

Dirac equation with spin symmetry for the modified Pöschl–Teller potential in D dimensions

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Abstract. We present solutions of the Dirac equation with spin symmetry for vector and scalar modified Pöschl–Teller potentials within the framework of an approximation of the centrifugal term. The relativistic energy spectrum is obtained using the Nikiforov–Uvarov method and the two-component spinor wave functions obtained are in terms of the Jacobi polynomials. It is found that there exist only positive energy states for bound states under spin symmetry, and the energy of a level with fixed value of n , increases with increase in dimension of space time and the potential range parameter α .

Keywords. Dirac equation; modified Pöschl–Teller potential; spin symmetry; Nikiforov–Uvarov method.

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1. Introduction

Solutions to relativistic equations play a very important role in many aspects of modern physics. In particular, the Dirac equation which describes the motion of a spin- $\frac{1}{2}$ particle has been used for solving many problems of nuclear and high-energy physics. The spin symmetry arises if the magnitude of the attractive Lorentz scalar potential $S(r)$ and the repulsive vector potential $V(r)$ are nearly equal, i.e. $S(r) \sim V(r)$, in the nuclei, while the pseudospin symmetry occurs when $S(r) \sim -V(r)$ [1–3]. The exact spin and pseudospin symmetries have been shown to correspond to the $SU(2)$ symmetries of the Dirac Hamiltonian [3]. The spin symmetry is relevant for mesons [4] and the pseudospin symmetry is used to explain deformed nuclei [5], superdeformation [6] and to establish an effective nuclear shell-model scheme [7–9]. Also, various potentials such as the Morse potential [10–12], Wood–Saxon potential [13], Coulomb and Hartmann potentials [14], Eckart potential [15,16], Pöschl–Teller potential [17,18] and the harmonic potential [19,20] have been studied within the framework of the spin and pseudospin symmetries.

Moreover, with interest in higher-dimensional theory, the multi-dimensional quantum-mechanical equations (relativistic and non-relativistic) have been solved with various physical potentials. For example, the D -dimensional Schrödinger equation has been studied with the Coulomb-like potential [21], pseudoharmonic potential [22], Hulthén potential [23] and Pöschl–Teller potential [24]. In addition, the D -dimensional relativistic Klein–Gordon and Dirac equations were studied with many exactly solvable models [25–30]. However, some physical potentials can only be solved exactly for the s -states. Unfortunately, the modified Pöschl–Teller potential is one of these potentials. For instance, a recent work [31] has presented the s -wave solutions of the Dirac equation with the Pöschl–Teller potential under the conditions of the exact spin symmetry and pseudospin symmetry. However, to extend the solutions of the modified Pöschl–Teller potential to any $\ell \neq 0$ state, some recent studies [17,24] have used a hyperbolic approximation for the centrifugal term to obtain the non-relativistic solutions of the modified Pöschl–Teller potential. In light of this, the present paper intends to extend the discussions on the relativistic Pöschl–Teller potential to D dimensions by presenting bound-state solutions of the D -dimensional Dirac equation with spin symmetry for Lorentz vector and scalar modified Pöschl–Teller potential using the Nikiforov–Uvarov method [32].

2. The Dirac equation in D dimensions

The D -dimensional Dirac equation with a scalar potential $V_s(r)$, a vector potential $V_v(r)$ and mass μ can be written in natural units $\hbar = c = 1$ as [27,33,34,36–38]

$$H\Psi(r) = E_{n,r,\kappa}\Psi(r) \quad \text{where} \quad H = \sum_{j=1}^D \hat{\alpha}_j p_j + \hat{\beta}[\mu + V_s(r)] + V_v(r). \quad (1)$$

Here $E_{n,r,\kappa}$ is the relativistic energy, and $\{\hat{\alpha}_j\}$ and $\hat{\beta}$ are Dirac matrices, which satisfy anti-commutation relations

$$\begin{aligned} \hat{\alpha}_j \hat{\alpha}_k + \hat{\alpha}_k \hat{\alpha}_j &= 2\delta_{jk} \mathbf{1} \\ \hat{\alpha}_j \hat{\beta} + \hat{\beta} \hat{\alpha}_j &= 0 \\ \hat{\alpha}_j^2 = \hat{\beta}^2 &= \mathbf{1} \end{aligned} \quad (2)$$

and

$$p_j = -i\partial_j = -i\frac{\partial}{\partial x_j}, \quad 1 \leq j \leq D. \quad (3)$$

The orbital angular momentum operators L_{jk} , the spinor operators S_{jk} and the total angular momentum operators J_{jk} can be defined as follows:

$$\begin{aligned} L_{jk} = -L_{kj} &= ix_j \frac{\partial}{\partial x_k} - ix_k \frac{\partial}{\partial x_j}, & S_{jk} = -S_{kj} &= i\hat{\alpha}_j \hat{\alpha}_k / 2, & J_{jk} &= L_{jk} + S_{jk}. \\ L^2 &= \sum_{j < k}^D L_{jk}^2, & S^2 &= \sum_{j < k}^D S_{jk}^2, & J^2 &= \sum_{j < k}^D J_{jk}^2, & 1 \leq j < k \leq D. \end{aligned} \quad (4)$$

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For a spherically symmetric potential, total angular momentum operator J_{jk} and the spin-orbit operator $\hat{K} = -\hat{\beta}(J^2 - L^2 - S^2 + (D-1)/2)$ commute with the Dirac Hamiltonian. For a given total angular momentum j , the eigenvalues of \hat{K} are $\kappa = \pm(j + (D-2)/2)$; $\kappa = -(j + (D-2)/2)$ for aligned spin $j = \ell + \frac{1}{2}$ and $\kappa = (j + (D-2)/2)$ for unaligned spin $j = \ell - \frac{1}{2}$.

Thus, we can introduce the hyperspherical coordinates [35] as

$$\begin{aligned} x_1 &= r \cos \theta_1 \\ x_\alpha &= r \sin \theta_1 \dots \sin \theta_{\alpha-1} \cos \phi, \quad 2 \leq \alpha \leq D-1 \\ x_D &= r \sin \theta_1 \dots \sin \theta_{D-2} \sin \phi, \end{aligned} \quad (5)$$

where the volume element of the configuration space is given as

$$\prod_{j=1}^D dx_j = r^{D-1} dr d\Omega, \quad d\Omega = \prod_{j=1}^{D-1} (\sin \theta_j)^{j-1} d\theta_j \quad (6)$$

with $0 \leq r < \infty$, $0 \leq \theta_k \leq \pi$, $k = 1, 2, \dots, D-2$, $0 \leq \phi \leq 2\pi$, such that the spinor wave functions can be classified according to the hyperradial quantum number n_r and the spin-orbit quantum number κ and can be written using the Pauli–Dirac representation

$$\Psi_{n_r, \kappa}(r, \Omega_D) = r^{-(D-1)/2} \begin{pmatrix} F_{n_r, \kappa}(r) Y_{jm}^\ell(\Omega_D) \\ i G_{n_r, \kappa}(r) Y_{jm}^{\tilde{\ell}}(\Omega_D) \end{pmatrix}, \quad (7)$$

where $F_{n_r, \kappa}(r)$ and $G_{n_r, \kappa}(r)$ are the radial wave functions of the upper- and the lower-spinor components respectively, $Y_{jm}^\ell(\Omega_D)$ and $Y_{jm}^{\tilde{\ell}}(\Omega_D)$ are the hyperspherical harmonic functions coupled with the total angular momentum j . The orbital and the pseudo-orbital angular momentum quantum numbers for spin symmetry ℓ and pseudospin symmetry $\tilde{\ell}$ refer to the upper and lower components respectively.

Substituting eq. (7) into eq. (1), and separating the variables, we obtain the following coupled radial Dirac equation for the spinor components:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} \right) F_{n_r, \kappa}(r) = [\mu + E_{n_r, \kappa} - \Delta(r)] G_{n_r, \kappa}(r), \quad (8)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} \right) G_{n_r, \kappa}(r) = [\mu - E_{n_r, \kappa} + \Sigma(r)] F_{n_r, \kappa}(r), \quad (9)$$

where $\Delta(r) = V_v(r) - V_s(r)$, $\Sigma(r) = V_v(r) + V_s(r)$ and $\kappa = \pm(2\ell + D - 1)/2$. Further details of the derivation can be obtained from refs [36–38]. Using eq. (8) as the upper component and substituting into eq. (9), we obtain the following second-order differential equations:

$$\begin{aligned} &\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} - [\mu + E_{n_r, \kappa} - \Delta(r)][\mu - E_{n_r, \kappa} + \Sigma(r)] \right. \\ &\quad \left. + \frac{\frac{d\Delta(r)}{dr} \left(\frac{d}{dr} + \frac{\kappa}{r} \right)}{[\mu(r) + E_{n_r, \kappa} - \Delta(r)]} \right] F_{n_r, \kappa}(r) = 0, \end{aligned} \quad (10)$$

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - [\mu + E_{n_r\kappa} - \Delta(r)][\mu - E_{n_r\kappa} + \Sigma(r)] - \frac{\frac{d\Sigma(r)}{dr} \left(\frac{d}{dr} - \frac{\kappa}{r} \right)}{[\mu(r) - E_{n_r\kappa} + \Sigma(r)]} \right] G_{n_r\kappa}(r) = 0. \tag{11}$$

We note that the energy eigenvalues in these equations depend on the angular momentum quantum number ℓ and dimension D . However, to solve these equations, we shall use an approximation for the centrifugal barrier and obtain the solutions using the Nikiforov–Uvarov method.

3. The Nikiforov–Uvarov method

Next, we give a brief description of the conventional Nikiforov–Uvarov method. A more detailed description of the method can be obtained in [32]. With an appropriate transformation $s = s(r)$, the one-dimensional Schrödinger equation can be reduced to a generalized equation of hypergeometric type which can be written as follows:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{12}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second-degree and $\tilde{\tau}(s)$ is at most a first-order polynomial. To find particular solution of eq. (12) by separation of variables, if one deals with

$$\psi(s) = \phi(s)y_{n_r}(s), \tag{13}$$

eq. (12) becomes

$$\sigma(s)y_{n_r}'' + \tau(s)y_{n_r}' + \lambda y_{n_r} = 0, \tag{14}$$

where

$$\sigma(s) = \pi(s) \frac{\phi(s)}{\phi'(s)} \tag{15}$$

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s), \quad \tau'(s) < 0, \tag{16}$$

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + t\sigma}, \tag{17}$$

and

$$\lambda = t + \pi'(s). \tag{18}$$

The polynomial $\tau(s)$ with the parameter s and prime factors show the differentials at first degree to be negative. However, determination of parameter t is essential in the calculation of $\pi(s)$. It is simply defined by setting the discriminate of the square root to zero [32].

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Therefore, one gets a general quadratic equation for t . The values of t can be used for calculating the energy eigenvalues using the following equation:

$$\lambda = t + \pi'(s) = -n_r \tau'(s) - \frac{n_r(n_r - 1)}{2} \sigma''(s). \quad (19)$$

Furthermore, the other part $y_{n_r}(s)$ of the wave function in eq. (12) is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation:

$$y_{n_r}(s) = \frac{B_{n_r}}{\rho(s)} \frac{d^{n_r}}{ds^{n_r}} [\sigma^{n_r}(s) \rho(s)], \quad (20)$$

where B_{n_r} is a normalizing constant and the weight function $\rho(s)$ must satisfy the condition [32]

$$(\sigma\rho)' = \tau\rho. \quad (21)$$

4. Solutions to the radial equations in D dimensions

The Lorentz vector $V_v(r)$ and scalar $V_s(r)$ modified Pöschl–Teller potential can be defined as follows [24,39–41]:

$$V_v(r) = -\frac{V_0}{\cosh^2(\alpha r)} \quad \text{and} \quad V_s(r) = -\frac{S_0}{\cosh^2(\alpha r)}, \quad (22)$$

where α is related to the range of the potential and V_0 and S_0 are the depths of the vector and scalar potentials respectively. Moreover, we can approximate the centrifugal terms as follows [17,24]:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\sinh^2(\alpha r)}. \quad (23)$$

Substituting eqs (22) and (23) into eqs (10) and (11), we have

$$\left[\frac{d^2}{dr^2} - \frac{\alpha^2 \kappa(\kappa + 1)}{\sinh^2(\alpha r)} - [\mu + E_{n_r, \kappa} - \Delta(r)][\mu - E_{n_r, \kappa} + \Sigma(r)] + \frac{\frac{d\Delta(r)}{dr} \left(\frac{d}{dr} + \frac{\kappa}{r} \right)}{[\mu(r) + E_{n_r, \kappa} - \Delta(r)]} \right] F_{n_r, \kappa}(r) = 0, \quad (24)$$

$$\left[\frac{d^2}{dr^2} - \frac{\alpha^2 \kappa(\kappa - 1)}{\sinh^2(\alpha r)} - [\mu + E_{n_r, \kappa} - \Delta(r)][\mu - E_{n_r, \kappa} + \Sigma(r)] - \frac{\frac{d\Sigma(r)}{dr} \left(\frac{d}{dr} - \frac{\kappa}{r} \right)}{[\mu(r) - E_{n_r, \kappa} + \Sigma(r)]} \right] G_{n_r, \kappa}(r) = 0, \quad (25)$$

where

$$\Delta(r) = \frac{S_0 - V_0}{\cosh^2(\alpha r)} \quad \text{and} \quad \Sigma(r) = \frac{-(V_0 + S_0)}{\cosh^2(\alpha r)}. \quad (26)$$

For spin symmetry, $V_v(r) \sim V_s(r)$, i.e. $\Delta(r) = V_v(r) - V_s(r) = C_1$ (a constant), which implies that $(d\Delta(r)/dr) = 0$. Thus, putting this into eq. (24), we have

$$\left[\frac{d^2}{dr^2} - \frac{\alpha^2 \kappa (\kappa + 1)}{\sinh^2(\alpha r)} - (\mu - E_{n_r, \kappa})(\mu + E_{n_r, \kappa} - C_1) + \frac{(V_0 + S_0)(E_{n_r, \kappa} + \mu - C_1)}{\cosh^2(\alpha r)} \right] F_{n_r, \kappa}(r) = 0. \tag{27}$$

If we take the transformation $s = \tanh^2(\alpha r)$, eq. (27) becomes

$$F''_{n_r, \kappa}(s) + \frac{1 - 3s}{2s(1 - s)} F'_{n_r, \kappa}(s) + \frac{1}{4s^2(1 - s)^2} [-\delta s^2 + (\delta + \gamma - \epsilon^2)s - \gamma] F_{n_r, \kappa} = 0, \tag{28}$$

where

$$\epsilon^2 = \frac{(\mu - E_{n_r, \kappa})(\mu + E_{n_r, \kappa} - C_1)}{\alpha^2},$$

$$\delta = \frac{(V_0 + S_0)(E_{n_r, \kappa} + \mu - C_1)}{\alpha^2}$$

and

$$\gamma = \kappa(\kappa + 1). \tag{29}$$

Comparing eqs (28) and (12) we can define the following:

$$\tilde{\tau}(s) = 1 - 3s, \quad \sigma(s) = 2s(1 - s)$$

and

$$\tilde{\sigma}(s) = -\delta s^2 + (\gamma + \delta - \epsilon^2)s - \gamma. \tag{30}$$

Inserting these into eq. (17), we have the following function:

$$\pi(s) = \frac{1 - s}{2} \pm \frac{1}{2} \sqrt{(1 + 4\delta - 8t)s^2 + (8t - 4(\gamma + \delta - \epsilon^2) - 2)s + 4\gamma + 1}. \tag{31}$$

The constant parameter t can be found by the condition that the discriminant of the expression under the square root has a double root, i.e., its discriminant is zero. Thus, the possible value function for each value of t is given as

$$\pi(s) = \frac{1 - s}{2} \pm \begin{cases} \frac{1}{2} [(-2\epsilon + \sqrt{1 + 4\gamma})s - \sqrt{1 + 4\gamma}] & \text{for } t = -\frac{1}{2}(\gamma - \delta + \epsilon^2) + \frac{1}{2}\epsilon\sqrt{1 + 4\gamma} \\ \frac{1}{2} [(2\epsilon + \sqrt{1 + 4\gamma})s - \sqrt{1 + 4\gamma}] & \text{for } t = -\frac{1}{2}(\gamma - \delta + \epsilon^2) - \frac{1}{2}\epsilon\sqrt{1 + 4\gamma} \end{cases}. \tag{32}$$

By Nikiforov–Uvarov method, we made an appropriate choice of the function $\pi(s) = \frac{1-s}{2} - \frac{1}{2} [(2\epsilon + \sqrt{1 + 4\gamma})s - \sqrt{1 + 4\gamma}]$ such that by eq. (19), the eigenvalue equation can be obtained to be

$$-\frac{1}{2}(\gamma - \delta + \epsilon^2) - \frac{1}{2}\epsilon\sqrt{1 + 4\gamma} - \frac{1}{2}(2\epsilon + \sqrt{1 + 4\gamma}) - \frac{1}{2} = n_r [4 + 2\epsilon + \sqrt{1 + 4\gamma}] + 2n_r(n_r - 1). \tag{33}$$

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Equation (33) can be written in the powers of ϵ as follows:

$$\epsilon^2 + \epsilon \left[2(2n_r + 1) + \sqrt{1 + 4\gamma} \right] + (\gamma - \delta) + \left[(1 + 2n_r) + \sqrt{1 + 4\gamma} \right] = 0, \quad (34)$$

such that we can obtain

$$-\epsilon^2 = -\frac{1}{4} \left[-2(2n_r + 1) - \sqrt{1 + 4\gamma} + \sqrt{1 + 4\delta} \right]^2, \quad (35)$$

from which we can obtain a rather complicated transcendental energy equation:

$$\begin{aligned} & (\mu - E_{n_r\kappa})(\mu + E_{n_r\kappa} - C_1) \\ &= \frac{\alpha^2}{4} \left[2(2n_r + 1) + (2\kappa + 1) - \frac{1}{\alpha} \sqrt{\alpha^2 + 4(V_0 + S_0)(E_{n_r\kappa} + \mu - C_1)} \right]^2. \end{aligned} \quad (36)$$

If we define a principal quantum number $n = 2n_r + \ell + 1$, eq. (36) becomes

$$(\mu - E_n)(\mu + E_n - C_1) = \frac{\alpha^2}{4} \left[2n + D - \frac{1}{\alpha} \sqrt{\alpha^2 + 4(V_0 + S_0)(E_n + \mu - C_1)} \right]^2, \quad (37)$$

where we have chosen $\kappa = (2\ell + D - 1)/2$ and $n = 1, 2, 3, \dots$. Some numerical values of the energy levels $E(\alpha, n, D)$ for some dimensions and excited states are given in table 1.

Table 1. The bound-state energy levels E_n are shown for spin symmetry. The numerical results show that the energy levels increase with both the dimensions D and the range parameter α .

		$E(\alpha, n, D)$			
		$C_1 = V_0 = S_0 = \mu = 1$			
D	n	$\alpha = 0.0001$	$\alpha = 0.001$	$\alpha = 0.005$	$\alpha = 0.01$
3	1	4.0032×10^{-8}	4.0326×10^{-6}	1.0442×10^{-4}	4.4016×10^{-4}
	2	9.0108×10^{-8}	9.1118×10^{-6}	2.4161×10^{-4}	1.1130×10^{-3}
	3	1.6026×10^{-7}	1.6271×10^{-5}	4.4842×10^{-4}	–
	4	2.5050×10^{-7}	2.5544×10^{-5}	–	–
	5	3.6087×10^{-7}	3.6973×10^{-5}	–	–
4	1	6.2562×10^{-8}	6.3142×10^{-6}	1.6530×10^{-4}	7.1490×10^{-4}
	2	1.2267×10^{-7}	1.2429×10^{-5}	3.3488×10^{-4}	–
	3	2.0287×10^{-7}	2.0641×10^{-5}	6.0121×10^{-4}	–
	4	3.0317×10^{-7}	3.0955×10^{-5}	–	–
	5	4.2361×10^{-7}	4.3513×10^{-5}	–	–
5	1	9.0108×10^{-8}	9.1118×10^{-6}	2.4161×10^{-4}	1.1131×10^{-3}
	2	1.6026×10^{-7}	1.6270×10^{-5}	4.4842×10^{-4}	–
	3	2.5050×10^{-7}	2.5541×10^{-5}	–	–
	4	3.6087×10^{-7}	3.6973×10^{-5}	–	–
	5	4.9139×10^{-7}	5.0616×10^{-5}	–	–

We now obtain the spinor components of the wavefunction for the spin symmetry using the Nikiforov–Uvarov method. By substituting $\pi(s)$ and $\sigma(s)$ into eq. (15) and solving the first-order differential equation, we have

$$\phi(s) = s^{(\kappa+1)/2}(1-s)^{\epsilon/2}. \tag{38}$$

Also using eq. (21), the weight function $\rho(s)$ can be obtained as

$$\rho(s) = \frac{1}{2}s^{(2\kappa-1)/2}(1-s)^\epsilon. \tag{39}$$

Substituting eq. (39) into the Rodrigues relation (20), we have

$$y_{n_r}(s) = B_{n_r} s^{-(2\kappa-1)/2}(1-s)^{-\epsilon} \frac{d^{n_r}}{ds^{n_r}} [s^{n_r+(2\kappa-1)/2}(1-s)^{n_r+\epsilon}]. \tag{40}$$

Therefore, we can write the upper component $F_{n_r,\kappa}(s)$ as

$$F_{n_r,\kappa}(s) = C_{n_r} s^{(\kappa+1)/2}(1-s)^{\epsilon/2} P_{n_r}^{((2\kappa-1)/2, \epsilon)}(1-2s), \tag{41}$$

where C_{n_r} is the normalization constant, and we have used the definition of the Jacobi polynomials [42], given as

$$P_n^{(a, b)}(s) = \frac{(-1)^n}{n!2^n(1-s)^a(1+s)^b} \frac{d^n}{ds^n} [(1-s)^{a+n}(1+s)^{b+n}]. \tag{42}$$

Using eq. (8) the lower-component can be obtained as follows:

$$G_{n_r,\kappa}(s) = A_1(s) P_{n_r}^{((2\kappa-1)/2, \epsilon)}(1-2s) + A_2(s) P_{n_r-1}^{(2\kappa+1)/2, \epsilon/2+1)}(1-2s), \tag{43}$$

where

$$A_1(s) = \frac{C_{n_r} \alpha s^{\kappa/2}(1-s)^{\epsilon/2} \left[\left(\frac{\kappa+1}{2}\right)(1-s) - \frac{\epsilon}{2}s \right] + C_{n_r} \frac{\alpha\kappa}{\tanh^{-1}(\sqrt{s})}}{\mu + E_{n_r,\kappa} - C_1}$$

and

$$A_2(s) = \frac{D_{n_r} \alpha s^{(\kappa+2)/2}(1-s)^{(\epsilon+2)/2}}{\mu + E_{n_r,\kappa} - C_1} \tag{44}$$

with constant D_{n_r} defined by

$$D_{n_r} = \frac{2\epsilon + 2n_r + 2\kappa + 1}{4} \times C_{n_r}. \tag{45}$$

Moreover, to compute the normalization constant C_{n_r} , it is easy to show that

$$\begin{aligned} & \int_0^\infty \left| r^{-\frac{(D-1)}{2}} F_{n_r,\kappa}(r) \right|^2 r^{D-1} dr \\ &= \int_0^\infty |F_{n_r,\kappa}(r)|^2 dr = \int_0^1 |F_{n_r,\kappa}(s)|^2 \frac{ds}{2\alpha\sqrt{s}(1-s)} = 1, \end{aligned} \tag{46}$$

where we have also used the substitution $s = \tanh^2(\alpha r)$. Putting eq. (41) into eq. (46) and using the following definition of the Jacobi polynomial [42]:

$$P_n^{(a,b)}(s) = \frac{\Gamma(n+a+1)}{n!\Gamma(1+a)} {}_2F_1\left(-n, a+b+n+1; 1+a; \frac{1-s}{2}\right), \tag{47}$$

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we arrived at

$$C_{n_r}^2 N_{n_r} \int_0^1 s^{\kappa+\frac{1}{2}} (1-s)^{\epsilon-1} [{}_2F_1(-n_r, \kappa + \epsilon + n_r + 1/2; \kappa + 1/2; s)]^2 ds = \alpha, \quad (48)$$

where $N_{n_r} = \frac{1}{2} \left[\frac{\Gamma(n_r + \kappa + 1/2)}{n_r! \Gamma(\kappa + 1/2)} \right]^2$ and ${}_2F_1$ is the hypergeometric function. Using the following series representation of the hypergeometric function

$${}_pF_q(a_1, \dots, a_p; c_1, \dots, c_q; s) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n s^n}{(c_1)_n \dots (c_q)_n n!} \quad (49)$$

we have

$$C_{n_r}^2 N_{n_r} \sum_{i=0}^{n_r} \sum_{j=0}^{n_r} \frac{(-n_r)_i (\kappa + \epsilon + n_r + 1/2)_i}{(\kappa + 1/2)_i i!} \frac{(-n_r)_j (\kappa + \epsilon + n_r + 1/2)_j}{(\kappa + 1/2)_j j!} \times \int_0^1 s^{\kappa+i+j+\frac{1}{2}} (1-s)^{\epsilon-1} ds = \alpha. \quad (50)$$

Hence, by the definition of the beta function, eq. (43) becomes

$$C_{n_r}^2 N_{n_r} \sum_{i=0}^{n_r} \sum_{j=0}^{n_r} \frac{(-n_r)_i (\kappa + \epsilon + n_r + 1/2)_i}{\kappa + 1/2)_i i!} \frac{(-n_r)_j (\kappa + \epsilon + n_r + 1/2)_j}{(\kappa + 1/2)_j j!} \times B\left(\kappa + i + j + \frac{3}{2}, \epsilon\right) = \alpha. \quad (51)$$

Using the relations $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ and the Pochhammer symbol $(a)_n = \Gamma(a + n)/\Gamma(a)$, eq. (51) can be written as

$$C_{n_r}^2 N_{n_r} \sum_{i=0}^{n_r} \frac{(-n_r)_i (\kappa + \epsilon + n_r + 1/2)_i (\kappa + \frac{3}{2})_i}{(\epsilon + \kappa + \frac{3}{2})_i (\kappa + 1/2)_i i!} \times \sum_{j=0}^{n_r} \frac{(-n_r)_j (\kappa + \epsilon + n_r + 1/2)_j (\kappa + i + \frac{3}{2})_j}{(\epsilon + \kappa + i + \frac{3}{2})_j (\kappa + 1/2)_j j!} = \frac{\alpha}{B(\kappa + \frac{3}{2}, \epsilon)}. \quad (52)$$

Lastly, eq. (52) can be used to compute the normalization constants for $n_r = 0, 1, 2, \dots$. In particular, for the ground state, i.e. $n_r = 0$, we have

$$C_0 = \sqrt{\frac{2\alpha}{B(\kappa + \frac{3}{2}, \epsilon)}}. \quad (53)$$

5. Concluding remarks

In conclusion, the solutions of the Dirac equation with spin symmetry for the modified Pöschl–Teller potential has been extended to a multidimensional case. The energy levels and the spinor components of the wave function were obtained using the Nikiforov–Uvarov method. We also obtained the normalization constants in the form of hypergeometric series.

Numerical results show that there are only positive-energy states for bound states with spin symmetry. Also, the energy levels with fixed value of n increase with the dimension of the space time and the potential range parameter α . Moreover, the existence of the degenerate states between $E(\alpha, n + 1, D)$ and $E(\alpha, n, D + 2)$ indicate that the energy levels can be completely determined using the ground state.

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