass. The former are arguably useful as illustrative models for red giants

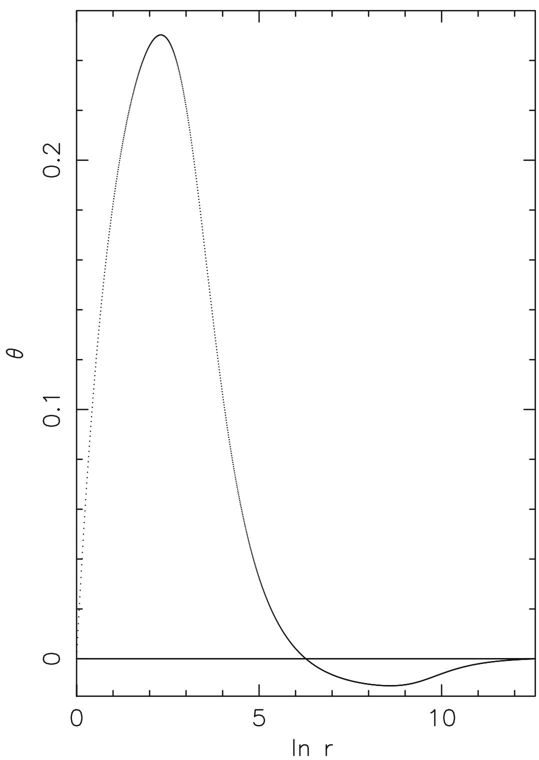

Figure 1. The $n = 5$ analytic model of Srivastava [3]. The model repeats every 4π horizontally, and is diminished (going to the right) by a factor of $e^{2\pi}$ in each cycle. The model applies to a conceivably astrophysical system only between the maximum of θ, where the mass coordinate is zero, and the first zero of θ beyond that, where the density is zero.

(which have a finite mass at a very small though not zero radius). But it is difficult to think of any application for a mathematical model with zero mass within a finite radius. Perhaps the central hole is filled with radiation at such a pressure as to support the surrounding shell, yet with negligible mass–density $E_{\rm rad}/c^2$.

3. The last word on stellar structure?

Chandrasekhar's title 'An Introduction ...' was of course a very modest statement. But there is no doubt that many astrophysicists nowadays believe that stellar structure and evolution are now fully understood. In fact the first time I heard someone say 'Stellar structure? Hasn't that all been done?' was in 1970, from the lips of Jerry Ostriker. So can we expect to see a book with the title I propose?

I think not. As soon as some problems appear to be understood, new problems appear. I believe we can roughly divide the physics of stars into 'zero-order' and 'first-order' effects. The zero-order effects are things like the equation of state, the self-gravity of Poisson's equation, nuclear energy production, radiative heat transport, convective heat transport and neutrino losses. We have expressions for all of these (a rather crude one in the case of convection) that we can feed into a computer code, and the results, comparing theoretical models with observed data on masses, radii, temperatures and luminosities, are generally reasonable.

But there are several effects that I would call first-order, although they are not necessarily small. Two of them are convective overshooting and mass loss by stellar wind. I do not think I should go into these in detail, but the former can alter the nuclear lifetime of a star by a factor of at least 1.5, and the latter can decrease the mass by much larger factors though generally rather late in a star's life. Computational models usually include both these effects nowadays, but also generally in a very *ad hoc* way: some rather arbitrary formula is chosen, and its strength varied in order to give the best agreement.

But even with impeccable zero-order physics and some plausible first-order physics, some problems remain that may require substantial lateral thinking. Any credible physics is going to tell us that if we have a wide binary (in the general galactic field and not in a dense cluster) that consists of two red giants, those two giants should be of rather similar mass; the rate of evolution depends strongly on mass, and if one component is (initially) say 50% more massive than the other it will already be a white dwarf when the secondary is a red giant, and alternatively if the primary is a red giant the secondary will still be very close to the zero-age main sequence. So what are we to make of the following:

OW Gem: (F2Ib-II + G8II; 30 + 35$R_\odot$; 5.8 + 3.9$M_\odot$; 1259 d; [4])

and

V643 Ori: (K2III + K7III; 16 + 22$R_\odot$; 3.4 + 2.0$M_\odot$; 52.4 d; [5])?

In these systems one giant is 50–70% more massive than the other. I would like to suggest that each system is the consequence of a merger of two stars within what was formerly a triple system. Consider

1 Gem: ((G6III + ?; 9.60 d) + K0III; 13.35 yr; [6])

and

V453 Cep: ((B9V + B9V; 1.185 d) + G8III; 54.72 d; [7]).

In 1 Gem the inner binary is very likely to merge when the G6III star fills its Roche lobe. The mass of its companion is not known, but is unlikely to be more than about half that of the giant, from the small mass function. After the merger we may have two giants with masses differing by $\sim$50%.

Until a few months ago I would have hesitated to put V643 Ori in the same category as OW Gem, because I would have thought it highly improbable that there would be triples with an outer period as short as 52 d. But then I heard about V453 Cep from Dr R E M Griffin. The inner pair in V453 Cep is not going to merge, because its mass ratio is close to unity; but the existence of this star (at 7th magnitude, presumably not a rare one-at-Galaxy object) suggests that a star like V643 Ori but with a period of $\sim$50 d is not improbable.

The reality of merger events has had a huge boost very recently. Tylenda *et al* [8] found that V1309 Sco was a contact binary until about three years ago, but with an orbital period ($\sim$1.4 d) that was shortening unusually rapidly; then in 2008 it had a 10-magnitude outburst in the infrared; and now it appears to be settling back as a single star. Mergers as the end-point of contact binary evolution were suggested by Robertson and Eggleton [9].

4. Three-dimensional modelling

For the last few years I have been working with colleagues on a three-dimensional code for doing hydrodynamics: we call this code `Djehuty` after the Egyptian god of astronomy – well, actually, astrology. We hope that this code will give some insight into convection, for instance, but for the time being it has mainly contributed to two topics involving red giants. Both of them are illustrated in figure 2.

Figure 2 [10] shows a cross-section through the interior of a star that is a red giant undergoing the almost-explosive ignition of helium in its electron-degenerate core. The black circle is the shell surrounding the core in which hydrogen is still burning to helium:

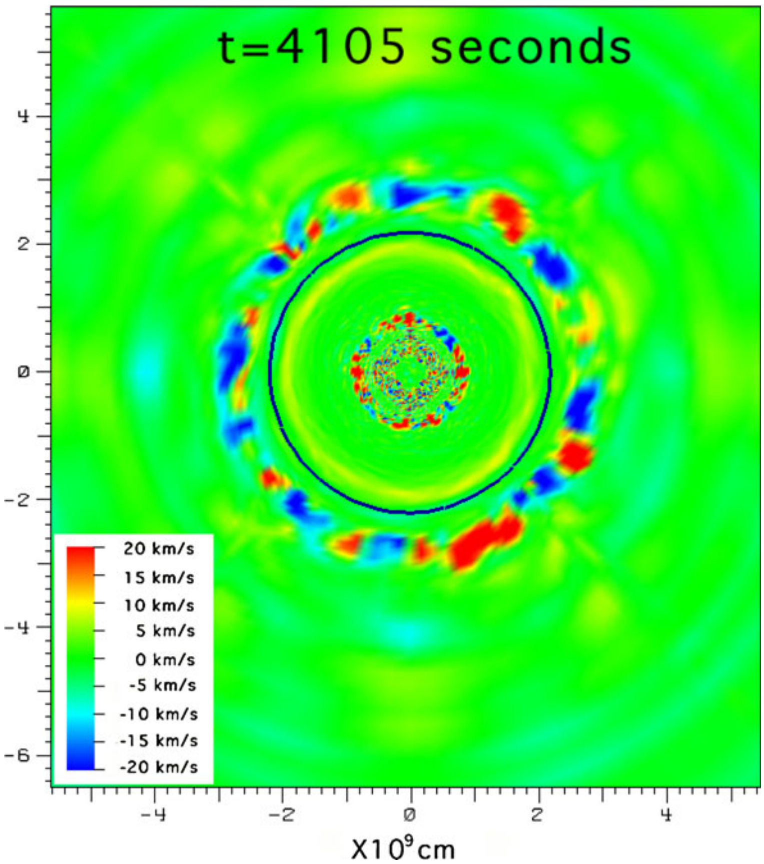

Figure 2. Motion in and near the helium core of a red giant undergoing the helium flash. Radial velocity is colour coded. In the central part of the core is the strong motion driven by the almost-explosive burning of helium (the 'helium flash'). The hydrogen-burning shell bounding the core is shown by a black circle. Outside this is an unexpected region of motion; this motion is driven by the molecular weight inversion caused by the burning of ^{3}He to ^{4}He plus protons.

it is very thin. Green represents zero (radial) velocity, and red and blue indicate outward and inward velocity respectively. The figure only shows a small fraction of the total star, whose overall radius is ∼3000 times the radius of the core.

Deep in the middle of the core is a multicoloured region where helium has ignited. The luminosity generated in this region is enormous, about 10^8 times the solar luminosity, but it hardly reaches the surface, even of the core, because it is absorbed by heating and lifting the degeneracy of the material locally. Nevertheless, it sets up some very vigorous convection. The significance of this is that it has often been debated how violent this ignition should be: could it actually run away and blow up the star? Our answer is an emphatic 'no'. We followed the evolution substantially further, and subject to various perturbations such as increasing the nuclear energy release by a large amount in a short time, and in an asymmetric manner; but when the perturbation was removed the evolution quickly returned to what it was before. Certainly the evolution is rapid, but we were unable to persuade the star to blow up in some spectacular fashion.

Figure 2 shows in addition something that was quite unexpected. Not far outside the black burning shell is a multicoloured region where some material is going up and some going down. We were worried that this might mean that our hydrodynamic algorithm was not working very well. But a close inspection showed that this motion was driven by a small inversion in the molecular weight. Normally the molecular weight increases inwards, and this contributes to hydrodynamic stability. But in a previously overlooked region there is burning of ^{3}He to ^{4}He, thus:

$$^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + {}^1\mathrm{H} + {}^1\mathrm{H},$$

which lowers the mean molecular weight from 3 to 2 – although of course the effect is diluted by the fact that there is only about 0.1% of ^{3}He in the whole mixture. But this small effect was enough to drive the motion seen in the figure.

A closer look at this issue showed us that this extra motion should have been present long before the helium in the core had ignited. In fact this extra motion explains a puzzle that has been around for 30 or 40 years. A star like the Sun is bound to produce, in its main sequence and early giant life, a percent or so of the isotopes ^{3}He and ^{13}C. Yet the former is not seen as much as expected, and the latter is often seen, in the more luminous red giants, at greater abundance than expected. The motion that we found explains both of these. The ^{3}He is consumed, and the ^{13}C produced, in much the same region: the multicoloured ring surrounding the thin shell in figure 2. But the instability there means that the composition there does not just change locally, but instead proceeds globally, because the motion once it has reached some kind of transient equilibrium gives mixing right up to the surface convection zone.

There are many ways in which 3D simulations can be used to improve what goes into the 1D, i.e. spherically symmetric, models that are most commonly used. Two that I am currently investigating are: (1) The amount of convective overshooting to be expected around main-sequence convective cores and (2) the amount of tidal friction to be expected in an eccentric binary orbit.

Both of these are what I loosely call first-order effects, and there are several more that ought to be capable of being analysed with a code such as `Djehuty`.

5. The red giant question

A question often asked is: why is it that stars at the end of their main-sequence lives always expand rather rapidly in radius, by a factor that might be 10 or 100, while the core at the same time shrinks? In about 1970 I asked this question of three leading workers in computational stellar evolution: Martin Schwarzschild, Icko Iben and Bohdan Paczyński. They all (independently) gave me much the same answer: what happens when you solve four simultaneous non-linear equations is unlikely to have some elementary explanation.

Some of the earliest light shed on this was from the work of Henrich and Chandrasekhar [11] and Schönberg and Chandrasekhar [12]. The latter work came up with a famous limit of 10%: this was the largest mass fraction that could be contained in an isothermal core, if the molecular weight in the core was twice that of the envelope. Some drastic change was necessary if in fact the hydrogen became exhausted in a core that contained more than 10% of the star's mass. This required the core to shrink until degeneracy became important, or until the helium in the core heated enough to ignite (or both, as in figure 2).

However, even though it is rather clear that the core must contract in response to central fuel exhaustion, it is by no means clear that the envelope must expand. In fact, one can see a counterexample. In the case of helium-burning 'main sequence' stars [13], stars of less than $0.8 M_\odot$, or more than $2.3 M_\odot$, hardly expand their envelopes at all even though their helium-exhausted carbon cores contract to white-dwarf radii.

Eggleton and Cannon [14] demonstrated that for a star to evolve to a configuration where the central density greatly exceeds the mean density (e.g. by factors of 10^{10} or more, as typical in red giants) it is necessary, but not sufficient, that the local polytropic index should approach close to, and perhaps exceed, the value 5. It need not be close to 5 over most of the star, but somewhere it must approach or exceed 5. Define X, the degree of central condensation, as

$$X \equiv \frac{4\pi R_\mathrm{s}^3 \rho_\mathrm{c}}{3M_\mathrm{s}}, \ \mathrm{c} = \text{centre}, \ \mathrm{s} = \text{surface}, \tag{7}$$

i.e. the ratio of the central density to mean density. If the maximum value of the local polytropic index $n(r)$ throughout the star is N, and $N < 5$, then we can prove that

$$X < X_N, \tag{8}$$

where X_N is the central condensation of a polytrope of index N. A reasonable approximation to X_N is

$$X_N \approx \frac{0.025}{\left(\frac{5}{6} - \frac{N}{N+1}\right)^3}. \tag{9}$$

For example, if in some computed model of a red giant $X = 2.5 \times 10^{10}$, then somewhere within the star the local polytropic index must exceed 4.9964. It follows from this that if we want to know why the star is a giant, it is necessary to look at those parts of the star where $n(r) \gtrsim 5$. In most red giants these parts are (a) where there is a steep molecular weight change, i.e. in the hydrogen burning (or helium burning) shell and (b) in the isothermal non-degenerate layer below the burning shell, if such a layer exists. The helium stars in some mass ranges do not become giants because either the change in molecular weight is not enough, or the isothermal non-degenerate layer below the burning shell is too thin (even non-existent).

The necessity for a μ-jump, along with a 'soft' region, i.e. one with n approaching 5 (or more), is aptly illustrated by an entirely analytic model consisting of an $n = 5$ core, a μ-jump of magnitude α, and an $n = 1$ surface layer. In such a model [15] we can show that for $\alpha = 3$ there is a maximum core-mass fraction, analogous to the Schönberg–Chandrasekhar limit, of $2/\pi$; for smaller α there is no limit, and for larger α there is a smaller limit which goes to zero like $1/\alpha^2$.

Both the examples above, the theorem proving that $X < X_N$ if $n(r) \leq N < 5$, and the theorem showing that the maximum core-mass fraction is $2/\pi$ in the piecewise polytrope with $n = 1$ or 5 and with molecular weight jump by a factor of 3, are provable analytically on the backs of (rather large) envelopes. I recommend them to you as exam questions that you can set for very enthusiastic potential theoretical astrophysicists, along with verifying Srivastava's [3] polytropic solution (eq. (6)).

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