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An analysis of the nonlinear equation $u_t = f(x,u)u_{xx} + g(x,u)u_x^2 + h(x,u)u_x + p(x,u)$

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Abstract. We use the method of preliminary group classification to analyse a particular form of the nonlinear diffusion equation in which the inhomogeneity is quadratic in u_x. The method yields an optimal system of one-dimensional subalgebras. As a result we obtain those explicit forms of the unknown functions f, g, h and p for which the equation admits additional point symmetries.

Keywords. Partial differential equations; symmetries, nonlinear diffusion equation.

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1. Introduction

One of the most significant properties of differential equations is the invariance of these equations under a particular group of transformations. When a differential equation is invariant under a Lie group of transformations, a reduction transformation will exist [1]. Sophus Marius Lie (1842–1899), the Norwegian mathematician, proposed a methodical process by which groups of symmetries of differential equations could be found [2]. He also provided a systematic method to search for these special group invariant solutions [1]. Thus, differential equations could be classified in terms of their symmetry groups, thereby identifying the set of equations that could be integrated or reduced to lower order equations by group theoretic algorithms.

One of Lie's many interests lay in the classification of partial differential equations (PDEs) according to their symmetries. He emphasized the idea of a complete group classification for an extensive class of linear and specific nonlinear second-order PDEs with two independent variables [3]. An integral part of the classification was based on the knowledge of the general solution of the determining equations as well as the utilization of equivalence transformations (arbitrary changes of the independent variables and linear transformations of the dependent ones). Unfortunately, the general solution of the determining equations could not always be found [4].

The method of preliminary group classification allows for the classification of all non-similar subalgebras of the algebra Lie generated through equivalence transformations of a considered differential equation [5]. This results in the explicit determination of functional forms which extend the existing principal Lie algebra and gives rise to the 'optimal system' of group invariant solutions from which all other solutions can be determined. This group classification method is based purely on algebraic manipulations contrary to the standard Lie algorithm wherein one is required to integrate differential equations.

The quasi-linear parabolic equation

$$u_t = [\Phi(u,x)]_{xx} + f(x)u^s u_x + g(x)u^m \tag{1}$$

is used to model a number of physical problems. These include the flow of liquids in porous media as well as the transport of thermal energy in plasma [6]. A complete group classification for various forms of this equation is easily available because of its strong relation to equations such as the nonlinear heat equation [7]. We shall consider a more general form of eq. (1), viz.

$$u_t = f(x,u)u_{xx} + g(x,u)u_x^2 + h(x,u)u_x + p(x,u) \tag{2}$$

which can be interpreted as a nonlinear diffusion equation with an inhomogeneity which is quadratic in u_x. (This study is a continuation of our programme to analyse different classes of nonlinear diffusion equations [8,9].) We shall show, using the method of preliminary group classification which was popularized by Ibragimov *et al* [3] and extended by Harin [4], that one can find group invariant solutions to eq. (2) by considering special forms of the arbitrary functions. Our approach will be systematic and will not reply any *ad hoc* assumptions. In all our working we assume that the reader is familiar with the Lie symmetry analysis of differential equations [2,10–12].

2. Preliminary group classification

Equation (2) will admit a one-dimensional Lie algebra with the basis

$$G_1 = \frac{\partial}{\partial t} \tag{3}$$

when the unknown functions are arbitrary. This is known as the principal Lie algebra.

One of the main differences between the standard Lie analysis and the method of preliminary group classification lies in the treatment of the unknown function. In the latter method, this function is treated as a differentiable variable. Adopting the notation $f(x,u) = f^1,\ g(x,u) = f^2,\ h(x,u) = f^3$ and $p(x,u) = f^4$, we rewrite eq. (2) as

$$u_t = f^1 u_{xx} + f^2 u_x^2 + f^3 u_x + f^4 \tag{4}$$

and search for a symmetry of the form

$$\begin{aligned} E = {} & \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \\ & + \mu^k(t,x,u,u_t,u_x,f^1,f^2)\frac{\partial}{\partial f^k}. \end{aligned} \tag{5}$$

The invariance conditions of eq. (4) are defined by

$$E^{[2]}\left(u_t - f^1 u_{xx} - f^2 u_x^2 - f^3 u_x - f^4\right) = 0 \tag{6}$$

$$E^{[2]}\left(f_t^k\right) = E^{[2]}\left(f_{u_t}^k\right) = E^{[2]}\left(f_{u_x}^k\right) = 0, \quad k = 1, \ldots, 4, \tag{7}$$

where

$$\begin{aligned} E^{[2]} &= E + \eta_1 \frac{\partial}{\partial u_t} + \eta_2 \frac{\partial}{\partial u_x} + \eta_{11} \frac{\partial}{\partial u_{tt}} + \eta_{22} \frac{\partial}{\partial u_{xx}} \\ &\quad + \omega_t^k \frac{\partial}{\partial f_t^k} + \omega_u^k \frac{\partial}{\partial f_u^k} + \omega_{u_t}^k \frac{\partial}{\partial f_{u_t}^k}. \end{aligned} \tag{8}$$

Solving eqs (6) and (7) yields the infinite continuous group of equivalence transformations generated by the infinitesimal operators

$$E_1 = \frac{\partial}{\partial t} \tag{9}$$

$$E_2 = t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g} - h\frac{\partial}{\partial h} - p\frac{\partial}{\partial p} \tag{10}$$

$$E_3 = b(x)\frac{\partial}{\partial x} + 2b'(x)f\frac{\partial}{\partial f} + 2b'(x)g\frac{\partial}{\partial g} + \left(b''(x)f + b'(x)h\right)\frac{\partial}{\partial h} \tag{11}$$

$$\begin{aligned} E_4 &= c(x,u)\frac{\partial}{\partial u} - \left(c_{uu}(x,u)f + c_u(x,u)g\right)\frac{\partial}{\partial g} \\ &\quad - \left(2c_{xu}(x,u)f + 2c_x(x,u)g\right)\frac{\partial}{\partial h} \\ &\quad + \left(c_u(x,u)p - c_{xx}(x,u)f - c_x(x,u)h\right)\frac{\partial}{\partial p} \end{aligned} \tag{12}$$

which can be verified [13] using `PROGRAM LIE` [14].

We also observe that the reflections

$$x \to -x, \qquad h \to -h, \tag{13}$$

$$u \to -u, \qquad g \to -g, \qquad p \to -p, \tag{14}$$

$$u \to -u, \qquad t \to -t, \qquad f \to -f, \qquad h \to -h \tag{15}$$

$$t \to -t, \qquad f \to -f, \qquad g \to -g, \qquad h \to -h, \qquad p \to -p, \tag{16}$$

leave eq. (2) invariant. These reflections will become significant later.

We consider the generators (9)–(12) on the space (x, u, f, g, h, p) as the functions $f,\ g,\ h$ and p depend only on the variables x and u. Thus,

$$Z = [-E_2] = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} + h\frac{\partial}{\partial h} + p\frac{\partial}{\partial p} \tag{17}$$

$$Y_b = [E_3] = b(x)\frac{\partial}{\partial x} + 2b'(x)f\frac{\partial}{\partial f} + 2b'(x)g\frac{\partial}{\partial g} + \left(b''(x)f + b'(x)h\right)\frac{\partial}{\partial h} \tag{18}$$

$$W_c = [E_4] = c(x,u)\frac{\partial}{\partial u} - \left(c_{uu}(x,u)f + c_u(x,u)g\right)\frac{\partial}{\partial g} - 2\left(c_{xu}(x,u)f + c_x(x,u)g\right)\frac{\partial}{\partial h} + \left(c_u(x,u)p - c_{xx}(x,u)f - c_x(x,u)h\right)\frac{\partial}{\partial p}. \tag{19}$$

The commutation relationships can be found in table 1. The Lie algebra must close and so we require either

$$W_{b(x)c_x(x,u)} = 0 \tag{20}$$

or

$$W_{b(x)c_x(x,u)} = W_{c(x,u)}. \tag{21}$$

We consider each separately.

2.1 *The Abelian case*

In eq. (20), it is clear that $c(x, u)$ depends only on the variable u as any other combination will result in the loss of at least one of the existing generators. Hence

$$W_c' = c(u)\frac{\partial}{\partial u} - \left(c''(u)f + c'(u)g\right)\frac{\partial}{\partial g} + c'(u)p\frac{\partial}{\partial p} \tag{22}$$

and the Lie algebra is $3A_1$ [15]. The respective automorphisms form a trivial subalgebra and will not provide any further information. The optimal system of one-dimensional

Table 1. Commutation table of eqs (17)–(19).

	Z	Y_b	W_c
Z	0	0	0
Y_b	0	0	$W_{b(x)c_x(x,u)}$
W_c	0	$-W_{b(x)c_x(x,u)}$	0

subalgebras can only be determined through a linear combination of all existing generators. As all the resultant vectors **e** will be invariant, it is necessary to avoid all linear combinations involving more than one generator. Thus, for

$$T_1 = Z, \quad T_2 = Y_a, \quad T_3 = W'_c \tag{23}$$

we obtain the optimal system of one-dimensional subalgebras [16], namely

$$T_3 = W'_c \qquad (M^{(1)}) \tag{24}$$

$$T_2 + \alpha T_3 = Y_b + \alpha W'_c \qquad (M^{(2)}), \quad -\infty < \alpha < \infty \tag{25}$$

$$T_1 + \beta T_2 + \alpha T_3 = Z + \beta Y_b + \alpha W'_c \quad (M^{(3)}),$$
$$-\infty < \alpha, \beta < \infty. \tag{26}$$

The results (24)–(26) of the optimal system of one-dimensional subalgebras and their additional operators are presented in tables 2 and 3, respectively [17].

2.2 *The non-Abelian case*

The requirement

$$W_{b(x)c_x(x,u)} = W_{c(x,u)} \tag{27}$$

leads to

$$\begin{aligned} c(x,u) &= b(x)c_x(x,u) \\ &= k(u)\exp\left(\int \frac{1}{b(x)}\mathrm{d}x\right) \\ &= k(u)R(x). \end{aligned} \tag{28}$$

Substituting eq. (28) into eq. (19)

$$\begin{aligned} W_{k(u)R(x)} &= k(u)R(x)\frac{\partial}{\partial u} - R(x)\left(k''(u)f + k'(u)g\right)\frac{\partial}{\partial g} \\ &\quad - 2R'(x)\left(k'(u)f + k(u)g\right)\frac{\partial}{\partial h} \\ &\quad - \left(k(u)R''(x)f + k(u)R'(x)h - k'(u)R(x)p\right)\frac{\partial}{\partial p}. \end{aligned} \tag{29}$$

To proceed with a determination of the optimal subgroups without too much difficulty, we follow [3] and set

$$b(x) = \frac{x}{n}, \tag{30}$$

when

$$R(x) = x^n. \tag{31}$$

Table 2. Classification of eq. (2) with respect to subalgebras (24)–(26) with arbitrary functions $\Phi(\lambda), \Gamma(\lambda), \delta(\lambda)$ and $\kappa(\lambda)$. Due to the generality of the g component we considered $c(u) = u^n$ in cases 3 and 6.

Case		λ	f	g	h	p
1	$M^{(1)}$	x	Φ	$(\Gamma - c'(u))/c(u)$	δ	$\kappa c(u)$
2	$M^{(2)}_{\alpha=0}$	u	$\Phi b(x)^2$	$\Gamma b(x)^2$	$b(x)\,(\delta + \Phi b'(x))$	κ
3	$M^{(2)}_{\alpha\neq 0}$	$\int \frac{\mathrm{d}x}{b(x)} - \frac{1}{\alpha}\int \frac{\mathrm{d}u}{c(u)}$	$\Phi b(x)^2$	$b(x)^2(\Gamma - n\Phi u^{n-1}(\alpha(1-n))^{n/(n-1)}) \times \exp\left(\frac{-\alpha}{b(x)} n u^{n-1}\mathrm{d}x\right)$	$b(x)\,(\delta + \Phi b'(x))$	$\kappa c(u)$
4	$M^{(3)}_{\alpha=0,\beta\neq 0}$	u	$\Phi b(x)^2 \exp\left(\int \frac{\mathrm{d}x}{\beta b(x)}\right)$	$\Gamma b(x)^2 \exp\left(\int \frac{\mathrm{d}x}{\beta b(x)}\right)$	$b(x)(\delta + \Phi b'(x)) \exp\left(\int \frac{\mathrm{d}x}{\beta b(x)}\right)$	$\kappa \exp\left(\int \frac{\mathrm{d}x}{\beta b(x)}\right)$
5	$M^{(3)}_{\alpha\neq 0,\beta=0}$	x	$\Phi \exp\left(\int \frac{\mathrm{d}u}{\alpha c(u)}\right)$	$c(u)\,(\Gamma - \Phi c'(u)) \exp\left(\int \frac{\mathrm{d}u}{\alpha c(u)}\right)$	$\delta \exp\left(\int \frac{\mathrm{d}u}{\alpha c(u)}\right)$	$\kappa c(u) \exp\left(\int \frac{\mathrm{d}u}{\alpha c(u)}\right)$
6	$M^{(3)}_{\alpha\neq 0,\beta\neq 0}$	$\int \frac{\mathrm{d}x}{\beta b(x)} - \int \frac{\mathrm{d}u}{\alpha c(u)}$	$\Phi b(x)^2 \exp\left(\int \frac{\mathrm{d}x}{\beta b(x)}\right)$	$b(x)^2\left(\Gamma - n\Phi u^{n-1}\left(\frac{\alpha(1-n)}{\beta}\right)^{n/(n-1)}\right) \times \exp\left(\int \frac{1}{\beta b(x)} - \frac{\alpha}{\beta b(x)} n u^{n-1}\mathrm{d}x\right)$	$b(x)(\delta + \Phi b'(x)) \exp\left(\int \frac{\mathrm{d}x}{\beta b(x)}\right)$	$\kappa c(u) \exp\left(\int \frac{\mathrm{d}x}{\beta b(x)}\right)$

Table 3. Additional operators relating to table 2.

Case	Ad. Op. W_2
1	$c(u)\frac{\partial}{\partial u}$
2	$b(x)\frac{\partial}{\partial x}$
3	$b(x)\frac{\partial}{\partial x} + \alpha c(u)\frac{\partial}{\partial u}$
4	$-t\frac{\partial}{\partial t} + \beta b(x)\frac{\partial}{\partial x}$
5	$-t\frac{\partial}{\partial t} + \alpha c(u)\frac{\partial}{\partial u}$
6	$-t\frac{\partial}{\partial t} + \beta b(x)\frac{\partial}{\partial x} + \alpha c(u)\frac{\partial}{\partial u}$

Besides, it is clear that $k(u)$ appears in the coefficient of $\partial/\partial u$ and can be absorbed by suitably redefining u. Thus, we set

$$k(u) = 1. \tag{32}$$

As a result of these simplifications, the Lie algebra is spanned by the following operators:

$$Z = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} + h\frac{\partial}{\partial h} + p\frac{\partial}{\partial p} \tag{33}$$

$$Y = x\frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f} + 2g\frac{\partial}{\partial g} + h\frac{\partial}{\partial h} \tag{34}$$

$$W_1 = x\frac{\partial}{\partial u} - 2g\frac{\partial}{\partial h} - h\frac{\partial}{\partial p} \tag{35}$$

$$W_2 = x^2\frac{\partial}{\partial u} - 4xg\frac{\partial}{\partial h} - 2\left(f + xh\right)\frac{\partial}{\partial p} \tag{36}$$

$$\vdots$$

$$W_n = x^n\frac{\partial}{\partial u} - 2nx^{n-1}g\frac{\partial}{\partial h} - n\left((n-1)x^{n-2}f + x^{n-1}h\right)\frac{\partial}{\partial p}. \tag{37}$$

It follows that

$$A_1 = 0 \tag{38}$$

$$A_Y = W_1\frac{\partial}{\partial W_1} + 2W_2\frac{\partial}{\partial W_2} + \cdots + nW_n\frac{\partial}{\partial W_n} \tag{39}$$

$$A_{W_1} = -W_1\frac{\partial}{\partial Y} \tag{40}$$

$$A_{W_2} = -2W_2\frac{\partial}{\partial Y} \tag{41}$$

$$\vdots$$

$$A_{W_n} = -nW_n\frac{\partial}{\partial Y}. \tag{42}$$

The automorphisms $A_1, \ldots, A_{W_n}$ generate a one-parameter group of linear transformations

$$A_Y\colon W_1' = a_2 W_1, \quad W_2' = a_2^2 W_2, \quad \cdots, W_n' = a_2^n W_n \tag{43}$$

$$A_{W_1}\colon Y_b' = Y + a_{2+1} W_1 \tag{44}$$

$$A_{W_2}\colon Y_b' = Y + 2a_{2+2} W_2 \tag{45}$$

$$A_{W_n}\colon Y_b' = Y + n a_{2+n} W_n \tag{46}$$

for $n = 1, \ldots, \infty$. Here, a_2 is a positive real parameter and a_1, a_{2+n} are real arbitrary parameters.

The product of the matrices associated with the infinitesimal operators A_i is therefore

$$\begin{aligned} M &= M_1(a_1) \times \cdots \times M_{2+n}(a_{2+n}) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & a_{2+1} & 2a_{2+2} & \cdots & na_{2+n} \\ 0 & 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_2^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2^n \end{pmatrix}. \end{aligned}$$

As expected, we can reduce our $(n+2)$-dimensional subspace to the three-dimensional subspace

$$Z = f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g} + h\frac{\partial}{\partial h} + p\frac{\partial}{\partial p} \tag{47}$$

$$Y = x\frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f} + 2g\frac{\partial}{\partial g} + h\frac{\partial}{\partial h} \tag{48}$$

$$W_n = x^n\frac{\partial}{\partial u} - 2nx^{n-1}g\frac{\partial}{\partial h} - n\left((n-1)x^{n-2}f + x^{n-1}h\right)\frac{\partial}{\partial p}, \tag{49}$$

where W_n contains $W_1, \ldots, W_n$.

Constructing the optimal system of one-dimensional subalgebras consists of finding all classes of the operators

$$U = e_1 Z + e_2 Y + \sum_{i=1}^{n} e_{i+2} W_i \tag{50}$$

Table 4. Contributions of trivial and nontrivial invariants e_1 and e_2 to eqs (52)–(54).

(i)	$e_1 = 0,$	$e_2 = 0$	$\longrightarrow$	W_n	
(ii)	$e_1 = 0,$	$e_2 \neq 0$	$\longrightarrow$	Y	
(iii)	$e_1 \neq 0,$	$e_2 = 0$	$\longrightarrow$	$Z,$	$Z + W_n$
(iv)	$e_1 \neq 0,$	$e_2 \neq 0$	$\longrightarrow$	$Z + \alpha Y$	$(\alpha \neq 0)$

Table 5. Classification of eq. (2) with respect to subalgebras (56)–(59) with arbitrary functions $\Phi(\lambda), \Gamma(\lambda), \delta(\lambda), \kappa(\lambda)$.

Case		λ	f	g	h	p	Ad. Op. W_2
1	$N^{(1)}$	u	Φx^2	Γx^2	δx	κ	$x\frac{\partial}{\partial x}$
2	$N^{(2)}$	x	Φ	Γ	$\delta - 2n\Gamma x^{-1}u$	$\left(2n^2\Gamma u + \kappa - n(n-1)\Phi\right) x^{-2}$	$x^n\frac{\partial}{\partial u}$
3	$N^{(3)}$	x	$\Phi\exp\left(x^{-n}u\right)$	$\Gamma\exp\left(x^{-n}u\right)$	$\left(\delta - 2n\Gamma x^{-1}u\right)\exp\left(x^{-n}u\right)$	$\exp\left(x^{-n}u\right)\left(n^2\Gamma x^{-2}u^2 + \kappa - n\delta x^{-1}u - n(n-1)\Phi x^{-2}u\right)$	$-t\frac{\partial}{\partial t} + x^n\frac{\partial}{\partial u}$
4	$N^{(4)}_{\alpha\neq 0}$	u	$\Phi x^{2+1/\alpha}$	$\Gamma x^{2+1/\alpha}$	$\delta x^{1+1/\alpha}$	$\kappa x^{1/\alpha}$	$-t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x}$

nonequivalent with respect to the group of inner automorphisms [4]. Thus, it is favourable to work with the coordinates of the decomposition, i.e. with the vector

$$\mathbf{e} = (e_1, \ldots, e_{n+2}). \tag{51}$$

The transposed matrix M^{T} of M gives rise to vector $\mathbf{e}$

$$\bar{e}_1 = e_1 \tag{52}$$

$$\bar{e}_2 = e_2 \tag{53}$$

$$\bar{e}_{2+n} = na_{n+2}e_2 + a_2^n e_{n+2}. \tag{54}$$

Equations (13)–(16) give rise to the single transformation:

$$W_n \rightarrow -W_n. \tag{55}$$

The components e_1 and e_2 are invariant and thus we examine each situation in which these selected components are zero or nonzero (see table 4).

Hence, the optimal system of one-dimensional subalgebras is given by

$$N^{(1)} = Y \tag{56}$$

$$N^{(2)} = W_n \tag{57}$$

$$N^{(3)} = Z + W_n \tag{58}$$

$$N^{(4)} = Z + \alpha Y, \tag{59}$$

where we have taken eq. (55) into account. The results relating to eqs (56)–(59) are presented in table 5.

We note that more specific forms of (2), i.e., when each function is set to zero in turn, do not affect the results obtained here [17].

3. Discussion

We successfully applied the method of preliminary group classification to the nonlinear diffusion equation

$$u_t = f(x,u)u_{xx} + g(x,u)u_x^2 + h(x,u)u_x + p(x,u) \tag{60}$$

in which the inhomogeneity was quadratic in u_x. This resulted in the determination of the functional forms extending the principal Lie algebra. This group classification method is based on purely algebraic manipulations, in contrast to the standard Lie algorithm wherein one is required to solve differential equations.

We illustrate how to use the results through a simple example. In its most general form one cannot find group invariant solutions to eq. (60) beyond those that are independent of t. However, if we take case 1 in table 5, eq. (60) becomes

$$u_t = \Phi(u)x^2u_{xx} + \Gamma(u)x^2u_x^2 + \delta(u)xu_x + \kappa(u) \tag{61}$$

which admits the symmetry

$$G_2 = x\frac{\partial}{\partial x}. \tag{62}$$

A combination of $G_1 + \alpha G_2$ yields the new independent variable as

$$q = x\mathrm{e}^{-\alpha t}, \tag{63}$$

where α is an arbitrary constant. Thus, we can reduce eq. (61) to

$$\Phi(u)u_{qq} + \Gamma(u)u_q^2 + (\delta(u) + \alpha)u_q + \kappa(u) = 0. \tag{64}$$

When

$$\delta(u) = -\alpha, \quad \kappa(u) = 0 \tag{65}$$

we can solve eq. (64) to obtain the quadrature

$$u_0 \int \exp\left(\int^u \frac{\Gamma(p)}{\Phi(p)}\mathrm{d}p\right)\mathrm{d}u = q - q_0, \tag{66}$$

where u_0 and q_0 are constants of integration. Hence, a group invariant solution to eq. (61) is given by

$$u_0 \int \exp\left(\int^u \frac{\Gamma(p)}{\Phi(p)}\mathrm{d}p\right)\mathrm{d}u = x\mathrm{e}^{-\alpha t} - q_0 \tag{67}$$

provided eq. (65) holds. One proceeds in a similar manner for the other cases of results provided in tables 2–5.

To our knowledge, the method of preliminary group classification has not previously been applied to this equation. However, many similar equations have received attention. First, this work extends our previous work [8] in which we analyse the equation

$$u_t = f(x, u_x)u_{xx} + g(x, u_x) \tag{68}$$

for which we need to set

$$f(x,u) = f(u), \quad g(x,u) = g(u), \quad h(x,u) = p(x,u) = 0. \tag{69}$$

Oron and Rosenau [18] and later Edwards [19] investigated the equation

$$u_t = K(u)u_{xx} + K'(u)u_x^2 + \delta\Phi'(u)u_x \tag{70}$$

which could be identified as a particular form of eq. (2) if and only if

$$\begin{aligned} f(x,u) &= K(u), \quad g(x,u) = K'(u), \\ h(x,u) &= \delta\Phi'(u), \quad p(x,u) = 0. \end{aligned} \tag{71}$$

The generality of our functional forms led to more general results in our analysis.

Gandarias [6] applied the direct Lie group formalism to deduce the symmetries of the porous medium equation

$$u_t = (u^n)_{xx} + q(x)u^m + k(x)u^s u_x. \tag{72}$$

This is a particular form of eq. (2) if and only if

$$f(x,u) = \lambda u^{n-1}, \qquad g(x,u) = \Phi u^{n-2},$$
$$h(x,u) = k(x)u^{s}, \qquad p(x,u) = q(x)u^{m}. \tag{73}$$

Because of certain errors in Gandarias paper we were not able to make a thorough comparison. For example, for an arbitrary m, she claimed this equation would admit a symmetry of the form $\partial/\partial x$. This is clearly not possible unless both $k(x)$ and $q(x)$ are constants.

Gandarias [1] also investigated the potential symmetries of the system

$$v_x = u \tag{74}$$

$$v_t = (u^n)_x + \frac{k(x)}{m}u^m. \tag{75}$$

She once again determined the forms of the arbitrary functions through a consistent application of the Lie group formalism. But a comparison was not possible due to certain errors in her analysis. This illustrates the importance of using the systematic approach of preliminary group classification as opposed to the direct Lie approach.

Khater *et al* [20] investigated the system

$$v_x = f(x)u \tag{76}$$

$$v_t = g(x)u^n u_x \tag{77}$$

for potential symmetries. Using Lie group theory they determined the point and potential symmetries for the chosen forms of f and g. A comparison of their results against ours is possible only if all functions in both systems are set to constants and $n = 0$, leaving no freedom for analysis.

Sophocleous [21,22] has recently studied in some detail the nonlinear diffusion equations of the form

$$u_t = [h(x)u^n u_x]_x, \tag{78}$$

from the point of view of both potential symmetries and Lie–Bäcklund symmetries. This class of equations is clearly more specific than our general form.

In [23], the nonlinear heat equation with quadratic inhomogeneities was also analysed. Their version of the equation is

$$f(x)u_t = (g(x)u^n u_x)_x + h(x)u^m. \tag{79}$$

It is clear that this equation is far more specific than ours. We need to set

$$f(x,u) = \frac{g(x)u^n}{f(x)}, \quad g(x,u) = \frac{ng(x)u^{n-1}}{f(x)},$$

$$h(x,u) = \frac{g'(x)u^n}{f(x)}, \quad p(x,u) = \frac{h(x)u^m}{f(x)} \tag{80}$$

in our equation to obtain eq. (79). Moreover, when we confined our analysis to point equivalence transformations, Vaneeva *et al* [23] also considered generalized extended and conditional equivalence groups. They also looked at the determination of conservation laws. All their results will apply to our equation when eq. (80) is taken into account.

In a series of papers, Lahno and Zhdanov [24,25] considered a class of nonlinear diffusion equations in which the nonlinearity was contained in the inhomogeneous term, namely

$$u_t = u_{xx} + g(t,x,u,u_x). \tag{81}$$

In a detailed analysis they applied the method of preliminary group classification to this equation and were able to obtain the forms of the function $g(t,x,u,u_x)$ for which the equation was invariant under one-, two-, three- and four-dimensional Lie algebras. In addition, a more general form of eq. (81), namely

$$u_t = f(t,x,u,u_x)u_{xx} + g(t,x,u,u_x) \tag{82}$$

was considered in [26]. As a result, the equivalence class was extended to equations invariant under five-dimensional Lie algebras. These equations are clearly outside the class of equations we considered. Their analysis however, cannot be used in our equation as our equations admit larger classes of equivalence transformations (indeed we have an infinite-dimensional Lie algebra) than their more general equation as pointed out in [27,28]. (Both these papers consider equations more specific than our equation.)

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