PRAMANA — journal of physics

Vol. 75, No. 4
October 2010
pp. 709–718

On complexly coupled modified KdV equations

AMITAVA CHOUDHURI
Department of Physics, Visva-Bharati University, Santiniketan 731 235, India
E-mail: amitava_ch26@yahoo.com

MS received 28 December 2009; accepted 8 June 2010

Abstract. We introduced complexly coupled modified KdV (ccmKdV) equations, which could be derived from a two-layer fluid model [Yang and Mao, *Chin. Phys. Lett.* **25**, 1527 (2008); Hu, *J. Phys. A: Math. Theor.* **43**, 185207 (2009)], and used the Miura transformation to construct expressions for their alternative Lax pair representations. We derived a Lagrangian-based approach to study the Hamiltonian structures of the ccmKdV equations and observed that the complexly coupled mKdV equations have an additional analytic structure. The coupled equations were characterized by two alternative Lagrangians not connected by a gauge term. We examined how the alternative Lagrangian descriptions of the system affect the bi-Hamiltonian structures.

Keywords. Complexly coupled modified KdV equations; analytic representation; Lax pair representation; bi-Hamiltonian structure.

PACS Nos 47.20.Ky; 42.81.Dp; 02.30.Jr

1. Introduction

Soliton theory was developed after Gardner, Greene, Kruskal and Miura (GGKM) [1] discovered the inverse scattering transform for the well-known Korteweg–de Vries (KdV) equation. This equation was formulated to explain the solitary water waves observed by John Scott Russell in the Edinburgh Glasgow Canal. It is a nonlinear equation in one spatial and one temporal (1+1) dimensions and admits soliton solution. The exact solvability of KdV equation implies that KdV equation possesses an infinite sequence of independent integrals which are in involution (two independent integrals are said to be in involution if their Poisson bracket commutes) [2]. The infinite number of conserved densities that generate flows which commute with the KdV flow give rise to the KdV hierarchy. Lax [3] showed that the KdV equation

$$u_t = u_{3x} - 6uu_x \tag{1}$$

satisfies the solvability condition for the system

$$L\psi = \lambda\psi \tag{2a}$$

and

$$\partial_t \psi = A\psi, \quad \partial_t = \frac{\partial}{\partial t} \tag{2b}$$

with

$$L = -\partial_x^2 + u, \tag{3a}$$

the so-called Schrödinger operator. Here $\mathcal{A}$ is a third-order linear operator written as

$$A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u. \tag{3b}$$

The existence of the solution $\psi = \psi(\lambda, x, t)$ for every constant λ is equivalent to

$$\partial_t L = AL - LA = [A, L]\,. \tag{4}$$

The results in (4) with (3a) and (3b) reduces to the KdV equation (1). Equation (4) is called the Lax representation and L and A are the linear operators, called Lax pair [3]. Any integrable equation can be written in the form given in (4). It is not always a straightforward task to find these operators for a given equation. In fact, no systematic procedure was devised to determine whether a given nonlinear partial differential equation could be represented as in (4). In the context of Lax's method, it is often said that L defines the original spectral problem while A represents an auxiliary spectral problem. The Lax pair representation holds good for all equations in the KdV hierarchy. The equations in the KdV hierarchy can also be generated by making use of the recursion operator

$$\Lambda = \partial_x^2 - 4u - 2u_x\partial_x^{-1}, \quad \partial_x = \frac{\partial}{\partial x} \tag{5}$$

in the differential relation

$$u_t = \Lambda^n u_x, \quad n = 0, 1, 2, 3, \ldots. \tag{6}$$

As one goes along the hierarchy, L remains unchanged but the differential operator associated with the auxiliary spectral problems changes. The identification of the KdV equation as an isospectral flow of the Schrödinger operator given in (3a) enables GGKM to devise a method for solving the KdV equation, called inverse scattering or inverse scattering transform (IST) mentioned before.

Soon after the Lax formulation was discovered, Zakharov and Faddeev [4] developed the Hamiltonian approach to integrability of nonlinear evolution equations in one spatial and one temporal (1+1) dimensions and, Gardner [5] interpreted the KdV equation as a completely integrable Hamiltonian system with ∂_x as the relevant Hamiltonian operator. A significant development in the Hamiltonian theory is due to Magri [6]. According to Magri [6], integrable Hamiltonian systems have an additional structure. They are bi-Hamiltonian, i.e. they are Hamiltonian with respect to two different compatible Hamiltonian operators ∂_x and $\left(\partial_x^3 - 4u\partial_x - 2u_x\right)$ such that

$$u_t = \partial_x \left(\frac{\delta H_n}{\delta u} \right) = \left(\partial_x^3 - 4u\partial_x - 2u_x \right) \left(\frac{\delta H_{n-1}}{\delta u} \right), \quad n = 1, 2, 3 \ldots. \tag{7}$$

Here $H_n = \int \mathcal{H}_n \, \mathrm{d}x$ with $\mathcal{H}_n$, the infinite conserved densities for the KdV equation. These conserved densities generate flows which commute with the KdV flow and as such give rise to an appropriate hierarchy. Traditionally, the expression for $\mathcal{H}_n$ is constructed using a mathematical formulation that does not make explicit reference to the Lagrangians of the equations in the hierarchy. A Lagrangian-based approach, however, can be used to identify $\mathcal{H}_n$ as the Hamiltonian density of the nth hierarchical equation in the KdV flow [7].

To ascertain whether the KdV equation was the only such equation possessing so many conserved densities, Miura [8] investigated the generalized KdV equation of the form

$$u_t = u_{3x} - 6u^s u_x \tag{8}$$

and found for $s = 1$ (KdV) and $s = 2$, there existed many conserved densities. For $s = 2$, eq. (8) reduces to an equation, the so-called modified KdV (mKdV) equation written in terms of variable v as

$$v_t = v_{3x} - 6v^2 v_x. \tag{9}$$

It has many applicative relevance. For example, mKdV equation is used to describe acoustic waves in anharmonic lattices and Alfvén waves in collisionless plasma. The recursion operator Λ for the mKdV equation [9]

$$\Lambda_m = \partial_x^2 - 4v^2 - 4v_x \partial_x^{-1} \cdot v \tag{10}$$

generates the mKdV hierarchy through the relation

$$v_t = \Lambda_m^n v_x, \quad n = 0, 1, 2, 3, \ldots. \tag{11}$$

It is interesting to note that the recursion operator given in (10) can be identified from (5) through the ingenious nonlinear transformation of Miura (the so-called Miura transformation) [8] given by

$$u = v_x + v^2, \quad v = v(x, t). \tag{12}$$

This transformation converts the KdV equation ($s = 1$) into a modified KdV (mKdV) ($s = 2$) equation given in (9). The Miura transformation (12), which relates solutions of the KdV and mKdV equations, may be derived by factorizing the scattering operator (3a) for the KdV equation. From this factorization, one may derive Wadati's [10] scattering problem which is used to solve the mKdV equation. The mKdV equation differs from the KdV equation only because of its cubic nonlinearity. The existence of infinite number of conserved densities of the mKdV equation implies the existence of infinite family of equations which are called the mKdV hierarchy. It is straightforward to obtain the equations in the mKdV hierarchy from those in the KdV hierarchy using Miura transformation. The Lax pair and the Hamiltonian structures of the equations in the mKdV hierarchy can be constructed from the corresponding results for the KdV equation. Miura transformation played an important role in this construction [11]. The mKdV equation was studied extensively because of its simplicity and physical significance [12]. Generalization of the mKdV equation to a multicomponent system or matrix equation was studied in [13]. Relatively recently, Iwao and Hirota [14] discussed a simple coupled version of the mKdV equation or the so-called coupled mKdV equation

$$\frac{\partial v_i}{\partial t} - \frac{\partial^3 v_i}{\partial x^3} + 6\left(\sum_{j,k=1}^{M} C_{jk} v_j v_k\right)\frac{\partial v_i}{\partial x} = 0, \quad i = 1, 2, \ldots M, \tag{13}$$

where the constant C_{jk} are set to be symmetric with respect to the subscripts $C_{jk} = C_{kj}$, without any loss of generality. The integrability of the equation was discussed by the bi-Hamiltonian [15] and Lax pair [16]. They derived the solution of the coupled version of the mKdV equation for $M = 1$ and 2.

We have sought to work with the complexly coupled modified KdV (ccmKdV) equations. The system of equations is a special case of the coupled variable coefficient modified KdV equation. The ccmKdV equation can be derived from a two-layer fluid model and can also be obtained from the mKdV equation by assuming its field complex. Recently, a number of works [17] was envisaged to study the analytical solutions of the ccmKdV equation. In this work we shall provide an analytic representation of the ccmKdV equation. By analytic representation we mean a representation of the system in terms of Euler–Lagrange's equations, i.e. the system of equations follow from an action principle. To that end we shall have to solve the associated inverse problem of variational calculus [18]. More specifically, we shall envisage a variational formulation for the coupled system implied by the ccmKdV equation. The approach to the present inverse problem is somewhat different from that used for higher mKdV equations [11]. Here the system of equations is of lowest order such that we can proceed simply by applying the Helmholtz theorem as realized in the calculus of differential forms [18]. We shall also provide Lax pair representation and construct Hamiltonian structures for the system of equations. We shall show that ccmKdV equations have two alternative Lagrangian representations with some interesting consequences on its Hamiltonian structures. The results presented are expected to serve as a useful test of integrability.

In §2 we introduce the ccmKdV equations and solve the inverse problem of variational calculus to construct expressions for the Lagrangian densities. We found that the system of equations followed from the action principle and as such could be obtained from appropriate Lagrangian densities via the so-called Euler–Lagrange equations. In §3 we shall derive their Lax pair representations. The corresponding Hamiltonian densities derived from Lagrangian densities constituted the conserved densities of the ccmKdV equations. We then used these Hamiltonian densities to study the Hamiltonian structures of the ccmKdV equations. Finally, in §4 we present some concluding remarks.

2. ccmKdV equations: Inverse variational problem

For $M = 2$, eq. (13) reduces two coupled equations of the form

$$\frac{\partial v_1}{\partial t} - \frac{\partial^3 v_1}{\partial x^3} + 6[C_{11}v_1^2 + C_{22}v_2^2 + (C_{12} + C_{21})v_1 v_2]\frac{\partial v_1}{\partial x} = 0 \tag{14a}$$

and

$$\frac{\partial v_2}{\partial t} - \frac{\partial^3 v_2}{\partial x^3} + 6[C_{11}v_1^2 + C_{22}v_2^2 + (C_{12} + C_{21})v_1 v_2]\frac{\partial v_2}{\partial x} = 0. \tag{14b}$$

If we put $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $v_1 = p$ and $v_2 = q$, eqs (14a) and (14b) take the form

$$p_t - p_{3x} + 6(p^2 - q^2 + 2ipq)p_x = 0 \tag{15a}$$

and

$$q_t - q_{3x} + 6(p^2 - q^2 + 2ipq)q_x = 0. \tag{15b}$$

By multiplying (15b) by i and adding with (15a) we obtain

$$(p + iq)_t - (p + iq)_{3x} + 6(p + iq)^2(p + iq)_x = 0. \tag{16}$$

Equation (16) is the well-known mKdV equation if we identify

$$(p + iq) = v. \tag{16$'$}$$

If we completely separate out the real and imaginary parts of eq. (16) we get the following equations:

$$p_t = p_{3x} - 6(p^2 - q^2)p_x + 12pqq_x \tag{17a}$$

and

$$q_t = q_{3x} - 6(p^2 - q^2)q_x - 12pqp_x. \tag{17b}$$

These are the complexly coupled modified KdV (ccmKdV) equations. This system of equations can be written in a simple matrix representation of the form

$$\eta_t = \eta_{3x} - \mathcal{M}\eta_x \tag{18}$$

with

$$\eta = \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} 6(p^2 - q^2) & -12pq \\ 12pq & 6(p^2 - q^2) \end{pmatrix}.$$

2.1 *Inverse variational problem*

To study the variational formulation of these equations, it will be convenient to work with velocity potential corresponding to $p(x,t)$ and $q(x,t)$. We thus write

$$p = -\psi_x \quad \text{and} \quad q = -\phi_x, \quad \psi = \psi(x,t), \quad \phi = \phi(x,t). \tag{19}$$

The potentials ψ and ϕ can be introduced into the problem by observing that (17a) and (17b) can be obtained from

$$\alpha_1 = p\mathrm{d}x + (p_{2x} - 2p^3 + 6pq^2)\mathrm{d}t, \tag{20a}$$

and

$$\alpha_2 = q\mathrm{d}x + (q_{2x} + 2q^3 - 6p^2 q)\mathrm{d}t \tag{20b}$$

and conditions for them to be closed are

$$\mathrm{d}\alpha_1 = 0 \quad \text{and} \quad \mathrm{d}\alpha_2 = 0. \tag{21}$$

From (21), using Poincaré Lemma [18], we write

$$\alpha_1 = -\mathrm{d}\psi \quad \text{and} \quad \alpha_2 = -\mathrm{d}\phi, \tag{22}$$

where ψ and ϕ are zero form or scalars. Combining (21) and (22), and using (20) we obtain

$$-\psi_x = p, \quad -\psi_t = (p_{2x} - 2p^3 + 6pq^2) \tag{23a}$$

and

$$-\phi_x = q, \quad -\phi_t = (q_{2x} + 2q^3 - 6p^2 q). \tag{23b}$$

From the integrability conditions of (23) we get

$$\psi_t = \psi_{3x} - 2\psi_x^3 + 6\psi_x\phi_x^2 \tag{24a}$$

and

$$\phi_t = \phi_{3x} + 2\phi_x^3 - 6\phi_x\psi_x^2. \tag{24b}$$

We have found that these equations follow from a variational principle

$$\delta I = 0, \quad I = \int \mathcal{L}\,\mathrm{d}x\,\mathrm{d}t \tag{25}$$

with two different Lagrangian densities given by

$$\mathcal{L}^{\mathrm{d}} = \frac{1}{2}(\psi_x\psi_t - \phi_x\phi_t) - \frac{1}{2}(\psi_x\psi_{3x} - \phi_x\phi_{3x}) + \frac{1}{2}\psi_x^4 + \frac{1}{2}\phi_x^4 - 3\psi_x^2\phi_x^2 \tag{26a}$$

and

$$\mathcal{L}^{\mathrm{i}} = \frac{1}{2}(\psi_x\phi_t + \phi_x\psi_t) - \frac{1}{2}(\psi_x\phi_{3x} + \phi_x\psi_{3x}) + 2\psi_x^3\phi_x - 2\psi_x\phi_x^3. \tag{26b}$$

The superscripts d and i on $\mathcal{L}$ stand for direct and indirect. We call $\mathcal{L}^{\mathrm{d}}$ a direct Lagrangian because $\mathcal{L}^{\mathrm{d}}$, when substituted in the Euler–Lagrange equations [9]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \eta_t}\right) - \frac{\delta \mathcal{L}}{\delta \eta} = 0, \tag{27}$$

with the variational derivative

$$\frac{\delta}{\delta\eta} = \frac{\partial}{\partial\eta} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial\eta_x}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial}{\partial\eta_{2x}}\right) - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial}{\partial\eta_{3x}}\right), \tag{28}$$

for $\eta = \psi$ and $\eta = \phi$ gives the equation of motion for ψ_t and ϕ_t given in (24a) and (24b) respectively, whereas substitution of $\mathcal{L}^{\mathrm{i}}$ in (27) with $\eta = \psi$ leads to equation for ϕ_t given in (24b) and conversely. This implies that the Lagrangian representation in (26b) is indirect [18]. Note that, these two Lagrangians in (26) are not connected by a gauge term and are called the alternative Lagrangian representations.

3. Lax pair representations and bi-Hamiltonian structures

We shall now study some interesting features of the Lax pair representations and bi-Hamiltonian structures of (17).

3.1 *Lax pair representation*

The Lax pair for the complexly coupled mKdV equations can be obtained by taking recourse to the use of Miura transformation in (12) in the following way. Writing L and A in the form

$$L = L_1 + iL_2 \tag{29a}$$

and

$$A = B_1 + iB_2 \tag{29b}$$

and using (3), (12) and (16′) we get

$$L_1 = -\partial_x^2 + p_x + p^2 - q^2, \tag{30a}$$

$$L_2 = q_x + 2pq, \tag{30b}$$

$$B_1 = 4\partial_x^3 - 6(p_x + p^2 - q^2)\partial_x - 3(p_{2x} + 2pp_x - 2qq_x) \tag{31a}$$

and

$$B_2 = -6(q_x + 2pq)\partial_x - 3(q_{2x} + 2pq_x + 2qp_x). \tag{31b}$$

It is interesting to see that the following slightly modified forms of Lax equations

$$\frac{\mathrm{d}L_1}{\mathrm{d}t} = [B_1, L_1] - [B_2, L_2] \tag{32a}$$

and

$$\frac{\mathrm{d}L_2}{\mathrm{d}t} = [B_1, L_2] + [B_2, L_1] \tag{32b}$$

yield the coupled equations in (17) and appear to have alternative Lax pair representations.

3.2 *Bi-Hamiltonian structure*

To derive the bi-Hamiltonian structure for the coupled equations (17) we have to find the Hamiltonian densities corresponding to the Lagrangian densities given in (26). The Hamiltonian densities in terms of original variables p and q are given by

$$\mathcal{H}^{\mathrm{d}} = \frac{1}{2}(pp_{2x} - qq_{2x}) - \frac{1}{2}p^4 - \frac{1}{2}q^4 + 3p^2q^2 \tag{33a}$$

and

$$\mathcal{H}^{\mathrm{i}} = \frac{1}{2}(p_{2x}q + pq_{2x}) - 2p^3q + 2pq^3. \tag{33b}$$

It is interesting to note that

$$\mathcal{H} = \mathcal{H}^{\mathrm{d}} + i\mathcal{H}^{\mathrm{i}} \tag{34}$$

represents the Hamiltonian density of the ccmKdV equation, such that it, via the canonical equation

$$(p + iq)_t = \partial_x \frac{\delta \mathcal{H}}{\delta(p + iq)}, \tag{35}$$

gives the coupled equations in (17). To realize the bi-Hamiltonian structure of the ccmKdV equation we consider equations obtained from the complex form of the linear equation in (11) in the mKdV hierarchy with $n = 0$. For these equations, the direct and indirect Hamiltonian densities

$$\mathcal{H}^{\mathrm{d}} = \frac{1}{2}(p^2 - q^2) \quad \text{and} \quad \mathcal{H}^{\mathrm{i}} = pq \tag{36}$$

give the total Hamiltonian densities

$$\mathcal{H} = \frac{1}{2}(p^2 - q^2) + ipq \tag{37}$$

The result for $\mathcal{H}$ when used in

$$(p+iq)_t = (\partial_x^3 - 4(p+iq)^2\partial_x - 4(p+iq)_x\partial_x^{-1}(p+iq)\partial_x)\frac{\delta\mathcal{H}}{\delta(p+iq)} \tag{38}$$

once again gives the coupled equations in (17). The first Hamiltonian operator ∂_x is the same as that of KdV and mKdV equations. The second Hamiltonian operator is constructed through the relation [9]

$$\Lambda_m\partial_x = (\partial_x^3 - 4(p+iq)^2\partial_x - 4(p+iq)_x\partial_x^{-1}(p+iq)\partial_x) \tag{39}$$

by putting the ansatz ($16'$) in mKdV's recursion operator given in (10). From (35) and (38) we see that as with the KdV and mKdV equations the ccmKdV equation possesses bi-Hamiltonian structure. This shows that the ccmKdV equations are integrable in the Liouville's sense [7].

4. Conclusion

The complexly coupled mKdV (ccmKdV) equations are a special case of the coupled variable coefficient modified KdV equation [17]. The ccmKdV equations can be derived from a two-layer fluid model and can also be obtained from the mKdV equation [17]. The ccmKdV equations satisfy positon, negaton and the complexiton solutions [17]. The nonlinear transformation of Miura (the so-called Miura transformation) is an aid to obtain the modified KdV (mKdV) equation from the KdV equation. We found that this transformation also provided an effective way to construct expressions for alternative Lax pair representations of the ccmKdV equations. As with the KdV equations, the bi-Hamiltonian structure of the mKdV equations were traditionally studied using involutive set of conserved Hamiltonian densities without explicit reference to their Lagrangians. A Lagrangian-based approach was used to realize the Hamiltonian and bi-Hamiltonian structures of ccmKdV equations. The ccmKdV equations have alternative Lagrangian representations and we used these alternative Lagrangians to construct the Hamiltonian and bi-Hamiltonian structures and examined how the alternative Lagrangian descriptions of the system affected the bi-Hamiltonian structure.

Acknowledgements

The author is deeply indebted to Prof. B Talukdar for drawing interest on the subject and for constructive discussions. The author would like to acknowledge the UGC, Government of India, for a project fellowship through grant No. F.32-39/2006(SR).

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