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Travelling wave solutions for $(N+1)$-dimensional nonlinear evolution equations

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Abstract. In this paper, we implement the exp-function method to obtain the exact travelling wave solutions of $(N+1)$-dimensional nonlinear evolution equations. Four models, the $(N+1)$-dimensional generalized Boussinesq equation, $(N+1)$-dimensional sine-cosine-Gordon equation, $(N+1)$-double sinh-Gordon equation and $(N+1)$-sinh-cosinh-Gordon equation, are used as vehicles to conduct the analysis. New travelling wave solutions are derived.

Keywords. Exp-function method; $(N+1)$-dimensional nonlinear equations; travelling wave solutions.

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1. Introduction

The nonlinear evolution equations have attracted the attention of many researchers because of their wide applications in various fields such as physics, fluid mechanics, biomathematics, chemical physics and other areas of science and engineering. The investigation of exact solutions for the nonlinear evolution equations is a particularly hot topic. To understand the nonlinear phenomena better as well as to apply them in real-life situations, it is important to find their exact solutions. Further, in recent years, much attention has been paid to study of solutions of nonlinear wave equations in low dimensions. But only little work is done on the high-dimensional equations. Motivated by this consideration, in this paper we deal with the travelling wave solutions of high-dimensional equations.

Recently, Yan [1] studied the $(N+1)$-dimensional generalized Boussinesq equation of the following form:

$$u_{tt} = u_{xx} + \lambda(u^n)_{xx} + u_{xxxx} + \sum_{j=1}^{N-1} u_{y_j y_j}, \tag{1}$$

where $\lambda \neq 0$ is a constant and $N > 1$ is an integer. When $n = 3$ and $N = 2$, eq. (1) represents the $(N+1)$-dimensional shallow water model, which has multiple soliton solutions [2].

Further, Wang *et al* [3] studied the following three $(N+1)$-dimensional nonlinear evolution equations:

$$\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} - \alpha\cos(u) - \beta\sin(2u) = 0, \tag{2}$$

$$\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} - \alpha\sinh(u) - \beta\sinh(2u) = 0, \tag{3}$$

$$\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} - \alpha\cosh(u) - \beta\sinh(2u) = 0. \tag{4}$$

Equations (2), (3) and (4) are called the $(N+1)$-dimensional sine-cosine-Gordon equation, double sinh-Gordon equation and sinh-cosinh-Gordon equation, respectively. Because of the wide applications of the $(N+1)$-dimensional equation in real-world problems, the search for exact solutions is of great importance and interest.

Different numerical methods, such as Jacobi elliptic function method [4], variational iteration method [5–7], tanh function method [8–11], homotopy perturbation method [12–15], direct algebraic method [16], manifold theory [17], integral method [18] and so on, have been proposed by various authors for solving nonlinear evolution equation. More recently, He and Wu [19] proposed a straightforward and concise method called the exp-function method to explore exact solutions of the modified KdV equation. This is a very powerful technique for solving nonlinear problems and its applications can be found in [20–25]. The purpose of this paper is to obtain new travelling wave solutions of higher-dimensional equations (eqs (2)–(4)) by applying the exp-function method. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

2. Solutions of $(N+1)$-dimensional generalized Boussinesq equation

To obtain the solution for eq. (1), we consider the transformation $u(x, y_1, y_2, \ldots, y_{N-1}, t) = u(\eta)$, $\eta = \tau(x + \sum_{j=1}^{N-1} y_j - ct)$ where $\tau \neq 0$ and $c \neq 0$. We can rewrite eq. (1) in the following nonlinear ordinary differential equation of the form

$$(N - c^2)u'' + \lambda(u^n)'' + \tau^2 u'''' = 0, \tag{5}$$

where the prime denotes derivative with respect to η. Integrating eq. (5) with respect to η and ignoring the constant of integration, we obtain

$$(N - c^2)u' + \lambda(u^n)' + \tau^2 u''' = 0. \tag{6}$$

Next we introduce the transformation $u^{n-1} = v$. Then we have

$$u' = \frac{1}{n-1} v^{[1/(n-1)-1]} v', \quad (u^n)' = \frac{n}{n-1} v^{[1/(n-1)]} v', \tag{7}$$

$$u''' = \frac{(n-2)(2n-3)}{(n-1)^3} v^{[1/(n-1)-3]} (v')^3 + \frac{3(2-n)}{(n-1)^2} v^{1/(n-1)-2} v' v'' + \frac{1}{n-1} v^{[1/(n-1)-1]} v'''. \tag{8}$$

Substituting (7) and (8) in eq. (6), we can rewrite the $(N+1)$-dimensional generalized Boussinesq eq. (5) in the following form:

$$(N - c^2)(n-1)^2 v^2 v' + \lambda n (n-1)^2 v^3 v' + \tau^2 (n-2)(2n-3)(v')^3 + 3\tau^2 (n-1)(2-n) v v' v'' + \tau^2 (n-1)^2 v^2 v''' = 0. \tag{9}$$

According to the exp-function method [19], we assume that the solution of eq. (8) can be expressed in the following form:

$$v(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{10}$$

where c, d, p and q are positive integers which are unknown to be determined further, a_n and b_m are unknown constants. To determine the values of c and p, we balance the linear term of the highest order in eq. (9) with the highest order nonlinear term. By simple calculation, we obtain $7p + 3c = 6p + 4c$ which gives $p = c$. Similarly, to determine the values of d and q, we balance the linear term of the lowest order in eq. (9) with the lowest order nonlinear term. We obtain $-(7q + 3d) = -(6q + 4d)$ which gives $q = d$.

We can freely choose the values of c and d, but the final solution does not strongly depend on values of c and d. For simplicity, we set $p = c = 1, b_1 = 1$ and $d = q = 1$. Then eq. (10) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{11}$$

Substituting eq. (11) in eq. (9) and using the Maple, equating to zero the coefficients of all powers of $\exp(n\eta)$ gives a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, \tau$ and c. Solving the systems of algebraic equations using Maple gives the following sets of nontrivial solution:

$$\left\{ a_1 = 0, a_0 = \frac{b_0 (n+1) \tau^2}{(n-1)^2 \lambda}, a_{-1} = 0, b_0 = \text{arb.}, b_{-1} = \frac{b_0^2}{4}, \tau = \text{arb.}, c = \pm \frac{\sqrt{\tau^2 + N(n-1)^2}}{n-1} \right\}, \tag{12}$$

where arb. is an arbitrary constant. Substituting eq. (12) in eq. (11) and using the transformation $u^{n-1} = v$, we obtain the following wave solution of eq. (1):

$$u(x, \tilde{y}, t) = \left(\frac{4 b_0 (n+1) \tau^2}{(n-1)^2 \lambda} \right)^{1/(n-1)} \times \left[\frac{1}{4 \exp(\eta) + 4 b_0 + b_0^2 \exp(-\eta)} \right]^{1/(n-1)}, \tag{13}$$

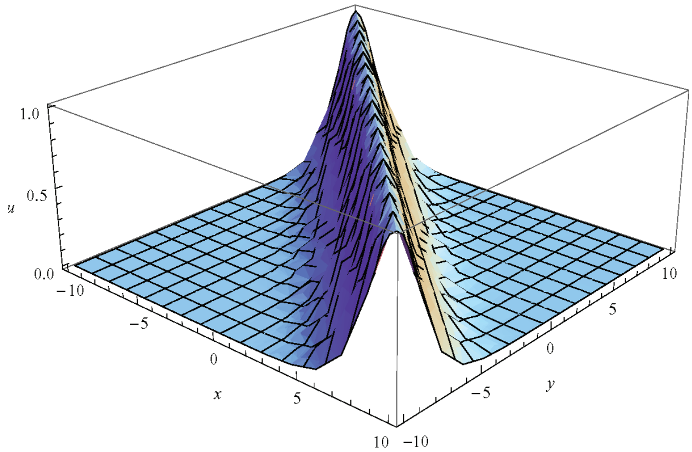

Figure 1. Solution of (13) is shown at $b_0 = 2, N = 2, n = 2, \lambda = 1, \tau = 1$.

where

$$\tilde{y} = y_1, y_2, \ldots, y_{N-1}$$

and

$$\eta = \tau\left(x + \sum_{j=1}^{N-1} y_j \pm \frac{\sqrt{\tau^2 + N(n-1)^2}}{n-1}t\right).$$

In particular, if we take $b_0 = 2, N = 2, n = 2, \lambda = 1$ and $\tau = 1$ in eq. (13) then we obtain the solution of eq. (1) in the form

$$u(x, y, t) = \frac{2}{1 + \cosh(x + y \pm \sqrt{3}t)}$$

and if we take $b_0 = 2, N = 2, n = 5, \lambda = 1$ and $\tau = 1$ then we have

$$u(x, y, t) = \left(\frac{2}{1 + \cosh(x + y \pm \frac{\sqrt{33}}{4}t)}\right)^{1/5}.$$

The behaviour of the obtained exact solution (13) is shown graphically (see figure 1).

If we choose $\lambda = 3, n = 2, N = 1, \tau = \sqrt{c^2 - 1}, c^2 > 1, b_0 = \pm 2$ then eq. (13) turns out to the following solutions as in [26]:

$$u(x, t) = \pm\frac{c^2 - 1}{2}\mathrm{sech}^2\left[\frac{\sqrt{c^2 - 1}}{2}(x \pm ct)\right]. \tag{14}$$

If we choose $\lambda = 3, n = 2, N = 1, \tau = k_1, b_0 = 2$, then from solution (13), we obtain the following solution which is given in [26]:

$$u(x, t) = \frac{k_1^2}{2}\mathrm{sech}^2\left[\frac{k_1}{2}(x \pm \sqrt{k_1^2 + 1t})\right]. \tag{15}$$

3. Solutions of $(N+1)$-dimensional sine-cosine-Gordon equation

We look for the travelling wave solutions by considering the transformation $u(x_1, x_2, \ldots, x_N, t) = u(\eta)$, $\eta = \tau(\sum_{j=1}^{N} x_j - ct)$, where $\tau \neq 0$ and $c \neq 0$. Using the above transformation, eq. (2) can be rewritten in the following form:

$$\tau^2(N - c^2)u'' - \alpha\cos(u) - \beta\sin(2u) = 0, \tag{16}$$

where the prime denotes derivative with respect to η. Next, let us consider the transformation $u = 2\tan^{-1} v$, then we obtain

$$u'' = \frac{2(v'' + v''v^2 - 2(v')^2 v)}{(1+v^2)^2}, \quad \cos(u) = \frac{1-v^2}{1+v^2}, \quad \sin(2u) = \frac{4v(1-v^2)}{(1+v^2)^2}. \tag{17}$$

By substituting eq. (17) in eq. (16), we can rewrite the $(N+1)$-dimensional sine-cosine-Gordon eq. (2) in the following form:

$$\begin{aligned} &2\tau^2(N-c^2)(1+v^2)v'' - 4\tau^2(N-c^2)v(v')^2 \\ &\quad +(v^2-1)(\alpha v^2 + 4\beta v + \alpha) = 0. \end{aligned} \tag{18}$$

Substituting eq. (11) in eq. (18) and then equating to zero the coefficients of all powers of $\exp(n\eta)$ yields a system of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, \tau$ and c. Solving the systems of algebraic equations using Maple gives the following sets of nontrivial solutions (see Appendix A)

$$\begin{aligned} \Bigg\{ &a_1 = 1, a_0 = \text{arb.}, a_{-1} = -\frac{a_0^2\alpha}{4(\alpha+2\beta)}, b_0 = -a_0, b_{-1} = -\frac{a_0^2\alpha}{4(\alpha+2\beta)}, \\ &\tau = \text{arb.}, c = \pm\frac{\sqrt{\alpha+2\beta+N\tau^2}}{\tau} \Bigg\}, \end{aligned} \tag{19}$$

Substituting (19) and (42) in eq. (11) and using the given transformation, we obtain the following travelling wave solutions of eq. (2):

$$u(\tilde{x}, t) = 2\tan^{-1}\left[\frac{\exp(\eta) + a_0 - \frac{a_0^2\alpha}{4(\alpha+2\beta)}\exp(-\eta)}{\exp(\eta) - a_0 - \frac{a_0^2\alpha}{4(\alpha+2\beta)}\exp(-\eta)}\right], \tag{20}$$

$$u(\tilde{x}, t) = -2\tan^{-1}\left[\frac{\exp(\eta) - a_0 - \frac{a_0^2\alpha}{4(\alpha-2\beta)}\exp(-\eta)}{\exp(\eta) + a_0 - \frac{a_0^2\alpha}{4(\alpha-2\beta)}\exp(-\eta)}\right], \tag{21}$$

where $\tilde{x} = x_1, x_2, \ldots, x_N$ and $\eta = \tau\left(\sum_{j=1}^{N} x_j \pm \frac{\sqrt{-\alpha+2\beta+N\tau^2}}{\tau} t\right)$. If we set $a_0 = 2\sqrt{(\alpha+2\beta)/\alpha}, N = 2, \tau = 1, \alpha = 2$ and $\beta = 1$ in eq. (20), then we have

$$u(x,y,t) = 2\tan^{-1}\left[1+\frac{2\sqrt{2}}{\sinh(x+y\pm\sqrt{2}t)-\sqrt{2}}\right], \tag{22}$$

and setting $a_0 = 1, N = 2, \tau = 1, \alpha = 2$ and $\beta = 1$ in eq. (20) we also have

$$u(x,y,t) = 2\tan^{-1}\left[1+\frac{16}{7\cosh(x+y\pm\sqrt{2}t)+9\sinh(x+y\pm\sqrt{2}t)-8}\right]. \tag{23}$$

Moreover, from eqs (11), (43), (45) and (46), we obtain the following travelling wave solutions of eq. (2):

$$u(\tilde{x},t) = -2\tan^{-1}\left[\frac{\frac{2\beta\mp\sqrt{4\beta^2-\alpha^2}}{\alpha}\exp(\eta)+a_0-A_1\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)}\right], \tag{24}$$

where

$$\eta = \tau\left(\sum_{j=1}^{N} x_j - ct\right)$$

and

$$a_0 = \frac{b_0(8\beta^2-\alpha^2\mp 4\beta\sqrt{4\beta^2-\alpha^2})}{\alpha(2\beta\mp\sqrt{4\beta^2-\alpha^2})}.$$

$$u(\tilde{x},t) = 2\tan^{-1}\left[\frac{\frac{-2\beta+\sqrt{4\beta^2-\alpha^2}}{\alpha}\exp(\eta)+a_0+a_1\exp(-\eta)}{\exp(\eta)+b_0-\frac{\alpha(a_0^2\alpha+b_0^2\alpha+4a_0b_0\beta)}{4(4\beta^2-\alpha^2)}\exp(-\eta)}\right], \tag{25}$$

where

$$\eta = \tau\left(\sum_{j=1}^{N} x_j \pm \frac{\sqrt{\alpha^2-4\beta^2+2\tau^2N\beta}}{\tau\sqrt{2\beta}}t\right)$$

and

$$a_1 = \frac{\alpha^2(a_0^2\alpha+b_0^2\alpha+4a_0b_0\beta)}{4(4\beta^2-\alpha^2)(2\beta\mp\sqrt{4\beta^2-\alpha^2})}.$$

Further, the behaviour of the obtained exact solutions (22) and (23) are shown graphically (see figures 2 and 3).

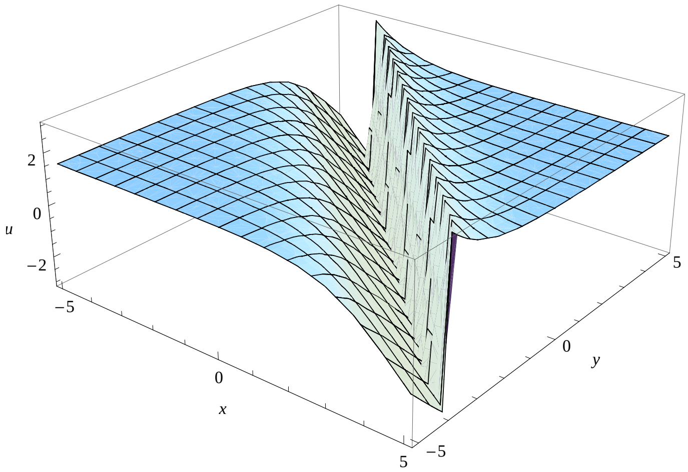

Figure 2. Graph of solution (22).

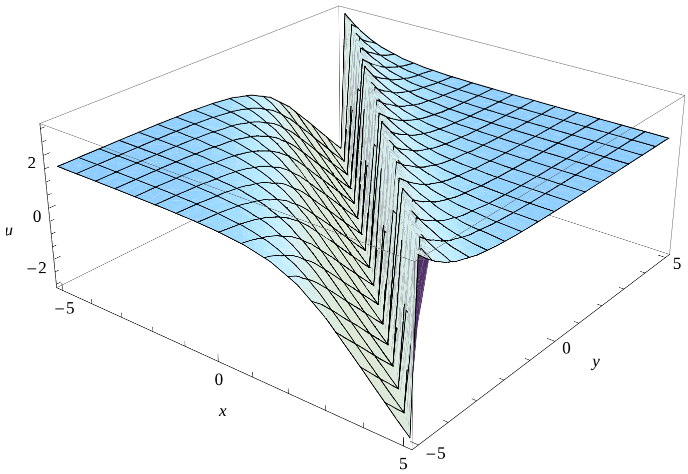

Figure 3. Graph of solution (23).

4. Solutions of $(N+1)$-dimensional double sinh-Gordon equation

By introducing the transformation $u(x_1, x_2, \ldots, x_N, t) = u(\eta)$, $\eta = \tau(\sum_{j=1}^{N} x_j - ct)$, where $\tau \neq 0$ and $c \neq 0$, we can convert eq. (3) into ordinary differential equation as

$$\tau^2(N - c^2)u'' - \alpha \sinh(u) - \beta \sinh(2u) = 0. \tag{26}$$

Further, consider the transformation $u = \ln v$. Then we have

$$u'' = \frac{v''v - (v')^2}{v^2}, \quad \sinh(u) = \frac{v - v^{-1}}{2}, \quad \sinh(2u) = \frac{v^2 - v^{-2}}{2}. \tag{27}$$

Substituting (27) in eq. (26), we can rewrite the $(N+1)$-dimensional double sinh-Gordon eq. (3) in the following form:

$$2\tau^2(N-c^2)(v''v-(v')^2)-\alpha(v^3-v)-\beta(v^4-1)=0. \tag{28}$$

Substitute eq. (11) in eq. (28) and using the Maple, equating to zero the coefficients of all powers of $\exp(n\eta)$ gives a set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}, \tau$ and c. Solving the systems of algebraic equations using Maple we obtain (see Appendix B)

$$\Big\{a_1=1, a_0=-b_0, a_{-1}=\frac{b_0^2\alpha}{4(\alpha+2\beta)}, b_0=\text{arb.}, b_{-1}=\frac{b_0^2\alpha}{4(\alpha+2\beta)},$$
$$\tau=\text{arb.}, c=\pm\frac{\sqrt{-\alpha-2\beta+N\tau^2}}{\tau}\Big\}, \tag{29}$$

Substituting (29) and (53) in eq. (11) and using the transformation, we obtain the following wave solutions of eq. (3):

$$u(\tilde{x},t)=\ln\left[\frac{\pm\exp(\eta)\mp b_0\pm\frac{b_0^2\alpha}{4(\alpha\pm2\beta)}\exp(-\eta)}{\exp(\eta)+b_0+\frac{b_0^2\alpha}{4(\alpha\pm2\beta)}\exp(-\eta)}\right], \tag{30}$$

where

$$\tilde{x}=x_1,x_2,\ldots,x_N$$

and

$$\eta=\tau\left(\sum_{j=1}^{N}x_j\pm\frac{\sqrt{\alpha-2\beta+N\tau^2}}{\tau}t\right).$$

$$u(\tilde{x},t)=\ln\left[\frac{\frac{-\alpha\pm\sqrt{\alpha^2-4\beta^2}}{2\beta}\exp(\eta)+a_1\exp(-\eta)}{\exp(\eta)+b_{-1}\exp(-\eta)}\right], \tag{31}$$

where

$$\eta=\tau\left(\sum_{j=1}^{N}x_j\pm\frac{\sqrt{-\alpha^2+4\beta^2+8\tau^2N\beta}}{2\tau\sqrt{2\beta}}t\right)$$

and

$$a_1=\mp\frac{b_{-1}(\alpha^2-4\beta^2\pm\alpha\sqrt{\alpha^2-4\beta^2})}{2\beta\sqrt{\alpha^2-4\beta^2}}.$$

$$u(\tilde{x},t) = \ln\left[\frac{\frac{-\alpha\pm\sqrt{\alpha^2-4\beta^2}}{2\beta}\exp(\eta)+\frac{b_0(2\beta^2-\alpha^2\pm\alpha\sqrt{\alpha^2-4\beta^2})}{\beta(\alpha\mp\sqrt{\alpha^2-4\beta^2})}+A_2\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)}\right], \tag{32}$$

where $\eta=\tau\left(\sum_{j=1}^{N}x_j-ct\right)$.

$$u(\tilde{x},t)=\ln\left[\frac{\frac{-\alpha\pm\sqrt{\alpha^2-4\beta^2}}{2\beta}\exp(\eta)+a_0+a_1\exp(-\eta)}{\exp(\eta)+b_0-\frac{\beta(a_0^2\beta+b_0^2\beta+a_0b_0\alpha)}{\alpha^2-4\beta^2}\exp(-\eta)}\right], \tag{33}$$

where

$$\eta=\tau\left(\sum_{j=1}^{N}x_j\pm\frac{\sqrt{-\alpha^2+4\beta^2+2\tau^2N\beta}}{\tau\sqrt{2\beta}}t\right)$$

and

$$a_1=\frac{2\beta^2(a_0^2\beta+b_0^2\beta+a_0b_0\alpha)}{(\alpha^2-4\beta^2)(\alpha\mp\sqrt{\alpha^2-4\beta^2})}.$$

5. Solutions of the $(N+1)$-dimensional sinh-cosinh-Gordon equation

To obtain the solutions for eq. (4), let us consider the transformation $u(x_1,x_2,\ldots,x_N,t)=u(\eta)$, $\eta=\tau(\sum_{j=1}^{N}x_j-ct)$, where $\tau\neq 0$ and $c\neq 0$, then we can rewrite eq. (4) in the following form:

$$\tau^2(N-c^2)u''-\alpha\cosh(u)-\beta\sinh(2u)=0. \tag{34}$$

We next introduce the transformation $u=\ln v$, then we get

$$u''=\frac{v''v-(v')^2}{v^2},\quad \cosh(u)=\frac{v+v^{-1}}{2},\quad \sinh(2u)=\frac{v^2-v^{-2}}{2}. \tag{35}$$

Substituting eq. (35) in eq. (34), we can rewrite the $(N+1)$-dimensional sinh-cosinh-Gordon eq. (4) in the following form:

$$2\tau^2(N-c^2)(v''v-(v')^2)-\alpha(v^3+v)-\beta(v^4-1)=0. \tag{36}$$

For simplicity, we set $p=c=1, a_0=0, b_1=1$ and $d=q=1$, then eq. (10) reduces to

$$v(\eta)=\frac{a_1\exp(\eta)+a_{-1}\exp(-\eta)}{\exp(\eta)+b_0+b_{-1}\exp(-\eta)}. \tag{37}$$

Substituting eq. (37) in eq. (36) and by the same manipulation as illustrated in the previous section, we obtain the following sets of nontrivial solutions (see Appendix C):

$$\left\{a_1 = \frac{-\alpha \pm \sqrt{\alpha^2+4\beta^2}}{2\beta}, a_{-1} = -\frac{b_{-1}(\alpha^2+2\beta^2 \mp \alpha\sqrt{\alpha^2+4\beta^2})}{\beta(\alpha \mp \sqrt{\alpha^2+4\beta^2})}, \right.$$
$$\left. b_0 = 0, b_{-1} = \text{arb.}, \tau = \text{arb.}, c = \text{arb.} \right\}. \tag{38}$$

Substituting eq. (38) in eq. (37) and using the transformation, we obtain the following wave solution of eq. (4):

$$u(\tilde{x},t) = \ln\left[\frac{\frac{-\alpha\pm\sqrt{\alpha^2+4\beta^2}}{2\beta}\exp(\eta) - \frac{b_{-1}(\alpha^2+2\beta^2\mp\alpha\sqrt{\alpha^2+4\beta^2})}{\beta(\alpha\mp\sqrt{\alpha^2+4\beta^2})}\exp(-\eta)}{\exp(\eta)+b_{-1}\exp(-\eta)}\right], \tag{39}$$

here $\tilde{x} = x_1, x_2, \ldots, x_N$ and $\eta = \tau(\sum_{j=1}^N x_j - ct)$. From eqs (37), (54) and (55), we obtain the following wave solutions:

$$u(\tilde{x},t) = \ln\left[\frac{\frac{-\alpha\pm\sqrt{\alpha^2+4\beta^2}}{2\beta}\exp(\eta) \mp \frac{b_{-1}(\alpha^2+4\beta^2\pm\alpha\sqrt{\alpha^2+4\beta^2})}{2\beta\sqrt{\alpha^2+4\beta^2}}\exp(-\eta)}{\exp(\eta)+b_{-1}\exp(-\eta)}\right], \tag{40}$$

where $\eta = \pm\frac{\sqrt{\alpha^2+4\beta^2}}{2\sqrt{2\beta(N-c^2)}}\left(\sum_{j=1}^N x_j - ct\right)$. From eqs (37), (56) and (57), we obtain the following wave solutions:

$$u(\tilde{x},t) = \ln\left[\frac{\frac{-\alpha\pm\sqrt{\alpha^2+4\beta^2}}{2\beta}\exp(\eta) + \frac{2b_0^2\beta^3}{(\alpha^2+4\beta^2)(\alpha\mp\sqrt{\alpha^2+4\beta^2})}\exp(-\eta)}{\exp(\eta)+\frac{b_0^2\beta^2}{\alpha^2+4\beta^2}\exp(-\eta)}\right], \tag{41}$$

where

$$\eta = \tau\left(\sum_{j=1}^N x_j \pm \frac{\sqrt{2\beta\tau^2 N - \alpha^2 - 4\beta^2}}{\tau\sqrt{2\beta}}t\right).$$

6. Conclusion

Travelling wave solutions are established for the $(N+1)$-dimensional evolution equations by using the exp-function method. Some of the obtained solutions are entirely new. The exp-function method is a promising method because it can establish a variety of solutions of distinct physical structures. The newly obtained solutions may be of importance while explaining some practical problems in physics.

Appendix A

$$\left\{a_1 = -1, a_0 = \text{arb.}, a_{-1} = \frac{a_0^2\alpha}{4(\alpha - 2\beta)}, b_0 = a_0, b_{-1} = -\frac{a_0^2\alpha}{4(\alpha - 2\beta)},\right.$$
$$\left.\tau = \text{arb.}, c = \pm\frac{\sqrt{-\alpha + 2\beta + N\tau^2}}{\tau}\right\}, \tag{42}$$

$$\left\{a_1 = \frac{-2\beta \pm \sqrt{4\beta^2 - \alpha^2}}{\alpha},\right.$$
$$a_0 = -\frac{b_0(8\beta^2 - \alpha^2 \mp 4\beta\sqrt{4\beta^2 - \alpha^2})}{\alpha(2\beta \mp \sqrt{4\beta^2 - \alpha^2})}, a_{-1} = A_1,$$
$$\left. b_0 = \text{arb.}, b_{-1} = \text{arb.}, \tau = \text{arb.}, c = \text{arb.}\right\}, \tag{43}$$

where

$$A_1 = -\frac{b_{-1}(128\beta^4 \mp 64\beta^3\sqrt{4\beta^2 - \alpha^2} - 32\alpha^2\beta^2 \pm 8\alpha^2\beta\sqrt{4\beta^2 - \alpha^2} + \alpha^4)}{\alpha(-6\alpha^2\beta \pm \alpha^2\sqrt{4\beta^2 - \alpha^2} + 32\beta^3 \mp 16\beta^2\sqrt{4\beta^2 - \alpha^2})}. \tag{44}$$

$$\left\{a_1 = \frac{-2\beta + \sqrt{4\beta^2 - \alpha^2}}{\alpha}, a_0 = \text{arb.},\right.$$
$$a_{-1} = \frac{\alpha^2(a_0^2\alpha + b_0^2\alpha + 4a_0b_0\beta)}{4(4\beta^2 - \alpha^2)(2\beta - \sqrt{4\beta^2 - \alpha^2})},$$
$$b_0 = \text{arb.}, b_{-1} = -\frac{\alpha(a_0^2\alpha + b_0^2\alpha + 4a_0b_0\beta)}{4(4\beta^2 - \alpha^2)},$$
$$\left.\tau = \text{arb.}, c = \pm\frac{\sqrt{\alpha^2 - 4\beta^2 + 2\tau^2 N\beta}}{\tau\sqrt{2\beta}}\right\}, \tag{45}$$

$$\left\{a_1 = \frac{-2\beta - \sqrt{4\beta^2 - \alpha^2}}{\alpha},\right.$$
$$a_0 = \text{arb.}, a_{-1} = \frac{\alpha^2(a_0^2\alpha + b_0^2\alpha + 4a_0b_0\beta)}{4(4\beta^2 - \alpha^2)(2\beta + \sqrt{4\beta^2 - \alpha^2})},$$
$$b_0 = \text{arb.}, b_{-1} = -\frac{\alpha(a_0^2\alpha + b_0^2\alpha + 4a_0b_0\beta)}{4(4\beta^2 - \alpha^2)}, \tau = \text{arb.},$$
$$\left. c = \pm\frac{\sqrt{\alpha^2 - 4\beta^2 + 2\tau^2 N\beta}}{\tau\sqrt{2\beta}}\right\}. \tag{46}$$

Appendix B

$$\begin{Bmatrix} a_1 = -1, a_0 = b_0, a_{-1} = -\dfrac{b_0^2\alpha}{4(\alpha-2\beta)}, b_0 = \text{arb.}, b_{-1} = \dfrac{b_0^2\alpha}{4(\alpha-2\beta)}, \\ \tau = \text{arb.}, c = \pm\dfrac{\sqrt{\alpha-2\beta+N\tau^2}}{\tau} \end{Bmatrix}, \tag{47}$$

$$\begin{Bmatrix} a_1 = \dfrac{-\alpha+\sqrt{\alpha^2-4\beta^2}}{2\beta}, a_0 = 0, a_{-1} = -\dfrac{b_{-1}(\alpha^2-4\beta^2+\alpha\sqrt{\alpha^2-4\beta^2})}{2\beta\sqrt{\alpha^2-4\beta^2}}, \\ b_0 = 0, b_{-1} = \text{arb.}, \tau = \text{arb.}, c = \pm\dfrac{\sqrt{-\alpha^2+4\beta^2+8\tau^2N\beta}}{2\tau\sqrt{2\beta}} \end{Bmatrix}, \tag{48}$$

$$\begin{Bmatrix} a_1 = -\dfrac{\alpha+\sqrt{\alpha^2-4\beta^2}}{2\beta}, a_0 = 0, a_{-1} = \dfrac{b_{-1}(\alpha^2-4\beta^2-\alpha\sqrt{\alpha^2-4\beta^2})}{2\beta\sqrt{\alpha^2-4\beta^2}}, \\ b_0 = 0, b_{-1} = \text{arb.}, \tau = \text{arb.}, c = \pm\dfrac{\sqrt{-\alpha^2+4\beta^2+8\tau^2N\beta}}{2\tau\sqrt{2\beta}} \end{Bmatrix}, \tag{49}$$

$$\begin{Bmatrix} a_1 = \dfrac{-\alpha\pm\sqrt{\alpha^2-4\beta^2}}{2\beta}, a_0 = \dfrac{b_0(2\beta^2-\alpha^2\pm\alpha\sqrt{\alpha^2-4\beta^2})}{\beta(\alpha\mp\sqrt{\alpha^2-4\beta^2})}, a_{-1} = A_2, \\ b_0 = \text{arb.}, b_{-1} = \text{arb.}, \tau = \text{arb.}, c = \text{arb.} \end{Bmatrix}, \tag{50}$$

where

$$A_2 = \frac{b_{-1}(2\beta^2-4\alpha^2\beta^2+\alpha^4\mp(\alpha^3-2\alpha\beta^2)\sqrt{\alpha^2-4\beta^2})}{\beta(3\alpha\beta^2-\alpha^3\pm(\alpha^2-\beta^2)\sqrt{\alpha^2-4\beta^2})}. \tag{51}$$

$$\begin{Bmatrix} a_1 = \dfrac{-\alpha+\sqrt{\alpha^2-4\beta^2}}{2\beta}, a_0 = \text{arb.}, \\ a_{-1} = \dfrac{2\beta^2(a_0^2\beta+b_0^2\beta+a_0b_0\alpha)}{(\alpha^2-4\beta^2)(\alpha-\sqrt{\alpha^2-4\beta^2})}, \\ b_0 = \text{arb.}, b_{-1} = -\dfrac{\beta(a_0^2\beta+b_0^2\beta+a_0b_0\alpha)}{\alpha^2-4\beta^2}, \\ \tau = \text{arb.}, c = \pm\dfrac{\sqrt{-\alpha^2+4\beta^2+2\tau^2N\beta}}{\tau\sqrt{2\beta}} \end{Bmatrix}, \tag{52}$$

$$\left\{a_1 = -\frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2\beta}, a_0 = \text{arb.}, \right.$$
$$a_{-1} = \frac{2\beta^2(a_0^2\beta + b_0^2\beta + a_0 b_0 \alpha)}{(\alpha^2 - 4\beta^2)(\alpha + \sqrt{\alpha^2 - 4\beta^2})},$$
$$b_0 = \text{arb.}, b_{-1} = -\frac{\beta(a_0^2\beta + b_0^2\beta + a_0 b_0 \alpha)}{\alpha^2 - 4\beta^2},$$
$$\left. \tau = \text{arb.}, c = \pm\frac{\sqrt{-\alpha^2 + 4\beta^2 + 2\tau^2 N\beta}}{\tau\sqrt{2\beta}} \right\}. \tag{53}$$

Appendix C

$$\left\{a_1 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta^2}}{2\beta}, a_{-1} = -\frac{b_{-1}(\alpha^2 + 4\beta^2 + \alpha\sqrt{\alpha^2 + 4\beta^2})}{2\beta\sqrt{\alpha^2 + 4\beta^2}}, \right.$$
$$\left. b_0 = 0, b_{-1} = \text{arb.}, \tau = \pm\frac{\sqrt{\alpha^2 + 4\beta^2}}{2\sqrt{2\beta(N - c^2)}}, c = \text{arb.} \right\}, \tag{54}$$

$$\left\{a_1 = -\frac{\alpha + \sqrt{\alpha^2 + 4\beta^2}}{2\beta}, a_{-1} = \frac{b_{-1}(\alpha^2 + 4\beta^2 - \alpha\sqrt{\alpha^2 + 4\beta^2})}{2\beta\sqrt{\alpha^2 + 4\beta^2}}, \right.$$
$$\left. b_0 = 0, b_{-1} = \text{arb.}, \tau = \pm\frac{\sqrt{\alpha^2 + 4\beta^2}}{2\sqrt{2\beta(N - c^2)}}, c = \text{arb.} \right\}, \tag{55}$$

$$\left\{a_1 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta^2}}{2\beta}, a_{-1} = \frac{2b_0^2\beta^3}{(\alpha^2 + 4\beta^2)(\alpha - \sqrt{\alpha^2 + 4\beta^2})}, \right.$$
$$\left. b_0 = 0, b_{-1} = \frac{b_0^2\beta^2}{\alpha^2 + 4\beta^2}, \tau = \text{arb.}, c = \pm\frac{\sqrt{2\beta\tau^2 N - \alpha^2 - 4\beta^2}}{\tau\sqrt{2\beta}} \right\}, \tag{56}$$

$$\left\{a_1 = -\frac{\alpha + \sqrt{\alpha^2 + 4\beta^2}}{2\beta}, a_{-1} = \frac{2b_0^2\beta^3}{(\alpha^2 + 4\beta^2)(\alpha + \sqrt{\alpha^2 + 4\beta^2})}, \right.$$
$$\left. b_0 = 0, b_{-1} = \frac{b_0^2\beta^2}{\alpha^2 + 4\beta^2}, \tau = \text{arb.}, c = \pm\frac{\sqrt{2\beta\tau^2 N - \alpha^2 - 4\beta^2}}{\tau\sqrt{2\beta}} \right\}. \tag{57}$$

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