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Application of the (G'/G)-expansion method for the Burgers, Burgers–Huxley and modified Burgers–KdV equations

H KHEIRI*, M R MOGHADDAM and V VAFAEI

Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

*Corresponding author. E-mail: h-kheiri@tabrizu.ac.ir

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Abstract. In this work, we present travelling wave solutions for the Burgers, Burgers–Huxley and modified Burgers–KdV equations. The (G'/G)-expansion method is used to determine travelling wave solutions of these sets of equations. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. It is shown that the proposed method is direct, effective and can be used for many other nonlinear evolution equations in mathematical physics.

Keywords. (G'/G)-expansion method; Burgers equation; Burgers–Huxley equation; modified Burgers–KdV equation; travelling wave solutions.

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1. Introduction

Most of the phenomena in real world can be described using nonlinear equations. In recent decades, many effective methods for obtaining exact solutions of nonlinear evolution equations (NLEEs), such as Painleve method [1], Jacobi elliptic function method [2], Hirota's bilinear method [3], the sine-cosine function method [4], the tanh-coth function method [5], the exp-function method [6], the homogeneous balance method [7] and so on have been presented.

Recently, Wang *et al* [8] proposed the (G'/G)-expansion method to find travelling wave solutions of NLEEs. Bekir [9] and Aslan [10] applied this method to obtain travelling wave solutions of some NLEEs. More recently, some authors [11,12] applied this method to improve and extend Wang *et al*'s work [8] to solve variable coefficient and high-dimensional equations. Zhang *et al* [13] devised an algorithm for using the method to solve nonlinear differential difference equations. Yu-Bin *et al* [14] modified the method to derive travelling wave solutions for Whitham–Broer–Kaup-Like equations. Zhang [15] solved the equations with the balance numbers which are not positive integers, by this method. For studying the Vakhnenko equation, Wen-An *et al* [16] presented a new function expansion method which can be thought of as the generalization of the (G'/G)-expansion method. Kheiri *et al* [17] applied this method for solving the combined and the double combined sinh-cosh-Gordon equations.

In this work, we apply the (G'/G)-expansion method to solve the Burgers, Burgers–Huxley and modified Burgers–KdV equations (mBKdV). The Burgers equation appears in various areas of applied mathematics, such as modelling of fluid dynamics, turbulence, boundary layer behaviour, shock wave formation and traffic flow. The Burgers–Huxley equation can be regarded as a model to describe the interaction between reaction mechanisms, convection effects and diffusion transports [18–20]. Many physical problems can be described by Burger–KdV and mBKdV equations. Typical examples are provided by the behaviour of long waves in shallow water and waves in plasmas. Mcintosh [21] demonstrated how to describe the average behaviour of travelling wave solution of mBKdV in the case of small dissipation.

2. Description of the (G'/G)-expansion method

We assume the given nonlinear partial differential equation for $u(x, t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{1}$$

where P is a polynomial in its arguments. The essence of the (G'/G)-expansion method can be presented in the following steps:

Step 1. Find travelling wave solutions of eq. (1) by taking $u(x, t) = U(\xi)$, $\xi = x - ct$ and transform eq. (1) to the ordinary differential equation

$$Q(U, U', U'', \ldots) = 0, \tag{2}$$

where prime denotes the derivative with respect to ξ.

Step 2. If possible, integrate eq. (2) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

Step 3. Introduce the solution $U(\xi)$ of eq. (2) in the finite series form

$$U(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i, \tag{3}$$

where a_i are real constants with $a_N \neq 0$ to be determined, N is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4}$$

where λ and μ are real constants to be determined.

Step 4. Determine N. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in eq. (2).

Step 5. Substituting (3) together with (4) into eq. (2) yields an algebraic equation involving powers of (G'/G). Equating the coefficients of each power of (G'/G) to zero gives a system of algebraic equations for a_i, λ, μ and c. Then, we solve the system with the aid of a computer algebra system (CAS), such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the solutions of eq. (4) are well known for us. So, we can obtain exact solutions of eq. (1).

3. Applications

In this section, we apply the (G'/G)-expansion method to solve the Burgers, Burgers–Huxley and modified Burgers–KdV equations.

3.1 *The Burgers equation*

The Burgers equation is presented as

$$u_t + uu_x = u_{xx}. \tag{5}$$

We make the transformation $u(x,t) = U(\xi)$, $\xi = x - ct$. Then we get

$$-cU' + UU' - U'' = 0, \tag{6}$$

where prime denotes the derivative with respect to ξ.

By one time integrating with respect to ξ, eq. (6) becomes

$$-cU + \frac{1}{2}U^2 - U' + D = 0, \tag{7}$$

where D is the integration constant. Balancing U' with U^2 gives $N = 1$. Therefore, we can write the solution of eq. (7) in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \qquad a_1 \neq 0. \tag{8}$$

By eqs (4) and (8) we derive

$$U^2(\xi) = a_1^2\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + a_0^2, \tag{9}$$

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu. \tag{10}$$

Substituting eqs (8)–(10) into eq. (7), setting the coefficients of $(G'/G)^i$ $(i = 0, 1, 2)$ to zero, we obtain a system of algebraic equations for a_0, a_1, c, λ and μ as follows:

$$\left(\frac{G'}{G}\right)^0: \quad -ca_0 + \frac{1}{2}a_0^2 + a_1\mu + D = 0,$$

$$\left(\frac{G'}{G}\right)^1: \quad -ca_1 + a_0a_1 + a_1\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^2: \quad \frac{1}{2}a_1^2 + a_1 = 0. \tag{11}$$

Solving this system by Maple gives

$$a_0 = -\lambda \pm \sqrt{\lambda^2 - 4\mu + 2D}, \quad a_1 = -2, \quad c = \pm\sqrt{\lambda^2 - 4\mu + 2D}. \tag{12}$$

Substituting the solution set (12) and the corresponding solutions of (4) into (8), we have the solutions of eq. (7) as follows:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions

$$\begin{aligned} U_1(\xi) &= \pm\sqrt{\lambda^2 - 4\mu + 2D} - \sqrt{\lambda^2 - 4\mu} \\ &\quad \times \left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi} \right), \end{aligned} \tag{13}$$

where $\xi = x \mp \sqrt{\lambda^2 - 4\mu + 2D}t$.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions

$$\begin{aligned} U_2(\xi) &= \pm\sqrt{\lambda^2 - 4\mu + 2D} - \sqrt{4\mu - \lambda^2} \\ &\quad \times \left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi} \right), \end{aligned} \tag{14}$$

where $\xi = x \mp \sqrt{\lambda^2 - 4\mu + 2D}t$.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions

$$U_3(\xi) = \pm\sqrt{2D} - \frac{2C_2}{C_1 + C_2\xi}, \tag{15}$$

where $\xi = x \mp \sqrt{2D}t$.

In solutions $U_i(\xi)$ $(i = 1, 2, 3)$, C_1 and C_2 are left as free parameters. It is obvious that hyperbolic, trigonometric and rational solutions were obtained by using the (G'/G)-expansion method, whereas only hyperbolic solutions were obtained in [18] and hyperbolic and trigonometric solutions in [19].

In particular, if we take $C_1 \neq 0$ and $C_2 = D = 0$, then U_1 becomes

$$U_1(\xi) = c\left(1 - \tanh\left(\frac{c}{2}\xi\right)\right), \tag{16}$$

and U_2 becomes

$$U_2(\xi) = c\left(1 + \tan\left(\frac{c}{2}\xi\right)\right). \tag{17}$$

We observe that the results (15)–(17) in Wazwaz [19] are particular cases of our results (13) and (14). Then our solutions are more general. The behaviour of exact travelling wave solutions of eq. (5) are shown in figure 1.

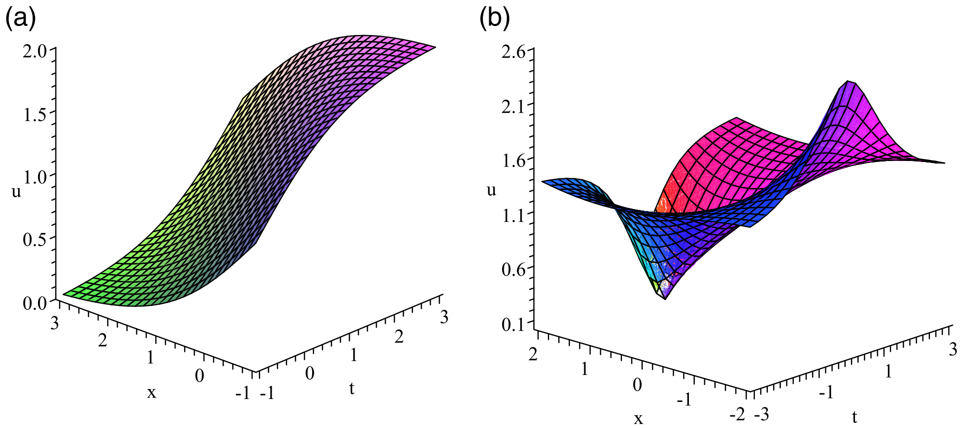

Figure 1. The graphs of exact travelling wave solutions of (**a**) eq. (16) and (**b**) eq. (17).

3.2 *The Burgers–Huxley equation*

Now, let us consider the following Burgers–Huxley equation in the form

$$u_t = u_{xx} + uu_x + u(k-u)(u-1), \quad k \neq 0. \tag{18}$$

We make the transformation $u(x,t) = U(\xi)$, $\xi = x - ct$. Then we get

$$cU' + UU' + U'' + U(k-U)(U-1) = 0, \tag{19}$$

where prime denotes the derivative with respect to ξ. Balancing U'' with U^3 gives $N = 1$. Therefore, we can write the solution of eq. (19) in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{20}$$

By using eqs (4) and (19) we have

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu, \tag{21}$$

$$U''(\xi) = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + (a_1\lambda^2 + 2a_1\mu)\left(\frac{G'}{G}\right) + a_1\lambda\mu, \tag{22}$$

$$U^2(\xi) = a_1^2\left(\frac{G'}{G}\right)^2 + 2a_0a_1\left(\frac{G'}{G}\right) + a_0^2, \tag{23}$$

$$U^3(\xi) = a_1^3\left(\frac{G'}{G}\right)^3 + 3a_0a_1^2\left(\frac{G'}{G}\right)^2 + 3a_0^2a_1\left(\frac{G'}{G}\right) + a_0^3. \tag{24}$$

Substituting eqs (20)–(24) into (19), setting coefficients of $(G'/G)^i$ $(i = 0, 1, 2, 3)$ to zero, we obtain a system of nonlinear algebraic equations a_0, a_1, c, λ and μ as follows:

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^0 : \; a_1\lambda\mu - a_0^3 + a_0^2 - ca_1\mu - a_0a_1\mu + ka_0^2 - ka_0 = 0,\\
&\left(\frac{G'}{G}\right)^1 : \; a_1\lambda^2 + 2a_1\mu + 2a_0a_1 - 3a_0^2a_1 - a_1^2\mu - ka_1 - ca_1\lambda - a_0a_1\lambda + 2ka_0a_1 = 0,\\
&\left(\frac{G'}{G}\right)^2 : \; a_1^2 + 3a_1\lambda - 3a_0a_1^2 - ca_1 - a_0a_1 - a_1^2\lambda + ka_1^2 = 0,\\
&\left(\frac{G'}{G}\right)^3 : \; 2a_1 - a_1^3 - a_1^2 = 0.
\end{aligned} \tag{25}$$

Solving this system by Maple gives

$$a_0 = \frac{\lambda+1}{2}, \quad a_1 = 1, \quad c = k-1, \quad \mu = \frac{\lambda^2-1}{4}, \tag{26}$$

$$a_0 = \frac{\lambda+k}{2}, \quad a_1 = 1, \quad c = 1-k, \quad \mu = \frac{\lambda^2-k^2}{4}, \tag{27}$$

$$a_0 = \frac{\lambda+k+1}{2}, \quad a_1 = 1, \quad c = -k-1, \quad \mu = \frac{\lambda^2-(k-1)^2}{4}, \tag{28}$$

$$a_0 = \frac{1}{2} - \lambda, \quad a_1 = -2, \quad c = \frac{1-4k}{2}, \quad \mu = \frac{1}{4}\left(\lambda^2 - \frac{1}{4}\right), \tag{29}$$

$$a_0 = \frac{k}{2} - \lambda, \quad a_1 = -2, \quad c = \frac{k-4}{2}, \quad \mu = \frac{1}{4}\left(\lambda^2 - \frac{k^2}{4}\right), \tag{30}$$

$$a_0 = \frac{k+1}{2} - \lambda, \quad a_1 = -2, \quad c = \frac{k+1}{2}, \quad \mu = \frac{1}{4}\left(\lambda^2 - \frac{(k-1)^2}{4}\right). \tag{31}$$

Substituting the solutions set (26)–(31) and the corresponding solutions of eq. (4) into eq. (20), we have the solutions of eq. (19) as follows:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions

$$U_1(\xi) = \frac{1}{2} + \frac{1}{2}\left(\frac{C_1 \sinh \frac{1}{2}\xi + C_2 \cosh \frac{1}{2}\xi}{C_1 \cosh \frac{1}{2}\xi + C_2 \sinh \frac{1}{2}\xi}\right), \tag{32}$$

where $\xi = x - (k-1)t$ and

$$U_2^{\pm}(\xi) = \frac{k}{2} + \frac{|k|}{2}\left(\frac{C_1 \sinh \frac{|k|}{2}\xi + C_2 \cosh \frac{|k|}{2}\xi}{C_1 \cosh \frac{|k|}{2}\xi + C_2 \sinh \frac{|k|}{2}\xi}\right), \tag{33}$$

where $\xi = x - (1-k)t$, the solution $U_2^+(\xi)$ (resp. $U_2^-(\xi)$) corresponds to $k > 0$ (resp. $k < 0$) and

$$U_3^{\pm}(\xi) = \frac{k+1}{2} + \frac{|k-1|}{2}\left(\frac{C_1 \sinh \frac{|k-1|}{2}\xi + C_2 \cosh \frac{|k-1|}{2}\xi}{C_1 \cosh \frac{|k-1|}{2}\xi + C_2 \sinh \frac{|k-1|}{2}\xi}\right), \tag{34}$$

where $\xi = x+(k+1)t$, the solution $U_3^+(\xi)$ (resp. $U_3^-(\xi)$) corresponds to $k \geq 1$ (resp. $k < 1$) and

$$U_4(\xi) = \frac{1}{2} - \frac{1}{2}\left(\frac{C_1 \sinh \frac{1}{4}\xi + C_2 \cosh \frac{1}{4}\xi}{C_1 \cosh \frac{1}{4}\xi + C_2 \sinh \frac{1}{4}\xi}\right), \tag{35}$$

where $\xi = x - \frac{1-4k}{2}t$, and

$$U_5^{\pm}(\xi) = \frac{k}{2} - \frac{|k|}{2}\left(\frac{C_1 \sinh \frac{|k|}{4}\xi + C_2 \cosh \frac{|k|}{4}\xi}{C_1 \cosh \frac{|k|}{4}\xi + C_2 \sinh \frac{|k|}{4}\xi}\right), \tag{36}$$

where $\xi = x - \frac{k-4}{2}t$, the solution $U_5^+(\xi)$ (resp. $U_5^-(\xi)$) corresponds to $k > 0$ (resp. $k < 0$) and

$$U_6^{\pm}(\xi) = \frac{k+1}{2} - \frac{|k-1|}{2}\left(\frac{C_1 \sinh \frac{|k-1|}{4}\xi + C_2 \cosh \frac{|k-1|}{4}\xi}{C_1 \cosh \frac{|k-1|}{4}\xi + C_2 \sinh \frac{|k-1|}{4}\xi}\right), \tag{37}$$

where $\xi = x - \frac{k+1}{2}t$, the solution $U_6^+(\xi)$ (resp. $U_6^-(\xi)$) corresponds to $k \geq 1$ (resp. $k < 1$).

When $\lambda^2 - 4\mu = 0$, according to eqs (28) and (31) we have $k = 1$. Hence, we obtain the rational function solutions

$$U_7(\xi) = 1 + \frac{C_2}{C_1 + C_2\xi}, \tag{38}$$

where $\xi = x + 2t$, and

$$U_8(\xi) = 1 - \frac{2C_2}{C_1 + C_2\xi}, \tag{39}$$

where $\xi = x - t$.

In solutions $U_i(\xi)$ $(i = 1, \ldots, 8)$, C_1 and C_2 are left as free parameters. It is obvious that hyperbolic and rational solutions are obtained by using the (G'/G)-expansion method, whereas only hyperbolic solutions were obtained in [19].

In particular, if we take $C_1 \neq 0$ and $C_2 = 0$, then U_i $(i = 1, \ldots, 6)$ become

$$U_1(\xi) = \frac{1}{2}\left(1 + \tanh\left(\frac{1}{2}\xi\right)\right), \tag{40}$$

$$U_2(\xi) = \frac{k}{2}\left(1 + \tanh\left(\frac{k}{2}\xi\right)\right), \tag{41}$$

$$U_3(\xi) = \frac{k+1}{2} + \frac{k-1}{2}\tanh\left(\frac{k-1}{2}\xi\right), \tag{42}$$

$$U_4(\xi) = \frac{1}{2}\left(1 - \tanh\left(\frac{1}{4}\xi\right)\right), \tag{43}$$

$$U_5(\xi) = \frac{k}{2}\left(1 - \tanh\left(\frac{k}{4}\xi\right)\right), \tag{44}$$

$$U_6(\xi) = \frac{k+1}{2} - \frac{k-1}{2}\tanh\left(\frac{k-1}{4}\xi\right). \tag{45}$$

For comparison, we observe that our solutions (32)–(37) include the solutions (40)–(42) of Wazwaz [19]. Therefore, our solutions contain the results of [19]. The behaviour of exact travelling wave solutions are shown in figure 2.

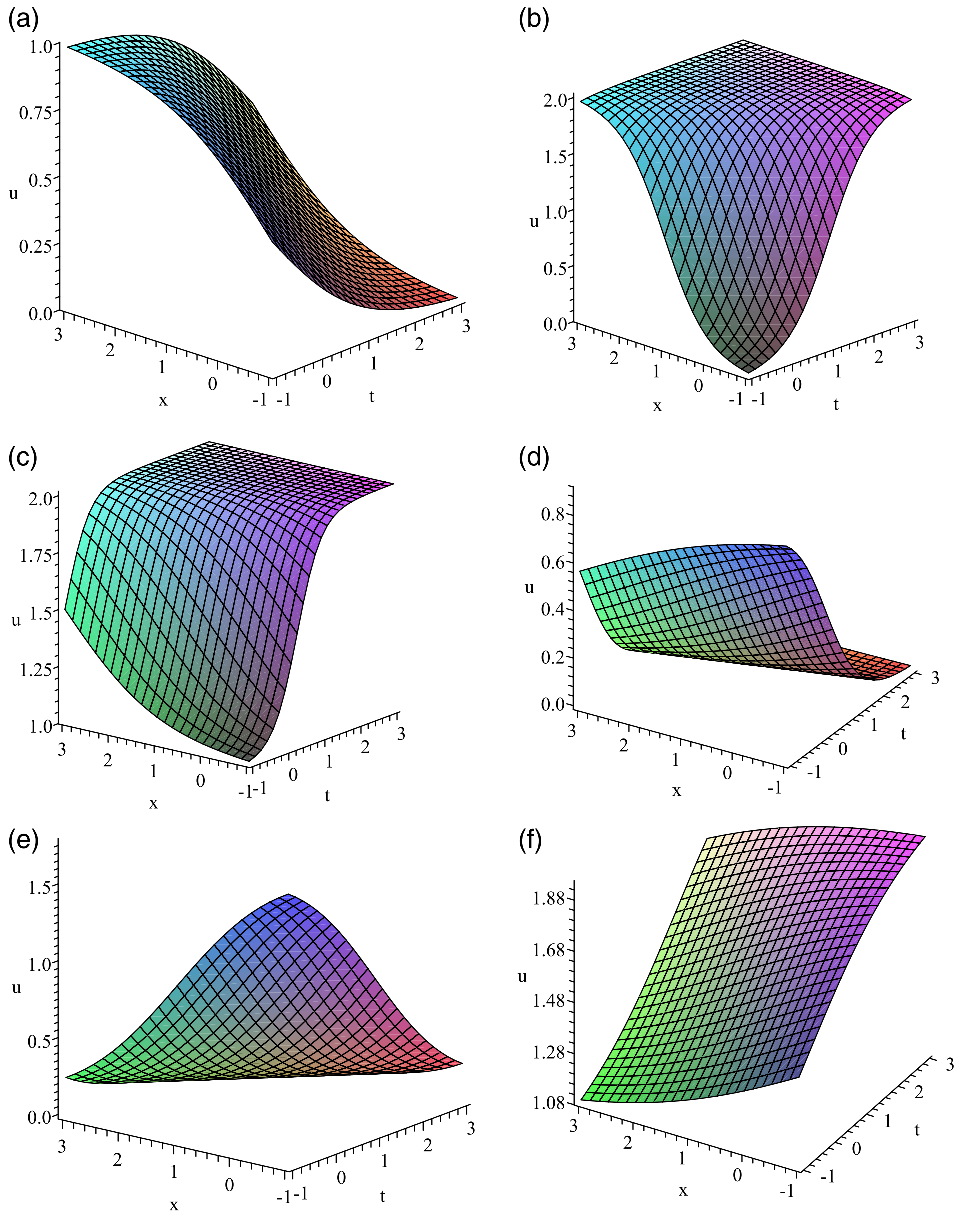

Figure 2. The graphs of exact travelling wave solutions of (**a**) eq. (40), (**b**) eq. (41), (**c**) eq. (42), (**d**) eq. (43), (**e**) eq. (44) and (**f**) eq. (45) with $k = 2$.

3.3 *The modified Burgers–KdV equation*

We next consider the modified Burgers–KdV equation

$$u_t + pu^2u_x + qu_{xx} - ru_{xxx} = 0, \tag{46}$$

where p, q and r are real constants. When $q = 0$, the modified Burgers–KdV equation reduces to the modified KdV equation. During the past several years, many have done research on travelling wave solution of the mBKdV equation. Mcintosh [21] demonstrated how to describe the average behaviour of travelling wave solution of eq. (46) during small dissipation. Jacobs and co-workers investigated the limit when r and q approached zero and the ratio r/q^2 remained constant, thus balancing the dissipation and dispersion in balance [22]. In the limit, it was shown that the travelling wave solutions of eq. (46) approach a shock wave solution. To determine the travelling wave solution of eq. (46), we make the transformation $u(x,t) = U(\xi)$, $\xi = x - ct$. Then we get

$$-cU' + pU^2U' + qU'' - rU''' = 0. \tag{47}$$

By integration with respect to ξ in eq. (47), we get

$$-cU + \frac{p}{3}U^3 + qU' - rU'' = 0. \tag{48}$$

Balancing U'' with U^3 gives $N = 1$. Therefore, we can write the solution of eq. (48) in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{49}$$

By using eqs (4) and (48) we have

$$U'(\xi) = -a_1\left(\frac{G'}{G}\right)^2 - a_1\lambda\left(\frac{G'}{G}\right) - a_1\mu, \tag{50}$$

$$U''(\xi) = 2a_1\left(\frac{G'}{G}\right)^3 + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + (a_1\lambda^2 + 2a_1\mu)\left(\frac{G'}{G}\right) + a_1\lambda\mu, \tag{51}$$

$$U^3(\xi) = a_1^3\left(\frac{G'}{G}\right)^3 + 3a_0a_1^2\left(\frac{G'}{G}\right)^2 + 3a_0^2a_1\left(\frac{G'}{G}\right) + a_0^3. \tag{52}$$

Substituting eqs (49)–(52) into (48), setting coefficients of $(G'/G)^i$ ($i = 0, 1, 2, 3$) to zero, we obtain a system of nonlinear algebraic equations a_0, a_1, c, λ and μ as follows:

$$\left(\frac{G'}{G}\right)^0: \quad -ca_0 + \frac{1}{3}pa_0^3 - qa_1\mu - ra_1\lambda\mu = 0, \tag{53}$$

$$\left(\frac{G'}{G}\right)^1: \quad -ca_1 + pa_0^2a_1 - qa_1\lambda - ra_1\lambda^2 - 2ra_1\mu = 0, \tag{54}$$

$$\left(\frac{G'}{G}\right)^2: \quad pa_0a_1^2 - qa_1 - 3ra_1\lambda = 0, \tag{55}$$

$$\left(\frac{G'}{G}\right)^3: \quad \frac{1}{3}pa_1^3 - 2ra_1 = 0. \tag{56}$$

Solving this system by Maple gives

$$a_0 = \pm\frac{q+3r\lambda}{\sqrt{6rp}}, \quad a_1 = \pm\sqrt{\frac{6r}{p}}, \quad c = \frac{2q^2}{9r},$$
$$\mu = \frac{1}{4}\left(\lambda^2 - \left(\frac{q}{3r}\right)^2\right), \quad rp > 0. \tag{57}$$

Substituting the solution set (57) and the corresponding solutions of (4) into (49), we have the solutions of eq. (48) as follows:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions

$$U_1^{\pm}(\xi) = \pm\sqrt{\frac{r}{6p}}\left(\frac{q}{r} + \left|\frac{q}{r}\right|\left(\frac{C_1\sinh\frac{1}{6}|\frac{q}{r}|\xi + C_2\cosh\frac{1}{6}|\frac{q}{r}|\xi}{C_1\cosh\frac{1}{6}|\frac{q}{r}|\xi + C_2\sinh\frac{1}{6}|\frac{q}{r}|\xi}\right)\right), \tag{58}$$

where $\xi = x - \frac{2q^2}{9r}t$, the solutions $U_1^+(\xi)$ (resp. $U_1^-(\xi)$) corresponds to $rq > 0$ (resp. $rq < 0$).

When $\lambda^2 - 4\mu = 0$, according to eq. (57) we have $q = 0$. Hence, we obtain the rational function solutions

$$U_2(\xi) = \pm\sqrt{\frac{6r}{p}}\frac{C_2}{C_1 + C_2\xi}, \tag{59}$$

where $\xi = x$, for modified KdV equation. In solutions $U_i(\xi)$ $(i = 1, 2)$, C_1 and C_2 are left as free parameters. It is obvious that hyperbolic and rational solutions are obtained by using the (G'/G)-expansion method.

Using different values for C_1, C_2, p, q and r we can obtain new solutions. For instance, if we take $C_1 \neq 0$, $C_2 = 0$ and $q = 6r$ then U_1^+ becomes

$$U_1^+(\xi) = \pm\sqrt{\frac{6r}{p}}\,(1 + \tanh\xi), \quad \xi = x - 8rt. \tag{60}$$

If we take $C_1 = 0$, $C_2 \neq 0$ and $q = 6r$ then U_1^+ becomes

$$U_1^+(\xi) = \pm\sqrt{\frac{6r}{p}}\,(1 + \coth\xi), \quad \xi = x - 8rt. \tag{61}$$

If we take $C_1 = 0$, $C_2 \neq 0$ and $q = 12r$, then U_1^+ becomes

$$U_1^+(\xi) = \pm\sqrt{\frac{6r}{p}}\,(2 + \tanh\xi + \coth\xi), \quad \xi = x - 32rt. \tag{62}$$

For comparison, we observe that our solution (58) includes the solutions (4.11)–(4.13) of Bekir [23]. Then our solutions are more general. It is worth noting that our rational solution (59) not derived in [23]. The behaviour of exact travelling wave solutions are shown in figure 3.

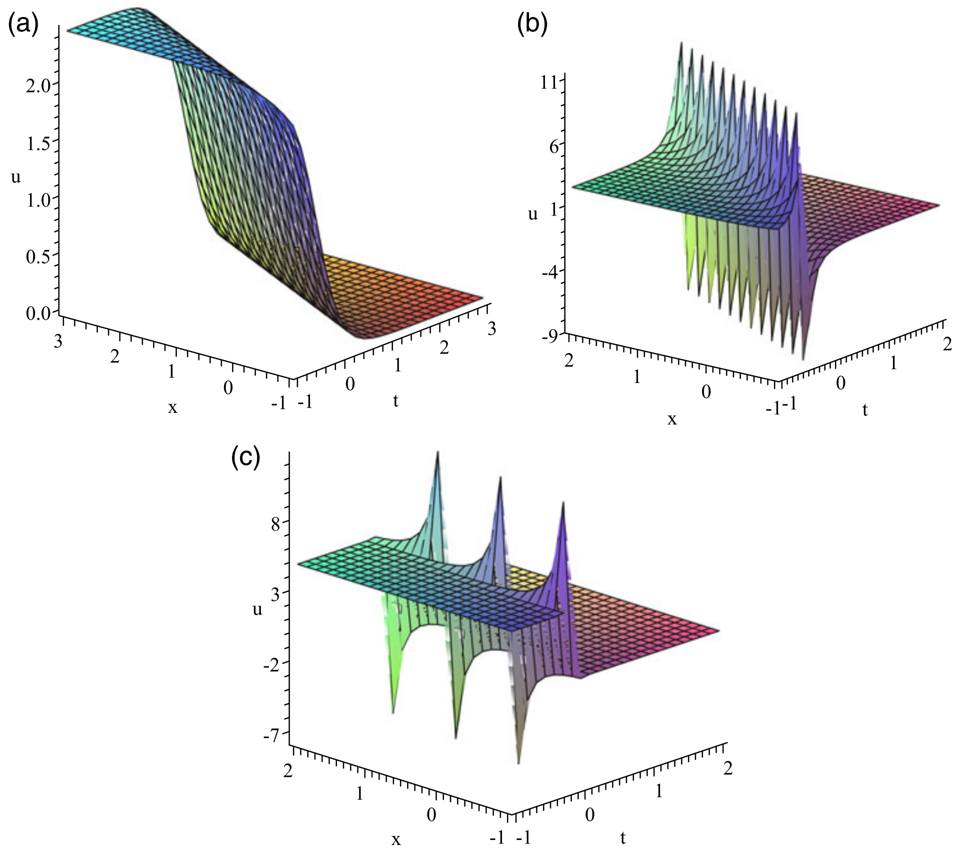

Figure 3. The graphs of exact travelling wave solutions of (**a**) eq. (60), (**b**) eq. (61) and (**c**) eq. (62) with $r = \frac{1}{4}$ and $p = 1$.

4. Conclusions

In this paper, an implementation of the (G'/G)-expansion method is given by applying it to three nonlinear equations to illustrate the validity and advantages of the method. As a result, hyperbolic function solutions, trigonometric function solutions and rational function solutions with parameters are obtained. The obtained solutions with free parameters may be important to explain some physical phenomena. The paper shows that the devised algorithm is effective and can be used for many other NLEEs in mathematical physics.

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