

## Eigenvalue spectra of a $\mathcal{PT}$ -symmetric coupled quartic potential in two dimensions

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**Abstract.** The Schrödinger equation was solved for a generalized  $\mathcal{PT}$ -symmetric quartic potential in two dimensions. It was found that, under a suitable ansatz for the wave function, the system possessed real and discrete energy eigenvalues. Analytic expressions for the energy eigenvalues and the eigenfunctions for the first four states were obtained. Some constraining relations among the wave function parameters rendered the problem quasi-solvable.

**Keywords.** Schrödinger equation; ground state; excited state;  $\mathcal{PT}$ -symmetry; eigenvalues; eigenfunctions.

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### 1. Introduction

In the last decade many interesting investigations on the  $\mathcal{PT}$ -symmetric quantum mechanics have generated renewed interest in the analysis of complex (non-Hermitian) potentials [1–7]. These studies showed that a non-Hermitian Hamiltonian could generate real and bounded eigenvalues except when  $\mathcal{PT}$ -symmetry was spontaneously broken in which case its complex eigenvalues should come in conjugate pairs. Therefore, it is now possible to investigate the eigenvalue spectra of a number of non-Hermitian Hamiltonian systems by imposing  $\mathcal{PT}$ -symmetric condition.

Most of these studies, however, are restricted to one-dimensional systems and their generalization in higher dimensions is needed for studying some nontrivial applications. With this motivation, recently we have studied a few two-dimensional complex systems [8–10] within the framework of an extended complex phase space. Here, with the same spirit, we found the quasi-exact solutions of the Schrödinger wave equation (SE) for a generalized  $\mathcal{PT}$ -symmetric complex quartic potential. In quantum mechanics, the quasi-exact solvable systems are those for which it is possible to find a finite number of (i.e. some specific or isolated) exact eigenvalues and the corresponding eigenfunctions in closed form. The non-Hermitian  $\mathcal{PT}$ -symmetric

quartic Hamiltonian, called ‘wrong sign’ Hamiltonian, is of the form

$$H = \frac{p^2}{2m} - gx^4, \quad g > 0, \quad (1)$$

and some of its variants were studied extensively in the past [7,11–15]. The Hamiltonian (1) is particularly interesting due to its analogy with a  $-\phi^4$  quantum field theory to model the dynamics of Higg’s sector of the Standard Model. But the main difficulty with (1) is that the wave functions do not vanish exponentially as  $|x| \rightarrow \infty$ , but oscillate on positive and negative real  $x$ -axes. However, when a cubic term  $i\lambda x^3$  is added to (1), the wave functions decay exponentially on real  $x$ -axis and also in the interior of the Stoke’s wedges. Thus, the presence of the imaginary cubic term in (1) enables one to calculate the  $\mathcal{C}$ -operator perturbatively which is essential to develop a consistent quantum theory to deal with non-Hermitian Hamiltonians.

Although there are various ways for obtaining complex Hamiltonians [16,17], in the present work, a complex quartic Hamiltonian is derived by choosing potential coupling parameters as complex. The same scheme of complexification was used in [18] also for studying a family of one-dimensional complex  $\mathcal{PT}$ -symmetric sextic potentials [10,19,20].

There are several methods for solving the SE for dynamical systems [21,22]. A technique known as the ansatz for the eigenfunction method, however, has been explored for obtaining ground and excited state energies of a variety of real potentials [23–26]. Very recently, Midya and Roy [27] investigated the quasi-exact solutions of the position-dependent mass Schrödinger equation for one-dimensional sextic potential using this technique. The same method was successfully used to obtain the eigenvalue spectra of a number of non-Hermitian complex potentials also [6,8–10,28].

The study of non-Hermitian Hamiltonians demands special attention because of their interesting applications in several areas of theoretical physics like superconductivity, population biology, quantum cosmology, condensed matter physics, quantum field theory, and so on [7].

The organization of the paper is as follows: in §2, a brief account of the main working steps of the eigenfunction ansatz method is presented. In §3, the expressions of the energy eigenvalues for the first four states for a general two-dimensional coupled quartic complex potential are obtained. Finally, the concluding remarks are presented in §4.

## 2. The method

Here, we describe the essential steps of the ansatz for the eigenfunction method for obtaining the solutions of the SE for two-dimensional systems.

The SE is written (for  $\hbar = m = 1$ ) as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2(E - V(x, y))\psi(x, y) = 0. \quad (2)$$

Next we make an ansatz for eigenfunction of the form

$$\psi(x, y) = \phi(x, y)e^{-g(x, y)}. \quad (3)$$

On substituting eq. (3) in eq. (2), we obtain

$$g_{xx} + g_{yy} - (g_x)^2 - (g_y)^2 + 2(E - V) + \frac{1}{\phi}(-2\phi_x g_x - 2\phi_y g_y + \phi_{xx} + \phi_{yy}) = 0, \quad (4)$$

where the subscripts indicate the differentiation with respect to the variables  $x$  and  $y$ .

From the above equation, it is clear that if the functions  $g$  and  $\phi$  are known for a given system, then rationalization of eq. (4) would directly provide the energy eigenvalues and eq. (3) would then act as the wave function for the system. However, the results for the ground state can be obtained by setting  $\phi(x, y) = \text{constant}$ . Therefore, for the ground state solutions, eq. (4) reduces to

$$g_{xx} + g_{yy} - (g_x)^2 - (g_y)^2 + 2(E - V) = 0. \quad (5)$$

Hence, to implement this scheme to solve SE for a specific potential, a few suitable forms of  $g(x, y)$  and  $\phi(x, y)$  are required. For polynomial type of potentials, these may be assumed polynomials as well.

So in the next section we consider a two-dimensional coupled quartic potential and find its eigenvalue spectra under a suitable ansatz for eigenfunction by solving eq. (4).

### 3. Solutions for $\mathcal{PT}$ -symmetric quartic potential

Let us consider a general two-dimensional coupled quartic complex potential

$$V(x, y) = a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{30}x^3 + a_{03}y^3 + a_{12}xy^2 + a_{21}x^2y + a_{22}x^2y^2 + a_{31}x^3y + a_{13}xy^3 + a_{40}x^4 + a_{04}y^4, \quad (6)$$

where  $a_{ij}$  are constants. The potential (6) will be  $\mathcal{PT}$ -symmetric, if  $a_{10}, a_{01}, a_{12}, a_{21}, a_{30}, a_{03} \in i\Re$  and  $a_{20}, a_{02}, a_{11}, a_{22}, a_{31}, a_{13}, a_{40}, a_{04} \in \Re$ .

For the present system, the ansatz for the function  $g(x, y)$  is made as

$$g(x, y) = \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 + \alpha_{02}y^2 + \alpha_{11}xy + \alpha_{12}xy^2 + \alpha_{21}x^2y + \alpha_{30}x^3 + \alpha_{03}y^3, \quad (7)$$

where the coefficients  $\alpha_{10}, \alpha_{01}, \alpha_{12}, \alpha_{21}, \alpha_{30}, \alpha_{03} \in i\Re$  and  $\alpha_{20}, \alpha_{02}, \alpha_{11} \in \Re$  are chosen to ensure the  $\mathcal{PT}$ -symmetry of the wave function.

Now, to solve the SE for the potential (6), in the following subsections, we assume the polynomial forms of  $\phi(x, y)$  for the first four states.

#### 3.1 Ground state solutions

For the ground state solution,  $\phi(x, y) = 1$ . Thus, using eqs.(6) and (7) in eq. (5) and on rationalization, we get the following set of algebraic equations

$$E = \alpha_{02} + \alpha_{20} - \frac{1}{2}(\alpha_{10}^2 + \alpha_{01}^2), \quad (8)$$

$$a_{01} = \alpha_{10}\alpha_{11} + 2\alpha_{02}\alpha_{01} - 3\alpha_{03} - \alpha_{21}, \quad (9)$$

$$a_{10} = \alpha_{01}\alpha_{11} + 2\alpha_{20}\alpha_{10} - 3\alpha_{30} - \alpha_{12}, \quad (10)$$

$$a_{12} = 4\alpha_{02}\alpha_{12} + 3\alpha_{11}\alpha_{03} + 2\alpha_{20}\alpha_{12} + 2\alpha_{11}\alpha_{21}, \quad (11)$$

$$a_{21} = 4\alpha_{20}\alpha_{21} + 3\alpha_{11}\alpha_{30} + 2\alpha_{02}\alpha_{21} + 2\alpha_{11}\alpha_{12}, \quad (12)$$

$$a_{11} = 2(\alpha_{20}\alpha_{11} + \alpha_{02}\alpha_{11} + \alpha_{21}\alpha_{10} + \alpha_{01}\alpha_{12}), \quad (13)$$

$$a_{22} = 2\alpha_{21}^2 + 2\alpha_{12}^2 + 3\alpha_{12}\alpha_{30} + 3\alpha_{21}\alpha_{03}, \quad (14)$$

$$a_{02} = 3\alpha_{03}\alpha_{01} + \alpha_{10}\alpha_{12} + 2\alpha_{02}^2 + \alpha_{11}^2/2, \quad (15)$$

$$a_{20} = \alpha_{21}\alpha_{01} + 3\alpha_{30}\alpha_{10} + 2\alpha_{20}^2 + \alpha_{11}^2/2, \quad (16)$$

$$a_{03} = 6\alpha_{02}\alpha_{03} + 2\alpha_{11}\alpha_{12}, \quad (17)$$

$$a_{30} = 6\alpha_{20}\alpha_{30} + 2\alpha_{11}\alpha_{21}, \quad (18)$$

$$a_{31} = 6\alpha_{21}\alpha_{30} + 2\alpha_{12}\alpha_{21}, \quad (19)$$

$$a_{13} = 6\alpha_{12}\alpha_{03} + 2\alpha_{12}\alpha_{21}, \quad (20)$$

$$a_{40} = (9\alpha_{30}^2 + \alpha_{21}^2)/2, \quad (21)$$

$$a_{04} = (9\alpha_{03}^2 + \alpha_{12}^2)/2, \quad (22)$$

among the potential coupling parameters, wave function parameters and the energy  $E$ . The above equations provide solutions of potential parameters  $a_{ij}$  in terms of the wave function parameters  $\alpha_{ij}$ . Therefore, for various choices of  $\alpha_{ij}$ , different  $a_{ij}$  will be obtained. In this way one can obtain the eigenvalue spectra of a family of complex quartic potentials in two dimensions. Note that the energy eigenvalue, eq. (8), is real as  $\alpha_{20}, \alpha_{02} \in \Re$ .

It is worth mentioning that the potential (6) essentially forms a two-dimensional quartic 'wrong sign' Hamiltonian because the potential coefficients  $a_{31}$ ,  $a_{13}$ ,  $a_{40}$  and  $a_{04}$  (eqs (19)–(22)), are negative numbers as  $\alpha_{12}$ ,  $\alpha_{21}$ ,  $\alpha_{03}$  and  $\alpha_{30}$  are assumed imaginary constants.

### 3.2 First excited state solution

Now we consider the second case, for which  $\phi(x, y)$  is given as

$$\phi(x, y) = x + y + \gamma_1, \quad (23)$$

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where  $\gamma_1$  is a complex constant. So, using eqs (6), (7) and (23) in eq. (4), we get a set of 17 equations out of which 12 equations are the same as given by eqs (11)–(22) and the remaining five equations are written as

$$E = \gamma_1(\alpha_{21} + a_{01} + 3\alpha_{03} - \alpha_{10}\alpha_{11} - 2\alpha_{10}\alpha_{02}) + \alpha_{20} + \alpha_{11} - \alpha_{10}^2 + 3\alpha_{02}, \quad (24)$$

$$E = \gamma_1(\alpha_{12} + a_{10} + 3\alpha_{30} - \alpha_{10}\alpha_{11} - 2\alpha_{10}\alpha_{20}) + \alpha_{02} + \alpha_{11} - \alpha_{10}^2 + 3\alpha_{20}, \quad (25)$$

$$E = (\alpha_{20} + \alpha_{02} - \alpha_{10}^2) - 2\alpha_{10}/\gamma_1, \quad (26)$$

$$a_{10} = -\alpha_{12} - \alpha_{21} - 6\alpha_{30} + \alpha_{11}\alpha_{01} + 2\alpha_{10}\alpha_{20}, \quad (27)$$

$$a_{01} = -\alpha_{12} - \alpha_{21} - 6\alpha_{03} + \alpha_{11}\alpha_{10} + 2\alpha_{01}\alpha_{02}. \quad (28)$$

Here, we get three relations for energy  $E$ . These expressions will provide unique and real energy eigenvalues when

$$\alpha_{01} = \alpha_{10}, \quad \alpha_{20} = \alpha_{02}, \quad \alpha_{12} = \alpha_{21} = -3\alpha_{30} = -3\alpha_{03}. \quad (29)$$

Thus under these restrictions, the energy eigenvalue  $E$  and the value of  $\gamma_1$  are computed as

$$E = 4\alpha_{20} + \alpha_{11} - \alpha_{10}^2, \quad (30)$$

$$\gamma_1 = \frac{-2\alpha_{10}}{\alpha_{11} + 2\alpha_{20}}. \quad (31)$$

Note that the energy is real and discrete as  $\alpha_{20}, \alpha_{11} \in \mathfrak{R}$ .

### 3.3 Second excited state solutions

For obtaining the energy eigenvalues of the second excited state of the system (6), we take the third choice of  $\phi(x, y)$  as

$$\phi(x, y) = x^2 + y^2 + a_3xy + a_2x + a_1y + a_0, \quad (32)$$

where  $a_0$  and  $a_3$  are real numbers while  $a_1$  and  $a_2$  are pure complex constants.

Again inserting eqs (6), (7) and (32) in eq. (4), we obtain a set of 20 algebraic equations. Out of these, six relations involving energy and two for  $a_{10}$  and  $a_{01}$  are different from eqs (11)–(22). The six relations involving  $E$  provide the energy equation as

$$E = 6\alpha_{20} + 2\alpha_{11} - (\alpha_{10}^2 + \alpha_{01}^2)/2, \quad (33)$$

under the same condition (29). The coefficients  $a_0, a_1, a_2$  and  $a_3$  are calculated as

$$a_0 = \frac{(2\alpha_{20} + \alpha_{11}) + (\alpha_{01} + \alpha_{10})^2}{(2\alpha_{20} + \alpha_{11})^2}, \quad (34)$$

$$a_1 = a_2 = \frac{\alpha_{10} + \alpha_{01}}{\alpha_{20} + \alpha_{11}/2}, \quad a_3 = 2. \quad (35)$$

Here also the energy eigenvalues are real and discrete.

### 3.4 Third excited state solutions

Finally, for obtaining the energy eigenvalues of the third excited state of system (6), the fourth choice of  $\phi(x, y)$  is made as

$$\begin{aligned} \phi(x, y) = & x^3 + y^3 + \beta_{21}x^2y + \beta_{12}xy^2 + \beta_{11}xy \\ & + \beta_{20}x^2 + \beta_{02}y^2 + \beta_{10}x + \beta_{01}y + \beta_0, \end{aligned} \quad (36)$$

where  $\beta_{11}$ ,  $\beta_{20}$  and  $\beta_{02}$  are the real numbers whereas  $\beta_{21}$ ,  $\beta_{12}$ ,  $\beta_{10}$ ,  $\beta_{01}$  and  $\beta_0$  are complex constants.

Inserting eqs (6), (7) and (36) in eq. (4), we obtain a set of 24 algebraic equations. Out of these, ten equations involving energy and two for  $a_{10}$  and  $a_{01}$  are different from eqs (11)–(22).

These ten relations involving  $E$  again reduce to a single energy equation

$$E = 8\alpha_{20} + \beta_{21}\alpha_{11} + \beta_{11}\alpha_{21} - 12\alpha_{30} - \alpha_{10}^2, \quad (37)$$

under the restriction (29). Further, the solutions of various  $\beta_{ij}$  in terms of  $\alpha_{ij}$  may be obtained by selecting  $\beta_{21} = \beta_{12}$ ,  $\beta_{20} = \beta_{02}$  and  $\beta_{10} = \beta_{01} = 1$ . The coefficients  $\beta_{20}$ ,  $\beta_{21}$ ,  $\beta_{11}$ ,  $\beta_0$  are given as

$$\beta_{20} = \frac{16\alpha_{20} + 2\alpha_{11}(8 + \alpha_{30}^{-1}) + 3}{12\alpha_{30} + 2\alpha_{10} - 18}, \quad \beta_{21} = 3. \quad (38)$$

$$\beta_{11} = \frac{-3}{4} - \frac{\alpha_{11}}{2\alpha_{30}} + \frac{9(16\alpha_{20} + 2\alpha_{11}(8 + \alpha_{30}^{-1}) + 3)}{4(6\alpha_{30} + \alpha_{10} - 9)}, \quad (39)$$

$$\beta_0 = \frac{2\alpha_{10} - 2\beta_{20}}{6\alpha_{20} + 3\alpha_{11} + \beta_{11}\alpha_{21} - 12\beta_{20}\alpha_{30}}. \quad (40)$$

Here also, the energy eigenvalue is real and discrete.

## 4. Conclusions

In this work, we have calculated the energy eigenvalues of a general two-dimensional  $\mathcal{PT}$ -symmetric coupled quartic potential using an ansatz for the wave function.

The SE is rationalized for suitable forms of  $g(x, y)$  and  $\phi(x, y)$ . The parameters of potential and  $\phi(x, y)$  are expressed in terms of  $g(x, y)$  parameters. This can provide a number of coupled quartic potentials for various choices of  $g(x, y)$  parameters  $\alpha_{ij}$ . The energy eigenvalues are computed for the first four states. By considering higher-order polynomials for  $\phi(x, y)$ , other higher excited state solutions for the quartic potential can be computed but expansion of algebra makes the calculations difficult. The eigenvalue spectra are found to be real and discrete. The solutions found in this study are exact with certain constraints on the potential parameters (see eq. (29)).

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