

## Entropy of the Kerr–Sen black hole

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MS received 8 July 2010; revised 13 October 2010; accepted 21 October 2010

**Abstract.** We study the entropy of Kerr–Sen black hole of heterotic string theory beyond semiclassical approximations. Applying the properties of exact differentials for three variables to the first law of thermodynamics, we derive the corrections to the entropy of the black hole. The leading (logarithmic) and non-leading corrections to the area law are obtained.

**Keywords.** Quantum aspects of black holes; thermodynamics; strings and branes.

**PACS Nos** 04.70.Dy; 04.70.Bw; 11.25.-w

### 1. Introduction

A semiclassical treatment of the quantum tunnelling approach to Hawking radiation gives some changes in the thermodynamical quantities. The quantum corrections to the Hawking temperature and the Bekenstein–Hawking area law have been studied for the Schwarzschild, Kerr and Kerr–Newman black holes [1–3] as well as other solutions [4–6].

It has been realized that the low-energy effective field theory describing string theory contains black hole solutions which can have properties which are qualitatively different from those that appear in ordinary Einstein gravity. Here we shall analyse the quantum corrections to the entropy of the Kerr–Sen black hole, which is an exact classical solution in the low-energy effective heterotic string theory with a finite amount of charge and angular momentum. To obtain the quantum corrections we use the criterion for exactness of differentials of black hole entropy, from the first law of thermodynamics with three parameters. We find that the leading correction term is logarithmic, while the other terms involve ascending powers of the inverse of the area.

In the quantum tunnelling approach, when a particle with positive energy tunnels out, it escapes to infinity and appears as Hawking radiation. But, when a particle with negative energy crosses inwards, it is absorbed by the black hole and as a result the mass of the black hole decreases. Therefore, the essence of the quantum tunnelling argument for Hawking radiation is the calculation of the imaginary part of the action. If we consider the action  $\mathcal{I}(r, t)$  and make an expansion in powers of  $\hbar$  we obtain

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$$\mathcal{I}(r, t) = \mathcal{I}_0(r, t) + \hbar \mathcal{I}_1(r, t) + \hbar^2 \mathcal{I}_2(r, t) + \dots \quad (1)$$

$$= \mathcal{I}_0(r, t) + \sum_i \hbar^i \mathcal{I}_i(r, t), \quad (2)$$

where  $\mathcal{I}_0$  gives the semiclassical value and the terms from  $O(\hbar)$  onwards are treated as quantum corrections. Banerjee and Majhi [7] showed that the correction terms  $\mathcal{I}_i$  were proportional to the semiclassical contribution  $\mathcal{I}_0$ . Since  $\mathcal{I}_0$  has the dimension  $\hbar$ , the proportionality constants should have the dimension of inverse of  $\hbar$ . In natural units ( $G = c = k_B = 1$ ), the Planck constant is of the order of the square of the Planck mass. Therefore, from dimensional analysis the proportionality constants have the dimension  $M^{-2i}$  where  $M$  is the mass of black hole, and the series expansion becomes

$$\mathcal{I}(r, t) = \mathcal{I}_0(r, t) + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \mathcal{I}_0(r, t) \quad (3)$$

$$= \mathcal{I}_0(r, t) \left( 1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} \right), \quad (4)$$

where  $\beta_i$ 's are dimensionless constant parameters. If the black hole has other macroscopic parameters such as angular momentum and electric charge, one can express this expansion in terms of the area of the event horizon, i.e. using the horizon radius  $r_H$  and the angular momentum  $a$ , as done in [1,4],

$$\mathcal{I}(r, t) = \mathcal{I}_0(r, t) \left( 1 + \sum_i \beta_i \frac{\hbar^i}{(r_H^2 + a^2)^i} \right). \quad (5)$$

This expansion will be used later to calculate the quantum corrections to the entropy of the Kerr–Sen black hole.

## 2. Entropy as an exact differential

To perform quantum corrections to the entropy of the black hole, we shall follow the analysis given in [12] for the BTZ black hole and in [1,4] for other black hole solutions. The first law of thermodynamics for the charged and rotating black holes is

$$dM = T dS + \Omega dJ + \Phi dQ, \quad (6)$$

where the parameters  $M$ ,  $J$  and  $Q$  are the mass, angular momentum and charge of the black hole, respectively, whereas  $T$ ,  $S$ ,  $\Omega$  and  $\Phi$  are the temperature, entropy, angular velocity and electrostatic potential, respectively. This equation can be rewritten as

$$dS(M, J, Q) = \frac{1}{T} dM - \frac{\Omega}{T} dJ - \frac{\Phi}{T} dQ, \quad (7)$$

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from which one can infer that, for  $dS$  to be an exact differential, the thermodynamical quantities must satisfy

$$\frac{\partial}{\partial J} \left( \frac{1}{T} \right) = \frac{\partial}{\partial M} \left( -\frac{\Omega}{T} \right), \quad (8)$$

$$\frac{\partial}{\partial Q} \left( \frac{1}{T} \right) = \frac{\partial}{\partial M} \left( -\frac{\Phi}{T} \right), \quad (9)$$

$$\frac{\partial}{\partial Q} \left( -\frac{\Omega}{T} \right) = \frac{\partial}{\partial J} \left( -\frac{\Phi}{T} \right). \quad (10)$$

If  $dS$  is an exact differential, we can write the entropy  $S(M, J, Q)$  in the integral form

$$\begin{aligned} S(M, J, Q) = & \int \frac{1}{T} dM - \int \frac{\Omega}{T} dJ - \int \frac{\Phi}{T} dQ - \int \left( \frac{\partial}{\partial J} \left( \int \frac{1}{T} dM \right) \right) dJ \\ & - \int \left( \frac{\partial}{\partial Q} \left( \int \frac{1}{T} dM \right) \right) dQ + \int \left( \frac{\partial}{\partial Q} \left( \int \frac{\Omega}{T} dJ \right) \right) dQ \\ & + \int \left( \frac{\partial}{\partial Q} \left( \int \left( \frac{\partial}{\partial J} \left( \int \frac{1}{T} dM \right) \right) dJ \right) \right) dQ. \end{aligned} \quad (11)$$

### 3. Standard entropy of the Kerr–Sen black hole

Sen [8,9] was able to find a charged, stationary, axially symmetric solution of the field equations by using target space duality, applied to the classical Kerr solution. The line element of this solution can be written, in generalized Boyer–Liquist coordinates, as

$$\begin{aligned} ds^2 = & - \left( 1 - \frac{2Mr}{\rho^2} \right) dt^2 + \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - \frac{4Mra \sin^2 \theta}{\rho^2} dt d\varphi \\ & + \left( r(r+r_\alpha) + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta d\varphi^2, \end{aligned} \quad (12)$$

where

$$\Delta = r(r+r_\alpha) - 2Mr + a^2, \quad (13)$$

$$\rho = r(r+r_\alpha) + a^2 \cos^2 \theta. \quad (14)$$

Here  $M$  is the mass of the black hole,  $a = J/M$  is the specific angular momentum of the black hole and the electric charge is given by

$$r_\alpha = \frac{Q^2}{M}. \quad (15)$$

Note that in the particular case of a static black hole, i.e.  $a = 0$ , the metric (12) coincides with the GMGHS solution [10] while in the particular case  $r_\alpha = 0$  reconstructs the Kerr solution.

The Kerr–Sen space has a spherical event horizon, which is the biggest root of the equation  $\Delta = 0$  and is given by

$$r_H = \frac{2M - r_\alpha + \sqrt{(2M - r_\alpha)^2 - 4a^2}}{2},$$

or in terms of the black hole parameters  $M$ ,  $Q$  and  $J$ ,

$$r_H = M - \frac{Q^2}{2M} + \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}}. \quad (16)$$

The area of the event horizon is given by

$$A = 4\pi (r_H^2 + a^2) = 8\pi M \left( M - \frac{Q^2}{2M} + \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}} \right). \quad (17)$$

Equation (16) tells us that the horizon disappears unless

$$|J| \leq M^2 - \frac{Q^2}{2}.$$

Therefore, the extremal black hole,  $|J| = M^2 - \frac{Q^2}{2}$ , has  $A = 8\pi|J|$ . The angular velocity at the horizon is given by

$$\Omega = \frac{J}{2M^2} \frac{1}{M - \frac{Q^2}{2M} + \sqrt{\left(M - \frac{Q^2}{2M}\right)^2 - \frac{J^2}{M^2}}}, \quad (18)$$

and the Hawking temperature is given by

$$T_H = \frac{\kappa \hbar}{2\pi} = \frac{\hbar \sqrt{(2M^2 - Q^2)^2 - 4J^2}}{4\pi M \left( 2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2} \right)}. \quad (19)$$

One can easily check that thermodynamical quantities for the Kerr–Sen black hole satisfy

$$\frac{\partial}{\partial J} \left( \int \frac{1}{T_H} dM \right) = -\frac{\Omega}{T_H}, \quad (20)$$

$$\frac{\partial}{\partial Q} \left( \int \frac{1}{T_H} dM \right) = -\frac{\Phi}{T_H}, \quad (21)$$

$$\frac{\partial}{\partial Q} \left( \int -\frac{\Omega}{T_H} dJ \right) = -\frac{\Phi}{T_H}. \quad (22)$$

Under these conditions, the integral form of the entropy (eq. (11)) reduces to

$$S_0(M, J, Q) = \int \frac{1}{T_H} dM, \quad (23)$$

and for the Kerr–Sen black hole this gives

$$S_0(M, J, Q) = \frac{4\pi}{\hbar} \int \frac{M \left( 2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2} \right)}{\sqrt{(2M^2 - Q^2)^2 - 4J^2}} dM, \quad (24)$$

$$S_0(M, J, Q) = \frac{\pi}{\hbar} \left( 2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2} \right), \quad (25)$$

which corresponds to the standard black hole entropy

$$S_0(M, J, Q) = \frac{A}{4\hbar} = \frac{\pi(r_H^2 + a^2)}{\hbar}. \quad (26)$$

#### 4. Quantum correction of the entropy

The expansion of the action given by (5) affects the Hawking temperature by introducing some correction terms [4,7]. Therefore, the temperature is now given by

$$T = T_H \left( 1 + \sum_i \beta_i \frac{\hbar^i}{(r_H^2 + a^2)^i} \right)^{-1}, \quad (27)$$

where  $T_H$  is the standard Hawking temperature and the terms with  $\beta_i$  are quantum correction terms to the temperature. It is easy to verify that the conditions which make  $dS$  an exact differential are satisfied when considering the new form of the temperature. Therefore, the entropy with correction terms is given by

$$S(M, J, Q) = \int \frac{1}{T} dM = \int \frac{1}{T_H} \left( 1 + \sum_i \beta_i \frac{\hbar^i}{(r_H^2 + a^2)^i} \right) dM, \quad (28)$$

or

$$S(M, J, Q) = \int \frac{1}{T_H} dM + \int \frac{\beta_1}{T_H} \frac{\hbar}{(r_H^2 + a^2)} dM + \int \frac{\beta_2}{T_H} \frac{\hbar^2}{(r_H^2 + a^2)^2} dM + \dots \quad (29)$$

This equation can be written as

$$S(M, J, Q) = S_0 + S_1 + S_2 + \dots, \quad (30)$$

where  $S_0$  is the standard entropy given by eq. (26) and  $S_1, S_2, \dots$  are quantum corrections. The first of these terms is

$$S_1 = \beta_1 \hbar \int \frac{1}{T_H(r_H^2 + a^2)} dM \quad (31)$$

$$= 4\pi\beta_1 \int \frac{M}{\sqrt{(2M^2 - Q^2)^2 - 4J^2}} dM. \quad (32)$$

Solving the integral, we obtain

$$S_1 = \pi\beta_1 \ln \left| 2 \left( 2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2} \right) \right|, \quad (33)$$

which can be written as

$$S_1 = \pi\beta_1 \ln |2(r_H^2 + a^2)|. \quad (34)$$

The following terms can be written, in general, as

$$S_j = \beta_j \hbar^j \int \frac{1}{T_H(r_H^2 + a^2)^j} dM \quad (35)$$

$$S_j = 4\pi\beta_j \hbar^{j-1} \int \frac{M dM}{(2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2})^{j-1} \sqrt{(2M^2 - Q^2)^2 - 4J^2}}. \quad (36)$$

By calculating the integral, we obtain

$$S_j = \frac{\pi\beta_j\hbar^{j-1}}{1-j} \left( 2M^2 - Q^2 + \sqrt{(2M^2 - Q^2)^2 - 4J^2} \right)^{1-j} \quad (37)$$

or

$$S_j = \frac{\pi\beta_j\hbar^{j-1}}{1-j} (r_H^2 + a^2)^{1-j} \quad (38)$$

for  $j > 1$ . Therefore, the entropy with the quantum corrections is

$$S(M, J, Q) = \frac{\pi(r_H^2 + a^2)}{\hbar} + \pi\beta_1 \ln|2(r_H^2 + a^2)| + \sum_{j>1} \frac{\pi\beta_j\hbar^{j-1}}{1-j} (r_H^2 + a^2)^{1-j}. \quad (39)$$

If  $Q = 0$ , we recover the corrections for the Kerr black hole found in [7]. On the other hand, if the angular momentum is also equal to zero we obtain the entropy for the Schwarzschild black hole ( $a = Q = 0$ ). Finally, if only the angular momentum vanishes (i.e.  $a = 0$ ), we get corrections for the GMGHS black hole [11].

Using eq. (17), with a redefinition of the  $\beta_i$ , we can write the entropy in terms of the area of the horizon as

$$S(M, J, Q) = \frac{A}{4\hbar} + \pi\beta_1 \ln|A| + \sum_{j>1} \frac{\pi\beta_j\hbar^{j-1}}{1-j} \left( \frac{A}{4\pi} \right)^{1-j}. \quad (40)$$

The first term in this expansion is the usual semiclassical entropy whereas the second term is the logarithmic correction found earlier for other geometries [12,13] using different methods. The value of the coefficients  $\beta_i$  can be evaluated using other approaches, such as the entanglement entropy calculation. Finally, note that the third term in the expansion is an inverse of area term similar to the one obtained by Modak [12] for the rotating BTZ black hole, for the charged BTZ black hole [13] and also in the general case studied by Akbar and Saifullah [4].

## 5. Conclusion

As is well known, the Hawking evaporation process can be understood as a consequence of quantum tunnelling in which some particles cross the event horizon. The positive energy particles tunnel out of the event horizon, whereas the negative energy particles crosses in, resulting in black hole evaporation. Using this analysis we have studied the quantum corrections to the entropy for Kerr–Sen black hole. With the help of the conditions for exactness of differential of entropy we obtain a power series for entropy. The first term is semiclassical, while the leading correction term is logarithmic as was found using other methods [12,13]. The other terms involve ascending powers of the inverse of the area. If the angular momentum become zero, we obtain results for the GMGHS black hole, in which case the power series involve just mass and electric charge [11]. This analysis shows that the quantum corrections to entropy obtained in [1,4,12] hold also for the black holes of string theory studied here.

### **Acknowledgement**

This work was supported by the Universidad Nacional de Colombia, Project Code 2010100.

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