PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 75, No. 3
September 2010
pp. 423–438

Slope and curvature of Isgur–Wise function using variationally improved perturbation theory in a quantum chromodynamics inspired potential model

BHASKAR JYOTI HAZARIKA[1,*] and D K CHOUDHURY[2]
[1]Department of Physics, Pandu College, Guwahati 781 012, India
[2]Department of Physics, Gauhati University, Guwahati 781 014, India
[*]Corresponding author. E-mail: bh53033@gmail.com; bjh_o6@rediffmail.com

MS received 9 January 2010; revised 12 April 2010; accepted 28 April 2010

Abstract. We used variationally improved perturbation theory (VIPT) in calculating the slope and curvature of Isgur–Wise (I–W) function with the Cornell potential $-\frac{4\alpha_s}{3r} + br + c$ instead of the usual stationary state perturbation theory as done earlier. We used $-(4\alpha_s/3r)$, i.e. the Coulombic potential, as the parent and the linear one, i.e. $br+c$ as the perturbed potential in the theory and calculated the slope and curvature of Isgur–Wise function including three states in the summation involved in the first-order correction to wave function in the method.

Keywords. Variationally improved perturbation theory; Isgur–Wise function; charge radii; convexity parameter.

PACS Nos **12.39.-x; 12.39.Jh; 12.39.Pn**

1. Introduction

The Isgur–Wise (I–W) function is a single unknown form factor which includes all the independent form factors occurring in weak decay amplitudes in the heavy quark limit because in the heavy quark limit, two additional symmetries appear in QCD which gives rise to a $SU(2N)$ symmetry called the heavy quark or Isgur–Wise (I–W) symmetry, where N is the number of quarks. The heavy quark symmetry enormously simplifies the analysis of semileptonic decays [1]. The I–W function and the relevant phenomenology are important topics in QCD as they act as a test for the correctness of any specified QCD-inspired model. Also, as the I–W function is related to the wave function directly, a correct estimation of the wave function is an essential tool to understand the decay processes and the relevant mechanism.

In recent years, a QCD-inspired quark model has been pursued by us [2,3] and I–W function has been calculated. In the model, the two-body Schrödinger equation was solved and first-order perturbed wave function for the ground state was obtained using the Dalgarno method [4]. Also in the model, the spin-independent ground state Fermi–Breit Hamiltonian with no contact term was considered [5] and the linear confinement was treated as perturbation keeping the Coulombic term as the parent one.

As an alternative approach one can use the variationally improved perturbation theory (VIPT) method [7] instead of the Dalgarno method in getting the wave function which combines the variational method and the perturbation theory.

The VIPT is a recent entry in the literature [6–8] which shows great promise regarding the use of approximation methods. The work by Aitchison and Dudek [6] inspired us to apply the method to the QCD-inspired model which had some limitations. Some of these limitations may be due to conventional perturbation technique. We know that the results of perturbation theory are expressed in terms of finite power series (in an expansion parameter which is taken to be very small) that seem to converge to the exact values when summed to higher order. After a certain order, however the results become increasingly worse since the series is usually divergent (being asymptotic). At this juncture, the variational method which estimates variationally optimized parameters (through energy minimization) helps in converting the divergent perturbation expansion to a convergent one which can be evaluated for large expansion parameters. We note that the variational method [4,9] is quite cumbersome as it is difficult to choose an appropriate trial wave function in terms of unknown parameter(s) which is later optimized to estimate the parameter(s). But in VIPT, we use a known wave function as the trial one (e.g. the $1s$ state H-atom wave function) and then optimize it to get the new parameter(s) (e.g. $\bar{\alpha}'_{10}$ in our case (eq. (16)) which make the perturbation series convergent. Further, we know that the perturbation theory is suitable to systems which have good unperturbed Hamiltonian, while variational method is robust even in cases where it is hard to determine a good unperturbed Hamiltonian. On the other hand, VIPT can be applied whether we have a good unperturbed Hamiltonian or not.

Question arises regarding the use of the Coulombic piece as the parent and linear part as the perturbed one of the total Cornell potential – that upto what distance this consideration is valid? Indeed, it was shown in ref. [6] that if $\langle r \rangle < r_0$ then the Coulomb base will perform better. Here $\langle r \rangle$ is the expectation value of the distance r which reasonably gives the size of a state (meson in this case) and r_0 is a point at which linear and Coulomb potentials become zero (figure 1 of Aitchison and Dudek [6]). Further, for low-lying mesons, i.e. $n = 1$, $l = 0$ (see eq. (8) of ref. [6]) the expectation value $\langle r \rangle$ is inversely proportional to the parameter $\alpha = 4\alpha_s/3$ (see eq. (12) of this work) for a given reduced mass μ. Using VIPT we get variably optimized $\bar{\alpha}'_{10}$ (see eq. (16) of this work) as the new parameter instead of α which assumes substantially larger value (see table 1 of this work) than α effectively making the 'linear term' weaker so that Coulombic piece becomes the parent. This ensures that the distance between the quarks is short enough to treat the binding effect mainly in terms of the Coulombic potential. Thus VIPT is a convenient and strong tool in treating the Coulombic potential as parent and linear potential as perturbed of the total Cornell potential.

It is evident from eq. (16) that $\bar{\alpha}'_{10}$ increases with the increase in α_s and greater values of $\bar{\alpha}'_{10}$ strongly support the binding effect mainly in terms of Coulombic potential. For the B-meson, the α_s values are small. It raises the question of applicability of the Coulombic part as the parent. However, the corresponding $\bar{\alpha}'_{10}$ values are sufficiently large to conform to the expectation $\langle r \rangle < r_0$ but not large enough to make the results of slope and curvature of the Isgur–Wise function compatible with the constraints as referred by Neubert [10].

This paper aims to apply the VIPT method to the QCD-inspired quark model [2,3] referred earlier and to calculate the I–W function, its slope and curvature. Using the same Hamiltonian and treating linear confinement as perturbation, we arrive at the hadronic wave function which enables us to calculate I–W function. Relativistic modification of the wave function [11,12] as well as the two-loop effect of strong coupling constant using V-scheme [13–16] are also taken into account.

The rest of the paper is organized as follows: Section 2 contains the formalism, §3 the result and calculation and §4 the discussion and conclusion.

2. Formalism

2.1 *Isgur–Wise function: Its slope and curvature*

The Isgur–Wise function is written as [1]

$$\xi(v_\mu \cdot v'_\mu) = \xi(y) \\ = 1 - \rho^2(y-1) + C(y-1)^2 + \cdots, \tag{1}$$

where

$$y = v_\mu \cdot v'_\mu \tag{2}$$

with v_μ and v'_μ being the four velocity of the heavy meson before and after the decay. The quantity ρ^2 is the slope of I–W function at $y = 1$ and known as charge radius:

$$\rho^2 = \left.\frac{\partial \xi}{\partial y}\right|_{y=1}. \tag{3}$$

The second-order derivative is the curvature of the I–W function known as convexity parameter:

$$C = \frac{1}{2}\left.\left(\frac{\partial^2 \xi}{\partial y^2}\right)\right|_{y=1}. \tag{4}$$

For the heavy–light flavour mesons, the I–W function can also be written as [3,17]

$$\xi\,(y) = \int_0^{+\infty} 4\pi r^2 \left|\psi\,(r)\right|^2 \cos pr\,\mathrm{d}r \tag{5}$$

where

$$p^2 = 2\mu\,(y-1)\,. \tag{6}$$

Here μ and ψ are respectively the reduced mass and wave function of the hadronic system.

2.2 *Variationally improved perturbation theory*

The VIPT method is not too old [6–8] and it combines two procedures, namely, stationary state perturbation theory and the variational method. We have the total Hamiltonian as

$$H = H_0 + H', \tag{7}$$

where H_0 is the parent Hamiltonian containing a physical parameter P (say) and H' is the perturbed Hamiltonian. The corresponding wave functions also contain P.

In VIPT,

$$P = P + P' - P', \tag{8}$$

where P' is the variational parameter such that

$$\begin{aligned} H &= H_{oP'} + H_o - H_{oP'} + H' \\ &= H_{oP'} + H'_{P'}. \end{aligned} \tag{9}$$

The parent Hamiltonian is now $H_{oP'}$ instead of H_o which depends on the variational parameter P' and $H'_{P'}$ is the new perturbed Hamiltonian instead of H' which also depends on P'. Correspondingly the wave functions will also change when P is replaced by P'. Now, one can treat these wave functions as trial wave functions with P' as the variational parameter and would find the value of P' which gives minimum value of energy corrected upto the first order. This will yeild variationally improved unperturbed wave function upon which the usual perturbation theory will be applied.

The wave function corrected up to the first order of jth state is given by [6]

$$\psi_j = \psi_j^{(0)} + \sum_{k \neq j} \frac{\int \psi_k^{(0)*} H'_{P'j} \psi_j^{(0)} \mathrm{d}v}{E_j^{(0)} - E_k^{(0)}}. \tag{10}$$

The energy corrected up to first order for the same state is

$$\begin{aligned} E_j &= \int \psi_j^{(0)*} H \psi_j^{(0)} \mathrm{d}v \\ &= \int \psi_j^{(0)*} (H_{oP'} + H'_{P'}) \psi_j^{(0)} \mathrm{d}v, \end{aligned} \tag{11}$$

where ψ_k and E_k are the wave function and energy eigenvalues of the kth state which are orthonormal to jth state. The superscript (0) is the zeroeth-order correction of the corresponding quantities.

With Cornell potential [18], we can have two possibilities to choose parent (and hence perturbed) Hamiltonian. In one, Coulombic one is the parent and in the other, linear one is the parent.

The summation in eq. (10) can include any number of kth states. In this work, terms upto three states in the summation are considered.

2.3 *Coulomb cum linear potential and wave functions using VIPT*

2.3.1 *With one term in the summation*

As explained earlier, variational parameter α' is used instead of the physical parameter $\alpha = 4\alpha_s/3$ (here Coulombic potential is the parent one). The Hamiltonian takes the form (eq. (9)):

$$\begin{aligned} H &= H_o + H' \\ &= -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} + br + c \\ &= -\frac{\nabla^2}{2\mu} - \frac{\alpha}{r} + br + c \\ &= -\frac{\nabla^2}{2\mu} - \frac{\alpha'}{r} + \frac{(\alpha' - \alpha)}{r} + br + c \\ &= H_{o\alpha'} + H'_{\alpha'}, \end{aligned} \tag{12}$$

where $\alpha = \alpha - \alpha' + \alpha'$. Now, $H_{o\alpha'} = -\frac{\nabla^2}{2\mu} - \frac{\alpha'}{r}$ is the parent Hamiltonian with α' and $H'_{\alpha'} = \frac{(\alpha'-\alpha)}{r} + br + c$ is the perturbed Hamiltonian with the same variational parameter α'. We notice that the physical parameter α is replaced by the variational parameter α'.

We consider j as $1s$ state ($n = 1$, $l = 0$) and in the summation of eq. (10), we consider only one kth state which is the $2s$ state ($n = 2$, $l = 0$).

The trial $1s$ state can be written (analogous to H-atom) with variational parameter α' as (this being the unperturbed wave function)

$$\psi_{10}^{(0)} = \frac{(\mu\alpha'_{10})^{3/2}}{\sqrt{\pi}} \mathrm{e}^{-\mu\alpha'_{10} r}, \tag{13}$$

where subscript 10 in α' indicates the quantum number (n, l) of the jth state.

We now find the value of α'_{10} which leads to minimum E_j given by (11) in the following way:

In the variational method, we are interested only in the 'r'-dependence of the Hamiltonian, and so c in $H'_{\alpha'}$ has no role to play in the calculation [4]. Using eqs (11), (12), (13)

$$E_{10}\,(\alpha'_{10}) = \frac{\mu\alpha'^2_{10}}{2} - \mu\alpha\alpha'_{10} + \frac{3b}{3\mu\alpha'_{10}}. \tag{14}$$

Minimization of eq. (14) gives

$$\alpha'^3_{10} - \alpha\alpha'^2_{10} - \frac{3b}{2\mu^2} = 0. \tag{15}$$

The solution of (15) is the required value of α'_{10} which gives minimum $E_{10}(\alpha'_{10})$ and we denote it by $\bar{\alpha}'_{10}$. Thus, unperturbed wave function in VIPT is

Table 1. $\bar{\alpha}'_{10}$ (eq. (15) for different mesons with α_s values under $\overline{\text{MS}}$ scheme.

Mesons	μ	α_s	$\alpha = 4\alpha_s/3$	$\bar{\alpha}'_{10}$
D	0.2761	0.39	0.52	1.7271
D_s	0.3648	0.39	0.52	1.4642
B	0.3100	0.22	0.2933	1.5104

Table 2. $\bar{\alpha}'_{10}$ (eq. (15)) for different mesons with α_s values in V-scheme.

Mesons	μ	α_s	$\alpha = 4\alpha_s/3$	$\bar{\alpha}'_{10}$
D	0.2761	0.693	0.924	1.9105
D_s	0.3648	0.693	0.924	1.6593
B	0.3100	0.261	0.348	1.531

$$\psi_{10}^{(0)}(\bar{\alpha}'_{10}) = \frac{(\mu\bar{\alpha}'_{10})^{3/2}}{\sqrt{\pi}} \mathrm{e}^{-\mu\bar{\alpha}'_{10}r}. \tag{16}$$

Here α'_{10} will be different for different mesons as solution of eq. (15) depends on μ and α with $b = 0.183$. We list the values of $\overline{\alpha}'_{10}$ in table 1 using known values of α_s under $\overline{\text{MS}}$ [3] and those in table 2 with α_s in V-scheme [13–16].

Now we consider the single kth state in the summation of eq. (10) which is the $2s$ state given by

$$\begin{aligned}\psi_k^{(0)}(\bar{\alpha}'_{10}) &= \psi_{20}^{(0)}(\bar{\alpha}'_{10}) \\ &= \frac{(\mu\bar{\alpha}'_{10})^{3/2}}{\sqrt{8\pi}} \mathrm{e}^{-\mu\bar{\alpha}'_{10}r/2}\left(1 - \frac{\mu\bar{\alpha}'_{10}r}{2}\right).\end{aligned} \tag{17}$$

Therefore, eq. (10) gives wave function corrected up to first order:

$$\psi_{10}(\bar{\alpha}'_{10}) = \psi_{10}^{(0)}(\bar{\alpha}'_{10}) + \frac{\int \psi_{20}^{(0)*}(\bar{\alpha}'_{10}) H'_{\bar{\alpha}'_{10}} \psi_{10}^{(0)}(\bar{\alpha}'_{10})\,\mathrm{d}v}{E_{10}^{(0)}(\bar{\alpha}'_{10}) - E_{20}^{(0)}(\bar{\alpha}'_{10})} \psi_{20}^{(0)}(\bar{\alpha}'_{10}). \tag{18}$$

The energy eigenvalues are given by

$$E_{n0}^{(0)}(\bar{\alpha}'_{10}) = -\frac{\mu\bar{\alpha}'^2_{10}}{2n^2}. \tag{19}$$

The summation in eq. (19) is dropped as we are considering single kth state. Also, we have $n = 1$ and 2, due to the single-state consideration in eq. (10). Carrying out the integration in (19) we find the wave function corrected up to the first order as

$$\begin{aligned}\psi_{10}(\bar{\alpha}'_{10}) &= \psi_{10}^{(0)}(\bar{\alpha}'_{10}) \\ &\quad - \frac{4\sqrt{\mu}}{3\sqrt{\pi}(\bar{\alpha}'_{10})^{1/2}}\left(\frac{4\mu\bar{\alpha}'_{10}(\alpha - \bar{\alpha}'_{10})}{27} - \frac{32b}{81\mu\bar{\alpha}'_{10}}\right) \\ &\quad \times \left(1 - \frac{\mu\bar{\alpha}'_{10}r}{2}\right)\mathrm{e}^{\mu\bar{\alpha}'_{10}r/2}.\end{aligned} \tag{20}$$

The relativistic version of (20) is [11,12]

$$\psi_{10,\mathrm{rel}}(\bar{\alpha}'_{10}) = \psi_{10}(\bar{\alpha}'_{10})[(r\mu\bar{\alpha}'_{10})^{-\epsilon}] \tag{21}$$

with

$$\epsilon = 1 - \sqrt{1 - \frac{4\alpha_s}{3}}. \tag{22}$$

The expressions for I–W function, charge radius and convexity parameter with confinement only (which corresponds to wave function given by eq. (20)) are

$$\xi_{S,\mathrm{conf}}(y) = 1 - \rho^2_{S,\mathrm{conf}}(y-1) + C_{S,\mathrm{conf}}(y-1)^2 + \cdots, \tag{23}$$

where the charge radius is

$$\rho^2_{S,\mathrm{conf}} = \frac{4\pi N_1^2}{\mu^3\bar{\alpha}'^5_{10}}\left[\frac{3c'^2_1}{4} + 84A^2 + \frac{1024c'_1 A}{243}\right], \tag{24}$$

and the convexity parameter is

$$C_{S,\mathrm{conf}} = \frac{4\pi N_1^2}{6\mu^3\bar{\alpha}'^7_{10}}\left[\frac{45c'^2_1}{8} + 5760A^2 + \frac{20\times 2^{12}c'_1 A}{3^6}\right]. \tag{25}$$

Here,

$$c'_1 = \frac{\mu\bar{\alpha}'_{10}}{\pi^{1/3}} \tag{26}$$

and

$$A = \frac{4\sqrt{\mu}}{3\sqrt{\pi}(\bar{\alpha}'_{10})^{1/2}}\left[\frac{4\mu\bar{\alpha}'_{10}(\alpha - \bar{\alpha}'_{10})}{27} - \frac{32b}{81\mu\bar{\alpha}'_{10}}\right]. \tag{27}$$

The subscript S refers to the single-state consideration in the summation of eq. (10). The normalization constant N_1 is given by

$$4\pi N_1^2 = \frac{1}{[(c'^2_1/4\mu^3\bar{\alpha}'^3_{10}) + (2A^2/\mu^3\bar{\alpha}'^3_{10})]}. \tag{28}$$

The respective relativistic versions are

$$\xi_{S,\mathrm{rel+conf}}(y) = 1 - \rho^2_{S,\mathrm{rel+conf}}(y-1) + C_{S,\mathrm{rel+conf}}(y-1)^2 + \cdots \tag{29}$$

$$\rho^2_{S,\mathrm{rel+conf}} = \frac{4\pi N'^2_1\Gamma(3-2\epsilon)(4-2\epsilon)(3-2\epsilon)}{\mu^3\bar{\alpha}'^5_{10}}\left[\frac{c'^2_1}{32} + X_1 + X_2\right] \tag{30}$$

and

$$C_{S,\mathrm{rel+conf}} = \frac{4\pi N_1'^2 \Gamma(3-2\epsilon)(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)}{6\mu^3\bar{\alpha}_{10}'^7} \times \left[\frac{c_1'^2}{128} + X_3 + X_4\right]. \tag{31}$$

Here the normalization constant N_1' is given by

$$4\pi N_1'^2 = \frac{\mu^3\bar{\alpha}_{10}'^3}{\Gamma(3-2\epsilon)\left[\frac{c_1'^2}{8} + X_5 + X_6\right]}. \tag{32}$$

All the functions $X_i(\epsilon), i = 1, 2, ..., 6$ are defined in the Appendix.

2.3.2 *With two terms in the summation*

In this step, we consider the $3s$ state ($n = 3$, $l = 0$) in addition to $2s$ state (as done in the single-term case). The $3s$ state with the variational parameter $\bar{\alpha}_{10}'$ is written as

$$\psi_{30}^{(0)}(\bar{\alpha}_{10}') = \frac{(\mu\bar{\alpha}_{10}')^{3/2}}{\sqrt{27\pi}}\left(1 - \frac{2\mu\bar{\alpha}_{10}' r}{3} + \frac{2\mu^2\bar{\alpha}_{10}'^2 r^2}{27}\right)\mathrm{e}^{-(\mu\bar{\alpha}_{10}' r/3)}. \tag{33}$$

By including this state, the summation and integration in (10) gives the wave function corrected upto the first order as

$$\psi_{10}(\bar{\alpha}_{10}') = \psi_{10}^{(0)}(\bar{\alpha}_{10}') - A\left(1 - \frac{\mu\bar{\alpha}_{10}' r}{2}\right)\mathrm{e}^{\mu\bar{\alpha}_{10}' r/2} + B\left(1 - \frac{-2\mu\bar{\alpha}_{10}' r}{3} + \frac{2\mu^2\bar{\alpha}_{10}'^2 r^2}{27}\right)\mathrm{e}^{-(\mu\bar{\alpha}_{10}' r/3)}, \tag{34}$$

where

$$B = \frac{\sqrt{\mu}}{\sqrt{\pi}(\bar{\alpha}_{10}')^{1/2}}\left[\frac{3\mu\bar{\alpha}_{10}'(\alpha - \bar{\alpha}_{10}')}{64} - \frac{27b}{256\mu\bar{\alpha}_{10}'}\right]. \tag{35}$$

The relativistic version is obtained by multiplying (34) by $(r\mu\bar{\alpha}_{10}')^{-\epsilon}$. The I–W function, charge radius and convexity parameter for the wave function (34) which is to be normalized are given by (i.e. with confinement only)

$$\xi_{D,\mathrm{conf}}(y) = 1 - \rho^2_{D,\mathrm{conf}}(y-1) + C_{D,\mathrm{conf}}(y-1)^2 + ..., \tag{36}$$

where the charge radius is

$$\rho^2_{D,\mathrm{conf}} = \frac{4\pi N_2^2}{\mu^3\bar{\alpha}_{10}'^5}\left[\frac{3c_1'^2}{4} + 84A^2 + \frac{1024c_1' A}{243} - \frac{3^4\times 211\times B^2}{4} + \frac{3^6\times 39\times c_1' B}{2^8} + \frac{6^6\times 69\times 16\times AB}{3\times 5^7}\right] \tag{37}$$

and the convexity parameter is

$$C_{D,\mathrm{conf}} = \frac{4\pi N_2^2}{6\mu^3\bar{\alpha}_{10}'^7}\left[\frac{45c_1'^2}{8} + 5760A^2 + \frac{20\times 2^{12}c_1'A}{3^6} \right.$$
$$\left. +414163\times B^2 + \frac{3^9\times 185\times c_1'B}{4^5} + \frac{6^9\times 24608\times AB}{3\times 5^9}\right] \quad (38)$$

with normalizaion constant N_2 given by

$$4\pi N_2^2 = \frac{1}{\left[\frac{c_1'^2}{4\mu^3\bar{\alpha}_{10}'^3} + \frac{2A^2}{\mu^3\bar{\alpha}_{10}'^3} + \frac{27B^2}{4\mu^3\bar{\alpha}_{10}'^3} + \frac{27c_1'B}{4\mu^3\bar{\alpha}_{10}'^3} - \frac{6^3\times 492\times AB}{5^5\mu^3\bar{\alpha}_{10}'^3}\right]}. \quad (39)$$

The subscript D refers to two terms in the summation. The respective relativistic versions of (36), (37) and (38) are

$$\xi_{D,\mathrm{rel+conf}}(y) = 1 - \rho^2_{D,\mathrm{rel+conf}}(y-1) + C_{D,\mathrm{rel+conf}}(y-1)^2 + \cdots, \quad (40)$$

where

$$\rho^2_{D,\mathrm{rel+conf}} = \frac{4\pi N_2'^2(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{\mu^3\bar{\alpha}_{10}'^5} \times\left[\frac{c_1'^2}{32} + X_1 + X_2 + \sum_{i=7}^{11} X_i\right] \quad (41)$$

and

$$C_{D,\mathrm{rel+conf}} = \frac{4\pi N_2'^2(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{6\mu^3\bar{\alpha}_{10}'^7} \times\left[\frac{c_1'^2}{128} + X_3 + X_4 + \sum_{i=12}^{16} X_i\right]. \quad (42)$$

The normalization constant N_2' is given as

$$4\pi N_2'^2 = \frac{\mu^3\bar{\alpha}_{10}'^3}{\Gamma(3-2\epsilon)\left[\frac{c_1'^2}{8} + X_5 + X_6 + \sum_{i=17}^{21} X_i\right]} \quad (43)$$

and $X_i(\epsilon)$, $i = 7, 8, \ldots, 21$ are defined in the Appendix.

2.3.3 *With three terms in the summation*

In addition to the $2s$ and $3s$ states, we now add the $4s$ state:

$$\psi_{40}^{(0)}(\bar{\alpha}_{10}') = \frac{(\mu\bar{\alpha}_{10}')^{3/2}}{\sqrt{2\pi}} \times\left(\frac{1}{4} - \frac{3\mu\bar{\alpha}_{10}'r}{16} + \frac{\mu^2\bar{\alpha}_{10}'^2r^2}{32} - \frac{\mu^3\bar{\alpha}_{10}'^3r^3}{8\times 96}\right)\mathrm{e}^{-(\mu\bar{\alpha}_{10}'r/4)}. \quad (44)$$

With the inclusion of this state, the first-order wave function now becomes

$$\psi_{10}(\bar{\alpha}'_{10}) = \psi^{(0)}_{10}(\bar{\alpha}'_{10}) - A\left(1 - \frac{\mu\bar{\alpha}'_{10}r}{2}\right)\mathrm{e}^{-(\mu\bar{\alpha}'_{10}r/2)}$$
$$+B\left(1 - \frac{2\mu\bar{\alpha}'_{10}r}{3} + \frac{2\mu^2\bar{\alpha}'^2_{10}r^2}{27}\right)\mathrm{e}^{-(\mu\bar{\alpha}'_{10}r/3)}$$
$$+D'\left(\frac{1}{4} - \frac{3\mu\bar{\alpha}'_{10}r}{16} + \frac{\mu^2\bar{\alpha}'^2_{10}r^2}{32} - \frac{\mu^3\bar{\alpha}'^3_{10}r^3}{8\times 96}\right)\mathrm{e}^{-(\mu\bar{\alpha}'_{10}r/4)}, \quad (45)$$

where

$$D' = \frac{(\mu\bar{\alpha}'_{10})^{3/2}}{\sqrt{\pi}}\left[\frac{36(\alpha - \bar{\alpha}'_{10})}{15625\bar{\alpha}'_{10}} - \frac{384b}{78125\mu^2\bar{\alpha}'^3_{10}}\right]. \quad (46)$$

As usual, the relativistic version of this wave function is obtained by multiplying the above expression by $(r\mu\bar{\alpha}'_{10})^{-\epsilon}$. Thus, with confinement only the I–W function is

$$\xi_{T,\mathrm{conf}}(y) = 1 - \rho^2_{T,\mathrm{conf}}(y-1) + C_{T,\mathrm{conf}}(y-1)^2 + \cdots, \quad (47)$$

where charge radius is

$$\rho^2_{T,\mathrm{conf}} = \frac{4\pi N_3^2}{\mu^3\bar{\alpha}'^5_{10}}\left[\frac{\rho^2_{D,\mathrm{conf}}\mu^3\bar{\alpha}'^5_{10}}{4\pi N_2^2} + 10368\times D'^2 - 2.51\times D'c'_1\right.$$
$$\left. -109.88\times D'A - 2558.46\times D'B\right] \quad (48)$$

and convexity parameter is

$$C_{T,\mathrm{conf}} = \frac{4\pi N_3^2}{6\mu^3\bar{\alpha}'^7_{10}}\left[\frac{C_{D,\mathrm{conf}}6\mu^3\bar{\alpha}'^7_{10}}{4\pi N_2^2} + 9123840\times D'^2\right.$$
$$\left. -19.32\times D'c'_1 - 3196.4\times D'A - 183755.94\times D'B\right] \quad (49)$$

with

$$4\pi N_3^2 = \frac{1}{\left[\frac{c'^2_1}{4\mu^3\bar{\alpha}'^3_{10}} + \frac{2A^2}{\mu^3\bar{\alpha}'^3_{10}} + \frac{27B^2}{4\mu^3\bar{\alpha}'^3_{10}} + \frac{27c'_1B}{4\mu^3\bar{\alpha}'^3_{10}} - \frac{6^3\times 492\times AB}{5^5\mu^3\bar{\alpha}'^3_{10}} + \frac{16D'^2}{\mu^3\bar{\alpha}'^3_{10}}\right]}. \quad (50)$$

Here, the subscript T refers to three terms in the summation.

The corresponding relativistic expressions are

$$\xi_{T,\mathrm{rel+conf}}(y) = 1 - \rho^2_{T,\mathrm{rel+conf}}(y-1) + C_{T,\mathrm{rel+conf}}(y-1)^2 + \cdots, \quad (51)$$

where

$$\rho^2_{T,\text{rel+conf}} = \frac{4\pi N_3'^2 (4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{\mu^3 \bar{\alpha}_{10}'^5} \left[\frac{c_1'^2}{32} + \sum_{i=22}^{29} X_i\right] \tag{52}$$

and

$$C_{T,\text{rel+conf}} = \frac{4\pi N_3'^2 (6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{6\mu^3 \bar{\alpha}_{10}'^7} \times \left[\frac{c_1'^2}{128} + \sum_{i=30}^{37} X_i\right]. \tag{53}$$

The normalization constant is given by

$$4\pi N_3'^2 = \frac{\mu^3 \bar{\alpha}_{10}'^3}{\Gamma(3-2\epsilon)[\frac{c_1'^2}{8} + \sum_{i=38}^{45} X_i]} \tag{54}$$

and the functions $X_i(\epsilon)$, $i = 21, 22, \ldots, 45$ are defined in the Appendix.

3. Calculation and results

We have listed the values of charge radius and convexity parameter of the calculated I–W function for various heavy–light flavour mesons in the present method considering single state, two states and three states in the summation occurred in VIPT with confinement and relativistic effect.

To set the tables we have used two sets of α_s values: one under $\overline{\text{MS}}$ scheme [3] and the other under V-scheme [13–16] at 'c' and 'b'-quark mass scale so that we get two sets of readings for the same quantities. Table 3 represents the numerical values of the parameters c_1', A, B, D' given by eqs (26), (27), (35) and (46) respectively with α_s under $\overline{\text{MS}}$ scheme while table 4 represents those values with α_s values under V-scheme. Similarly, tables 5–7 give charge radius and convexity parameter for different combinations of states with α_s values under $\overline{\text{MS}}$ scheme whereas tables 8–10 give the same quantities with α_s values under V-scheme. The values of $\bar{\alpha}_{10}'$ are taken from tables 1 and 2.

In table 11, we record the predictions of ρ^2 and C for the present model [19] using Dalgarno method [4] while in table 12, we refer to the predicted values of ρ^2 and C for different models [19–31]. In table 11, only one set of result is shown for the D-, D_s-mesons while two sets are shown for B-meson taken from the tables 1, 2 and 4 of ref. [19] to show the preference of higher α_s values for this meson. Specifically, it is seen that for $\alpha_s = 0.261$, as computed in the V-scheme at b-quark scale, the predictions overshoot the predictions of other models (table 12) by two orders of magnitude. However, for $\alpha_s = 0.60$, the results are comparable. In ref. [19] such an enhanced value of α_s was attributed to the necessity of potentially large flavour-dependent higher-order effects beyond $O(\alpha_s^3)$ in the V-scheme [14–16].

An analysis of tables 5–10 shows that relativistic effects invariably reduce the values of ρ^2 and C so as to bring them close to the predictions of other models. This feature further improves as we take two and three terms in the summation of eq. (10).

Table 3. Various parameters with α_s values under $\overline{\mathrm{MS}}$ scheme.

Mesons	α_s	c_1'	A	$B \times 10^{-2}$	$D' \times 10^{-4}$
D	0.39	0.33	−0.0712	0.304	5.055
D_s	0.39	0.37	−0.0800	0.340	5.600
B	0.22	0.32	−0.0820	0.350	5.770

Table 4. Various parameters with α_s values in the V-scheme.

Mesons	α_s	c_1'	A	$B \times 10^{-2}$	$D' \times 10^{-4}$
D	0.693	0.36	−0.0613	0.3166	4.345
D_s	0.693	0.42	−0.0660	0.3400	4.650
B	0.261	0.33	−0.0800	0.4100	5.670

Table 5. Charge radius and convexity parameter with single term in eq. (10) under $\overline{\mathrm{MS}}$ scheme.

Mesons	$\rho^2_{S,\mathrm{conf}}$	$C_{S,\mathrm{conf}}$	$\rho^2_{S,\mathrm{rel+conf}}$	$C_{S,\mathrm{rel+conf}}$
D	3.73	13.92	2.197	5.61
D_s	5.06	26.18	2.530	10.54
B	5.83	29.08	4.132	18.72

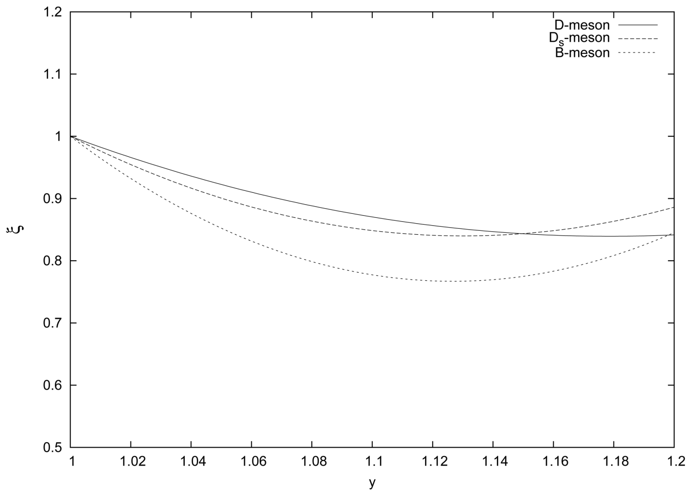

Figure 1. Variation of Isgur–Wise function $\xi(y)$ vs. velocity transfer ratio 'y' with three terms in the summation of eq. (10) (see table 7).

Correspondingly, the graphs which show the variation of I–W function $\xi(y)$ vs. velocity transfer ratio 'y' consist of two figures out of which the first one (i.e. figure 1) correspond to $\overline{\mathrm{MS}}$ scheme and the last one (i.e. figure 2) to V-scheme.

Table 6. Charge radius and convexity parameter with two terms in eq. (10) under $\overline{\mathrm{MS}}$ scheme.

Mesons	$\rho^2_{D,\mathrm{conf}}$	$C_{D,\mathrm{conf}}$	$\rho^2_{D,\mathrm{rel+conf}}$	$C_{D,\mathrm{rel+conf}}$
D	2.84	9.37	1.83	5.184
D_s	3.90	17.72	2.50	9.776
B	4.14	18.55	3.72	14.92

Table 7. Charge radius and convexity parameter with three terms in eq. (10) under $\overline{\mathrm{MS}}$ scheme.

Mesons	$\rho^2_{T,\mathrm{conf}}$	$C_{T,\mathrm{conf}}$	$\rho^2_{T,\mathrm{rel+conf}}$	$C_{T,\mathrm{rel+conf}}$
D	2.83	9.15	1.80	5.04
D_s	3.88	17.28	2.46	9.45
B	4.13	18.10	3.68	14.53

Table 8. Charge radius and convexity parameter with single term in eq. (10) under V-scheme.

Mesons	$\rho^2_{S,\mathrm{conf}}$	$C_{S,\mathrm{conf}}$	$\rho^2_{S,\mathrm{rel+conf}}$	$C_{S,\mathrm{rel+conf}}$
D	2.19	6.22	0.433	0.525
D_s	2.62	9.55	0.560	0.850
B	5.43	26.26	3.570	15.270

Table 9. Charge radius and convexity parameter with two terms in eq. (10) under V-scheme.

Mesons	$\rho^2_{D,\mathrm{conf}}$	$C_{D,\mathrm{conf}}$	$\rho^2_{D,\mathrm{rel+conf}}$	$C_{D,\mathrm{rel+conf}}$
D	1.82	4.57	0.432	0.524
D_s	2.28	7.31	0.550	0.840
B	3.60	16.20	3.160	12.320

Table 10. Charge radius and convexity parameter with three terms in eq. (10) under V-scheme.

Mesons	$\rho^2_{T,\mathrm{conf}}$	$C_{T,\mathrm{conf}}$	$\rho^2_{T,\mathrm{rel+conf}}$	$C_{T,\mathrm{rel+conf}}$
D	1.79	4.36	0.430	0.516
D_s	2.25	6.98	0.545	0.815
B	3.55	15.43	3.120	11.770

Table 11. Predictions of the slope and curvature of the I–W function with $b = 0.183$ GeV^2, $A_0 = 1$ and $c = 1$ GeV in V-scheme for the model of ref. [19] with relativistic and confinement effect.

Mesons	α_s	$\rho^2_{\rm rel+conf}$	$C_{\rm rel+conf}$
D	0.625	1.136	5.377
D_s	0.625	1.083	3.583
B	0.261	128.1	5212
	0.600	1.329	7.2

Table 12. Predictions of the slope and curvature of the I–W function in various models.

Model	Value of ρ^2	Value of curvature C
Le Youanc *et al* [20]	≥ 0.75	–
Le Youanc *et al* [21]	≥ 0.75	≥ 0.47
Rosner [28]	1.66	2.76
Mannel [29,30]	0.98	0.98
Pole ansatz [31]	1.42	2.71
MIT bag model [27]	2.35	3.95
Simple quark model [26]	1	1.11
Skryme model [24]	1.3	0.85
QCD sum rule [25]	0.65	0.47
Relativistic three-quark model [23]	1.35	1.75
Infinite momentum frame quark model [22]	3.04	6.81
Neubert [10]	0.82±0.09	–

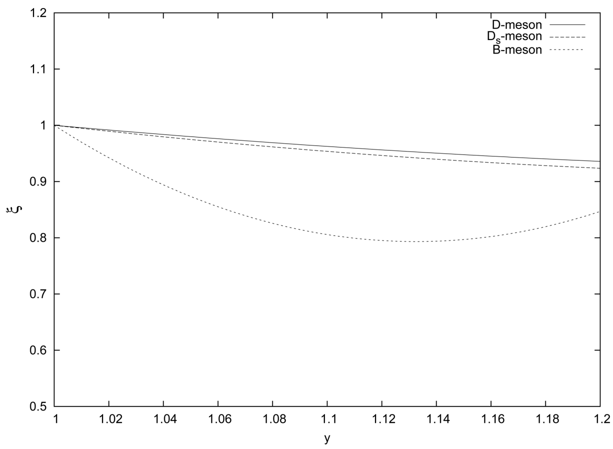

Figure 2. Variation of Isgur–Wise function $\xi(y)$ vs. 'y' with three terms in the summation of eq. (10) (see table 10).

4. Discussion and conclusion

In this paper, we have calculated the slope and curvature of the I–W function using VIPT method in the QCD-inspired quark model [3,13,19]. In this approach, we notice that with the inclusion of more states in the summation of eq. (10), the results come closer to the predictions of the other models [19–31]. We have seen from the results that the slope and curvature agree quite well with the values and bounds of other models in table 12 for D- and D_s-mesons but not as expected for B-meson. This is due to the low value of α_s for B-meson. Such a feature was earlier noticed in ref. [19] too, suggesting the necessity of higher-order effects beyond $O(\alpha_s^3)$ in V-scheme.

We also note that eqs (24), (25), (30), (31), (37), (38), (41), (42), (48), (49), (52) and (53) along with (28), (32), (39), (43), (50) and (54) of the text contain several large numerical factors appearing to be divergent compared to the leading order term which is contrary to the expectation of perturbation theory. However, a careful study reveals that actually it is not so.

As an illustration, the correct leading order term in eq. (24) with $b = 0$, $\bar{\alpha}'_{10} = \alpha$ becomes $\rho^2_{S,\mathrm{conf,LO}} = 3/\alpha^2 = 27/16\alpha_s^2$; which for $\alpha_s = 0.693$ is $\sim$3.51 not far away from the results of table 8. Similar analysis can be done for the other equations as well.

It will also be interesting to explore if the linear potential as parent incorporating more terms in the correction for wave function can improve the results of the present analysis as far as B-meson is concerned. Such an investigation is currently under progress.

Appendix

$$X_1 = A^2 \left(1 + \frac{(6-2\epsilon)(5-2\epsilon)}{4} - (5-2\epsilon)\right) \tag{A1}$$

$$X_2 = 64c'_1 A \left(\frac{(5-2\epsilon)}{729} - 1243\right) \tag{A2}$$

$$X_3 = A^2 \left(1 + \frac{(8-2\epsilon)(7-2\epsilon)}{4} - (7-2\epsilon)\right) \tag{A3}$$

$$X_4 = 256c'_1 A \left(-\frac{1}{2187} + \frac{(7-2\epsilon)}{6561}\right) \tag{A4}$$

$$X_5 = A^2 \left[1 + \frac{(4-2\epsilon)(3-2\epsilon)}{4} - (3-2\epsilon)\right]. \tag{A5}$$

Rest of the equations can be obtained from the authors on request.

Acknowledgement

The authors thank N S Bordoloi, S Kalita, Z H Devi, C Thakuria, B R Baruah and A Baruah for their technical support. The authors also thank F M Fernandez and I J R Aitchison for inspiring correspondence.

References

[1] N Isgur and M B Wise, *Phys. Lett.* **B232**, 113 (1989)
[2] D K Choudhury, P Das, D D Goswami and J N Sharma, *Pramana – J. Phys.* **44**, 519 (1995)
[3] D K Choudhury and N S Bordoloi, *Int. J. Mod. Phys.* **A15**, 3667 (2000)
[4] A K Ghatak and S Lokanathan, in: *Quantum mechanics* (McGraw Hill, 1997) pp. 291
[5] A D Rujula, H Georgi and S L Glashow, *Phys. Rev.* **D12**, 147 (1975)
[6] I J R Aitchison and J J Dudek, *Eur. J. Phys.* **23**, 605 (2002)
[7] S K You, K J Jeon, C K Kim and K Nahm, *Eur. J. Phys.* **19**, 179 (1998)
[8] F M Fernandez, *Eur. J. Phys.* **24**, 289 (2003)
[9] Y B Ding, X Q Li and P N Shen, *Phys. Rev.* **D60**, 074010 (1999)
[10] M Neubert, *Int. J. Mod. Phys.* **A11**, 4173 (1996)
[11] J J Sakurai, in: *Advanced quantum mechanics* (Addison-Wesley Publishing Company, Massachusetts, 1986) p. 128
[12] C Itzykson and J Zuber, in: *Quantum field theory* (International Student Edition, McGraw Hill, Singapore, 1986) p. 79
[13] D K Choudhury and N S Bordoloi, *Mod. Phys. Lett.* **A17(29)**, 1909 (2002)
[14] M Peter, *Phys. Rev. Lett.* **78**, 603 (1997); *Nucl. Phys.* **B501**, 471 (1997)
[15] Y Schroeder, *Phys. Lett.* **B447**, 321 (1999)
[16] Y Schroeder, *Nucl. Phy. Proc. Suppl.* **86**, 525 (2000)
[17] F E Close and A Wambach, *Nucl. Phys.* **B412**, 169 (1994)
[18] Riazuddin and Fiyyazuddin, in: *A modern introduction to particle physics* (Allied Publishers Limited, 2000) p. 256
[19] D K Choudhury and Bordoloi, *Mod. Phys. Lett.* **A26**, 443 (2009)
[20] A Le Yaouanc, L Oliver, O Pene and J C Raynal, *Phys. Lett.* **B365**, 319 (1996)
[21] A Le Yaouanc, L Oliver and J C Raynal, *Phys. Rev.* **D69**, 094022 (2004)
[22] B König, J G Körner, M Krämer and P Kroll, *Phys. Rev.* **D56**, 4282 (1997)
[23] M A Ivanov, V E Lyubouvitskij, L G Körner and P Kroll, *Phys. Rev.* **D56**, 348 (1997)
[24] E Jenkins, A Manohar and M B Wise, *Nucl. Phys.* **B396**, 38 (1996)
[25] Y B Dai, C S Huang, M K Huang and C Liu, *Phys. Lett.* **B387**, 379 (1996)
[26] B Holdom, M Sutherland and J Mureika, *Phys. Rev.* **D49**, 2359 (1994)
[27] M Sadzikowski and K Zalewski, *Z. Phys.* **C59**, 667 (1993)
[28] J L Rosner, *Phys. Rev.* **D42**, 3732 (1990)
[29] T Mannel, W Roberts and Z Ryzak, *Phys. Rev.* **D44**, R18 (1991)
[30] T Mannel, W Roberts and Z Ryzak, *Phys. Lett.* **B255**, 593 (1993)
[31] M Neubert, *Phys. Lett.* **B264**, 455 (1991)