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An irrational trial equation method and its applications

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Abstract. An irrational trial equation method was proposed to solve nonlinear differential equations. By this method, a number of exact travelling wave solutions to the Burgers–KdV equation and the dissipative double sine-Gordon equation were obtained. A more general irrational trial equation method was discussed, and many exact solutions to the Fujimoto–Watanabe equation were given.

Keywords. Trial equation method; exact solution; Burgers–KdV equation; dissipative double sine-Gordon equation; Fujimoto–Watanabe equation.

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1. Introduction

It is important to find the travelling wave solutions of nonlinear evolution equations. Some powerful methods such as inverse scattering method, Painleve analysis and Hirota bilinear method are introduced and developed [1]. Some direct expansion methods are proposed and applied to give exact solutions to nonlinear differential equations (see, for example, refs [2–4] and references therein). The dynamic system approach has been used to analyse the existence of travelling wave solutions to some nonlinear equations (see, for example, ref. [5] and references therein). Based on the decomposition of differential operator, an interesting and powerful approach, namely factorization method [6–8] has been introduced to deal with nonlinear differential equations. Recently, in a series of papers [9–12], Liu proposed the trial equation method which is different from those direct methods. Liu's key idea is that exact solution to a differential equation can be given by solving an integration. For example, consider a differential equation of u. We always assume that its exact solution satisfies a solvable equation $u' = F(u)$. Therefore, our task is just to find the function F. Liu has obtained a number of exact solutions to many nonlinear differential equations when $F(u)$ is a polynomial or a rational function. However, for some nonlinear ordinary differential equations with rank inhomogeneous, we cannot find a polynomial $F(u)$ or a rational function $F(u)$. Therefore, we need a new trial equation method to solve these kinds of equations. In the present paper, we take F as an irrational function, and hence propose a new trial equation method. As an application, we give some exact solutions to the Burgers–KdV equation [13–15]

$$u_t + \delta u u_x + \beta u_{xx} + \gamma u_{xxx} = 0, \tag{1}$$

and the dissipative double sine-Gordon equation [16]

$$c^2 u_{xx} - u_{tt} - r u_t = \alpha_1 \sin u + \alpha_2 \sin(2u). \tag{2}$$

Finally, we also discuss a more general irrational trial equation method, and use it to give a number of exact solutions to the Fujimoto–Watanabe equation [17]

$$u_t = u^3 u_{xxx} + 3u^2 u_x u_{xx} + 3\alpha u^2 u_x. \tag{3}$$

2. Irrational trial equation method

We consider the following nonlinear partial differential equation:

$$N(u, u_t, u_{tt}, \ldots, u_x, u_{xx}, \ldots, u_{tx}, \ldots) = 0. \tag{4}$$

Under the travelling wave transformation

$$u = u(\xi), \quad \xi = kx + \omega t, \tag{5}$$

eq.(4) becomes the following ordinary differential equation:

$$P(u, u', u'', \ldots) = 0, \tag{6}$$

where the prime means the differentiation with respect to ξ. Sometimes, by integration, the order of eq. (6) can be reduced. Now, our method can be described as follows:

Step 1. Take an irrational trial equation

$$u' = \sum_{i=0}^{k_1} a_i u^i + \left(\sum_{i=0}^{k_2} b_i u^i\right) \sqrt{\sum_{i=0}^{k_3} c_i u^i}, \tag{7}$$

where $a_0, \ldots, a_{k_1}$, $b_0, \ldots, b_{k_2}$ and $c_0, \ldots, c_{k_3}$ are the constants to be determined.

By eq. (7), we derive the following equation:

$$u'' = \left(\sum_{i=1}^{k_1} i a_i u^{i-1}\right)\left(\sum_{i=0}^{k_1} a_i u^i\right) + \left(\sum_{i=0}^{k_2} b_i u^i\right)\left(\sum_{i=1}^{k_2} i b_i u^{i-1}\right)\left(\sum_{i=0}^{k_3} c_i u^i\right)$$

$$+\frac{1}{2}\left(\sum_{i=0}^{k_2} b_i u^i\right)^2 \left(\sum_{i=1}^{k_3} i c_i u^{i-1}\right)$$
$$+\frac{1}{2}\left(\sum_{i=0}^{k_1} a_i u^i\right)\left(\sum_{i=0}^{k_2} b_i u^i\right)\left(\sum_{i=1}^{k_3} i c_i u^{i-1}\right)\left(\sum_{i=0}^{k_3} c_i u^i\right)^{-1/2}$$
$$+\Bigg[\left(\sum_{i=1}^{k_2} i b_i u^{i-1}\right)\left(\sum_{i=0}^{k_1} a_i u^i\right)$$
$$+\left(\sum_{i=1}^{k_1} i a_i u^{i-1}\right)\left(\sum_{i=0}^{k_2} b_i u^i\right)\Bigg]\sqrt{\left(\sum_{i=0}^{k_3} c_i u^i\right)}, \quad (8)$$

and other derivation terms such as u''', and so on.

Step 2. Substituting u', u'' and other derivation terms into eq. (6) yields the following expression:

$$G(u) + H(u)\sqrt{\sum_{i=0}^{k_3} c_i u^i} = 0, \quad (9)$$

where $G(u)$ and $H(u)$ are two polynomials of u. According to the balance principle, we can obtain the relation of k_1, k_2 and k_3 or their values.

Step 3. Taking concrete values of k_1, k_2 and k_3, and letting all coefficients of $G(u)$ and $H(u)$ to be zero yield a system of nonlinear algebraic equations. Solving the system, we obtain the values of $a_0, \ldots, a_{k_1}$, $b_0, \ldots, b_{k_2}$ and $c_0, \ldots, c_{k_3}$.

Step 4. Integrating eq. (7) gives the solutions of u.

3. Applications

Example 1. *The Burgers–KdV equation* (1)

Under the travelling wave transformation and integration, the Burgers–KdV equation (1) becomes

$$u'' + \frac{\beta}{\gamma k}u' = -\frac{\delta}{2k^2\gamma}u^2 - \frac{\omega}{k^3\gamma}u + D, \quad (10)$$

where D is an arbitrary constant. We denote $A = \beta/\gamma k, B = -(\delta/2k^2\gamma)$ and $C = -(\omega/k^3\gamma)$. Substituting eqs (7) and (8) into eq. (10) and using the balance principle, it follows that $2k_2 + k_3 - 1 = 2$ and $2k_1 - 1 < 2$. Then we obtain $k_1 = k_2 = k_3 = 1$ or $k_1 = 0, k_2 = k_3 = 1$.

In the case of $k_1 = k_2 = k_3 = 1$, eq. (7) becomes

$$u' = a_1 u + a_0 + (b_1 u + b_0)\sqrt{c_1 u + c_0}, \quad (11)$$

where a_i, b_i, c_i are the parameters to be determined, for $i = 0, 1$. Furthermore, from eq. (11), we have

$$u'' = \left\{ a_1 + b_1\sqrt{c_1 u + c_0} + \frac{c_1(b_1 u + b_0)}{2\sqrt{c_1 u + c_0}} \right\} \times \{a_1 u + a_0 + (b_1 u + b_0)\sqrt{c_1 u + c_0}\}. \quad (12)$$

Substituting u' and u'' into eq. (10) yields

$$G(u) + H(u)\sqrt{c_1 u + c_0} = 0, \quad (13)$$

where

$$G(u) = \left(Ab_1c_1 + \frac{5}{2}a_1b_1c_1 \right) u^2 + \left((A + 2a_1)b_1c_0 + Ab_0c_1 + \frac{3}{2}a_1b_0c_1 + \frac{3}{2}a_0b_1c_1 \right) u + (A + a_1)b_0c_0 + a_0b_1c_0 + \frac{1}{2}a_0b_0c_1, \quad (14)$$

$$H(u) = \left(\frac{3}{2}b_1^2c_1 - B \right) u^2 + (2b_1c_1b_0 + b_1^2c_0 + a_1^2 + a_1A - C)u + b_1b_0c_0 + \frac{1}{2}c_1b_0^2 + a_0a_1 + a_0A - D. \quad (15)$$

In order to give these parameters, let $G(u) \equiv 0, H(u) \equiv 0$, and hence we get a system of algebraic equations

$$\frac{3}{2}b_1^2c_1 - B = 0, \quad (16)$$

$$2b_1c_1b_0 + b_1^2c_0 + a_1^2 + a_1A - C = 0, \quad (17)$$

$$b_1b_0c_0 + \frac{1}{2}c_1b_0^2 + a_0a_1 + a_0A - D = 0, \quad (18)$$

$$Ab_1c_1 + \frac{5}{2}a_1b_1c_1 = 0, \quad (19)$$

$$(A + 2a_1)b_1c_0 + Ab_0c_1 + \frac{3}{2}a_1b_0c_1 + \frac{3}{2}a_0b_1c_1 = 0, \quad (20)$$

$$(A + a_1)b_0c_0 + a_0b_1c_0 + \frac{1}{2}a_0b_0c_1 = 0. \quad (21)$$

By solving the algebraic equations (16)–(21), we have

$$a_0 = -\frac{12A}{5B} - \frac{AC}{5B} - \frac{6A^3}{250B}, \quad a_1 = -\frac{2A}{5}, \quad b_1 = -2,$$
$$b_0 = -\frac{C}{B} - \frac{6A^2}{25B}, \quad c_1 = \frac{B}{6}, \quad c_0 = 1 + \frac{C}{12} + \frac{A^2}{100}, \quad A = \pm 10. \quad (22)$$

Furthermore, solving eq. (11) gives the solutions to the Burgers–KdV equation (1),

$$u_1 = -\frac{3\beta^2}{25\delta\gamma}\left\{\frac{4\exp(\mp 4(\frac{\beta}{10\gamma}x+\omega t-\xi_0))}{(1\mp\exp(\mp 2(\frac{\beta}{10\gamma}x+\omega t-\xi_0)))^2} - 2 \mp \frac{250\omega\gamma^2}{3\beta^3}\right\} \quad (23)$$

and

$$u_2 = -\frac{3\beta^2}{25\delta\gamma}\left\{\frac{4\exp(\pm 4(-\frac{\beta}{10\gamma}x+\omega t-\xi_0))}{(1\pm\exp(\pm 2(-\frac{\beta}{10\gamma}x+\omega t-\xi_0)))^2} - 2 \pm \frac{250\omega\gamma^2}{3\beta^3}\right\}, \quad (24)$$

where ω and ξ_0 are two arbitrary constants.

In the case of $k_1 = 0$, $k_2 = k_3 = 1$, the corresponding results of eq. (1) are included as special cases in the solutions (23) and (24).

Example 2. *The dissipative double sine-Gordon equation* (2)

Under the travelling wave transformation, eq. (2) becomes

$$(c^2k^2-\omega^2)u'' - r\omega u' = \alpha_1 \sin u + \alpha_2 \sin(2u). \quad (25)$$

Take a transformation

$$v = \sin u. \quad (26)$$

Correspondingly, we have

$$u = \arcsin v, \quad (27)$$

$$\sin 2u = 2v\sqrt{1-v^2}, \quad (28)$$

$$u' = \frac{v'}{\sqrt{1-v^2}}, \quad (29)$$

$$u'' = \frac{v''}{\sqrt{1-v^2}} + \frac{v(v')^2}{(\sqrt{1-v^2})^3}. \quad (30)$$

Substituting eqs (27)–(30) into eq. (25) yields

$$\frac{(c^2k^2-\omega^2)v''}{\sqrt{1-v^2}} + \frac{(c^2k^2-\omega^2)v(v')^2}{(\sqrt{1-v^2})^3} - \frac{r\omega v'}{\sqrt{1-v^2}} = \alpha_1 v + 2\alpha_2 v\sqrt{1-v^2}. \quad (31)$$

By our trial equation method, when $k^2 = (2r^2\omega^2\alpha_2 + \omega^2\alpha_1^2)/(c^2\alpha_1^2)$, we have the trial equation

$$v' = -\frac{\alpha_1}{r\omega}v\sqrt{1-v^2}. \quad (32)$$

Integrating eq. (32), we have

$$\tan\left(\frac{1}{2}\arcsin v\right) = \pm\exp\left(\pm\frac{\alpha_1}{r\omega}(\xi-\xi_0)\right), \tag{33}$$

where ξ_0 is an arbitrary constant. Substituting eq. (27) into eq. (33), we obtain the solutions of the dissipative double sine-Gordon equation (2)

$$u = \pm 2\arctan\left(\exp\left(\pm\left(\frac{\sqrt{2r^2\alpha_2+\alpha_1^2}}{rc}x \pm \frac{\alpha_1}{r}t - \xi_0\right)\right)\right), \tag{34}$$

where ξ_0 is an arbitrary constant.

4. Discussion

In our new trial equation method, the irrational trial equation (7) can be replaced by the following more general form:

$$u' = \sum_{i=0}^{k_1} a_i u^i + \left(\sum_{i=0}^{k_2} b_i u^i\right)\sqrt{\frac{\sum_{i=0}^{k_3} c_i u^i}{\sum_{i=0}^{k_4} d_i u^i}}, \tag{35}$$

where $a_0,\ldots,a_{k_1}$, $b_0,\ldots,b_{k_2}$, $c_0,\ldots,c_{k_3}$, $d_0,\ldots,d_{k_4}$ are the constants to be determined. Therefore, we can give a more general irrational trial equation method as follows.

Step 1. Take an irrational trial equation (35). Correspondingly, we derive the following equation:

$$\begin{aligned} u'' = {} & \left(\sum_{i=1}^{k_1} i a_i u^{i-1}\right)\left(\sum_{i=0}^{k_1} a_i u^i\right) + \left(\sum_{i=0}^{k_2} b_i u^i\right)\left(\sum_{i=1}^{k_2} i b_i u^{i-1}\right)\frac{(\sum_{i=0}^{k_3} c_i u^i)}{(\sum_{i=0}^{k_4} d_i u^i)} \\ & + \frac{(\sum_{i=0}^{k_2} b_i u^i)^2[(\sum_{i=1}^{k_3} i c_i u^{i-1})(\sum_{i=0}^{k_4} d_i u^i) - (\sum_{i=0}^{k_3} c_i u^i)(\sum_{i=1}^{k_4} i d_i u^{i-1})]}{2(\sum_{i=0}^{k_4} d_i u^i)^2} \\ & + \frac{\left\{\begin{array}{r} (\sum_{i=0}^{k_1} a_i u^i)(\sum_{i=0}^{k_2} b_i u^i)[(\sum_{i=1}^{k_3} i c_i u^{i-1})(\sum_{i=0}^{k_4} d_i u^i) \\ -(\sum_{i=0}^{k_3} c_i u^i)(\sum_{i=1}^{k_4} i d_i u^{i-1})] \end{array}\right\}}{2(\sum_{i=0}^{k_4} d_i u^i)^2} \\ & \times \left(\frac{(\sum_{i=0}^{k_3} c_i u^i)}{(\sum_{i=0}^{k_4} d_i u^i)}\right)^{-1/2} + \left[\left(\sum_{i=1}^{k_2} i b_i u^{i-1}\right)\left(\sum_{i=0}^{k_1} a_i u^i\right)\right. \\ & \left. + \left(\sum_{i=1}^{k_1} i a_i u^{i-1}\right)\left(\sum_{i=0}^{k_2} b_i u^i\right)\right]\sqrt{\frac{\sum_{i=0}^{k_3} c_i u^i}{\sum_{i=0}^{k_4} d_i u^i}}, \end{aligned} \tag{36}$$

and other derivation terms such as u''', and so on.

Step 2. Substituting u',u'' and other derivation terms into eq. (6) yields the following equation:

$$G(u)+H(u)\sqrt{\frac{\sum_{i=0}^{k_3} c_i u^i}{\sum_{i=0}^{k_4} d_i u^i}}=0, \tag{37}$$

where $G(u)$ and $H(u)$ are two polynomials of u. According to the balance principle, we can obtain the relation of k_1, k_2, k_3 and k_4 or their values.

Step 3. Taking concrete values of k_1, k_2, k_3 and k_4, and letting all coefficients of $G(u)$ and $H(u)$ to be zero yield a system of nonlinear algebraic equations. Solving the system, we get the values of $a_0, \ldots, a_{k_1}$, $b_0, \ldots, b_{k_2}, c_0, \ldots, c_{k_3}$ and $d_0, \ldots, d_{k_4}$.

Step 4. Integrating eq. (35) gives the solutions of u.

As an application, we consider the Fujimoto–Watanabe equation (3). Under the travelling wave transformation, eq. (3) becomes

$$\omega u' = 3k\alpha u^2 u' + 3k^3 u^2 u' u'' + k^3 u^3 u'''. \tag{38}$$

By Steps 1–4 in the section, we can take $k_1 = k_2 = 0, k_3 = 3$ and $k_4 = 2$. Then we have $c_3 = -2\alpha/k^2, c_1 = -2\omega/k^3, d_2 = 1, d_1 = d_0 = 0$, and c_0 and c_2 are two arbitrary constants. We take $a_0 = 0, b_0 = 1$ for simplicity. Then we have the trial equation

$$u' = \sqrt{\frac{c_3 u^3 + c_2 u^2 + c_1 u + c_0}{u^2}}. \tag{39}$$

Integrating eq. (39), we obtain the solutions to the Fujimoto–Watanabe equation (3) as follows:

$$\begin{aligned}\pm(\xi-\xi_0) &= \frac{\sqrt{2}k}{\sqrt{-\alpha}}\\ &\times\left(\sqrt{u-\alpha_2} - \frac{\alpha_1}{\sqrt{\alpha_2-\alpha_1}}\arctan\frac{\sqrt{u-\alpha_2}}{\sqrt{\alpha_2-\alpha_1}}\right),\\ &\alpha_2 > \alpha_1,\end{aligned} \tag{40}$$

$$\begin{aligned}\pm(\xi-\xi_0) &= \frac{\sqrt{2}k}{\sqrt{-\alpha}}\\ &\times\left(\sqrt{u-\alpha_2} - \frac{\alpha_1}{2\sqrt{\alpha_1-\alpha_2}}\ln\left|\frac{\sqrt{u-\alpha_2}-\sqrt{\alpha_1-\alpha_2}}{\sqrt{u-\alpha_2}+\sqrt{\alpha_1-\alpha_2}}\right|\right),\\ &\alpha_1 > \alpha_2,\end{aligned} \tag{41}$$

$$\pm(\xi-\xi_0) = \frac{\sqrt{2}k}{\sqrt{-\alpha}}\left(\sqrt{u-\alpha_1} - \frac{\alpha_1}{\sqrt{u-\alpha_1}}\right) \tag{42}$$

$$\begin{aligned}\pm(\xi-\xi_0) = \frac{2\sqrt{-2\alpha(\alpha_1-\alpha_2)}}{k}\Big\{&\alpha_1 F(\varphi,l) - (\alpha_1-\alpha_3)E(\varphi,l)\\ &+(\alpha_1-\alpha_3)\tan\varphi\sqrt{1-l^2\sin^2\varphi}\Big\}, \quad \alpha_1>\alpha_2>\alpha_3,\end{aligned} \tag{43}$$

where $l^2 = (\alpha_2 - \alpha_3)/(\alpha_1 - \alpha_3)$, $F(\varphi, l) = \int_0^\varphi \mathrm{d}\phi / \sqrt{1 - l^2 \sin^2 \phi}$, $E(\varphi, l) = \int_0^\varphi \sqrt{1 - l^2 \sin^2 \phi}\, \mathrm{d}\phi$, α_1, α_2 and α_3 are the roots of the polynomial equation $u^3 - \frac{c_2 k^2}{2\alpha} u^2 + \frac{\omega}{k\alpha} u - \frac{c_0 k^2}{2\alpha} = 0$. Obviously, the solutions (40)–(42) are the elementary function solutions, and the solutions (43) are the elliptic function solutions. We can also take other values of k_1, k_2, k_3 and k_4 and deal with these cases similarly.

5. Conclusion

In the paper, we proposed a new irrational trial equation method and used it to obtain some exact travelling wave solutions to the Burgers–KdV equation and the dissipative double sine-Gordon equation. We also discussed a more general irrational trial equation method. As an application, we gave a number of exact solutions to the Fujimoto–Watanabe equation. The proposed method could also be applied to other nonlinear differential equations such as BBM–Burgers equation, Fisher equation, and so on.

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