PRAMANA
— journal of
physics

Vol. 74, No. 6
June 2010
pp. 875–882

Characterization of the Lovelock gravity by Bianchi derivative

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MS received 30 November 2009; revised 15 February 2010; accepted 19 February 2010

Abstract. We prove the theorem: The second-order quasilinear differential operator as a second-rank divergence-free tensor in the equation of motion for gravitation could always be derived from the trace of the Bianchi derivative of the fourth-rank tensor, which is a homogeneous polynomial in curvatures. The existence of such a tensor for each term in the polynomial Lagrangian is a new characterization of the Lovelock gravity.

Keywords. Gauss–Bonnet gravity; Lovelock gravity; higher derivative; higher dimensions; Bianchi derivative.

PACS Nos 04.20.-q; 04.20.Cv; 04.50.Gh; 11.10.-z; 11.15.-q

The Bianchi differential identity involving the purely antisymmetric derivative of a derivative ($D^2 = 0$) was famously interpreted by John Wheeler [1] as, *boundary of boundary is zero*. The familiar examples are the curl of the gradient and the divergence of the curl being zero. The former signifies a scalar while the latter a vector field. However, its contraction in the two cases is vacuous and hence does not lead to a non-trivial statement [1a]. When we go beyond a vector to a tensor field, it becomes interesting.

Gravity is the universal force which means it links to all particles unmindful of their mass being non-zero or zero. Its linkage to massless particles leads to the profound realization that it could only be described by the curvature of space-time [2,3]. This means that the dynamics of gravity resides in space-time curvature which must fully and entirely determine it. That is, the gravitational dynamics must follow from the curvature which is described by the fourth-rank Riemann curvature tensor (it is defined as $A^i{}_{;lk} - A^i{}_{;kl} = R^i{}_{mlk}A^m$, a generalized 'curl'). It involves second and square of the first derivatives of the space-time metric, g_{ab}. The Bianchi identity is given by the antisymmetric derivative of $R^i{}_{mlk}$,

$$R^i{}_{m[lk;n]} = 0. \tag{1}$$

If the gravitational dynamics has to follow from the curvature, it has to follow from this identity which is the only available geometric relation. The only thing we can

do to it is to contract the available indices which do lead, unlike for scalar and vector, to a non-vacuous relation,

$$G^a{}_{b;a} = 0, \quad G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \tag{2}$$

where R_{ab} is the Ricci tensor, the contraction of Riemann, while R is the trace of Ricci. Now the trace (contraction) of the Bianchi identity yields a non-trivial differential identity from which we can deduce the following relation:

$$G_{ab} = \kappa T_{ab} - \Lambda g_{ab}, \quad T^a{}_{b;a} = 0, \tag{3}$$

where T_{ab} is the second-rank symmetric tensor with vanishing divergence and κ and Λ are constants. The left-hand side of the equation is a second-order differential operator on the metric g_{ab}. For this equation to describe dynamics of gravity, the tensor T_{ab} should describe the source/charge for gravity which should also be universal. It should be shared by all particles and hence T_{ab} should represent energy–momentum distribution. Thus we obtain the Einstein equation for gravitation which entirely follows from the space-time curvature. We have however two constants of which one (κ) is to be determined by experimentally measuring the strength of the force and is identified with Newton's constant, $\kappa = -8\pi G/c^2$. Why is there a new constant Λ which though arises in the equation as naturally as the energy–momentum tensor, T_{ab}? It is perhaps because of the absence of fixed space-time background which exists for the rest of physics and the new constant may be a signature of this fact. It is the universal character of gravity which makes space-time itself dynamic. The force-free state would however be characterized by homogeneity and isotropy of space and homogeneity of time which will in general be described by space-time of the constant curvature and not necessarily of zero curvature. The new constant Λ is the measure of the constant curvature of space-time and it identifies the most general space-time for force-free state. It may in some deep and fundamental sense be related to the basic structure of space-time.

We also know that the complete contraction of Riemann gives the scalar curvature, R, the Einstein–Hilbert Lagrangian which on variation leads to the divergence-free Einstein tensor, G_{ab}, and subsequently the Einstein equation. We thus have the two distinct but equivalent derivations for the gravitational dynamics. The former is simply driven by the geometry while the latter is in the spirit of every dynamical equation following from an action. It is always possible to write an action constructed from Riemann curvature for higher derivative gravity and derive the corresponding equation of motion. Similarly, is it possible to derive an analogue of G_{ab}, a divergence-free differential operator from the Bianchi derivative of the higher-order curvature polynomial? This is the question we wish to address and show that the answer is in affirmative. It would give yet another characterization of the Lovelock gravity.

We believe that gravitational dynamics to follow from the space-time curvature should be a general principle which should be true in general for higher-order theories as well. Thus arises the question of going beyond the linear order in Riemann. Let us consider the quadratic tensor,

$$\mathcal{R}_{abcd} = R_{abmn} R^{mn}{}_{cd} + \alpha R_{[a}{}^m R_{b]mcd} + \beta R R_{abcd}, \tag{4}$$

where α, β are constants. We consider the Bianchi derivative, $\mathcal{R}_{ab[cd;e]}$ which on contraction gives

$$g^{ac}g^{bd}\mathcal{R}_{ab[cd;e]} = (-2\mathcal{R}_e{}^c + \mathcal{R}\delta_e{}^c)_{;c} \tag{5}$$

where $\mathcal{R}_{ac} = g^{bd}\mathcal{R}_{abcd}$ and $\mathcal{R} = g^{ab}\mathcal{R}_{ab}$. It turns out that for $\alpha = 4$ and $\beta = 1$, we obtain

$$\begin{aligned}\mathcal{R}^{cd}{}_{[cd;e]} &= (-2\mathcal{R}_e{}^c + \mathcal{R}\delta_e{}^c)_{;c} \\ &= \left(-H_e{}^c + \frac{1}{2}L_{\mathrm{GB}}\delta_e{}^c\right)_{;c}.\end{aligned} \tag{6}$$

That is

$$\mathcal{R}^{cd}{}_{[cd;e]} - \frac{1}{2}(L_{\mathrm{GB}}\delta_e{}^c)_{;c} = -H^c{}_{e_{;c}} = 0. \tag{7}$$

The tensor H_{ab} is divergence-free, $H_{a;b}^{\ \ b} = 0$, and is given by

$$\begin{aligned}H_{ab} = 2(&RR_{ab} - 2R_a{}^m R_{bm} - 2R^{mn}R_{ambn} \\ &+R_a{}^{mnl}R_{bmnl}) - \frac{1}{2}L_{\mathrm{GB}}g_{ab}.\end{aligned} \tag{8}$$

It results from the variation of the well-known Gauss–Bonnet Lagrangian $L_{\mathrm{GB}} = R^2_{abcd} - 4R^2_{ab} + R^2$ where we have written $R^2_{abcd} = R_{abcd}R^{abcd}$. That is, we can write

$$H_{ab} = 2\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}g_{ab}, \tag{9}$$

where $\mathcal{R} = L_{\mathrm{GB}}$.

Though H_{ab} can be written in terms of $\mathcal{R}_{abcd}$, it does not follow directly from it as G_{ab} does from R_{abcd}. We note that Bianchi derivative vanishes only for the Riemann curvature signifying the fact that it can be written as a generalized curl of a vector. No other tensor will have vanishing Bianchi derivative. However, to find the analogue of G_{ab}, we do not require the vanishing of Bianchi derivative of the quadratic tensor $\mathcal{R}_{abcd}$, instead vanishing of its trace would suffice. We see that even the trace of Bianchi derivative does not vanish but is equal to $\frac{1}{2}(L_{\mathrm{GB}}\delta_e{}^c)_{;c}$. It suggests that the curvature polynomial should also include a term involving its own trace which would make no contribution in the linear case. Before we do that, let us write $\mathcal{R}_{abcd}$ and H_{ab} for a general order n in the Lovelock polynomial [4]. In the context of higher derivative gravity theories, a unified scheme of writing Lagrangian is given [5] as an invariant product, $Q^{abcd}R_{abcd}$ where Q^{abcd} has the same symmetry properties as the Riemann tensor. It is constructed from metric and Riemann curvature and has vanishing divergence, $Q^{abcd}{}_{;c} = 0$. For the Lovelock Lagrangian, it is required to be a homogeneous function of the Riemann curvature. Since $\mathcal{R}_{abcd}$ is also a homogeneous polynomial in Riemann curvature for a given order, it is possible to write

$$\mathcal{R}_{abcd} = Q_{ab}{}^{mn}R_{mncd}, \tag{10}$$

where Q_{abcd} is in general given for nth order by [5],

$$Q^{ab}{}_{cd} = \delta^{aba_1b_1\ldots a_nb_n}_{cdc_1d_1\ldots c_nd_n} R_{a_1b_1}{}^{c_1d_1}\ldots R_{a_nb_n}{}^{c_nd_n}. \tag{11}$$

Let us verify this for the quadratic Gauss–Bonnet case where Q_{abcd} could explicitly be written as [5]

$$Q_{abcd} = R_{abcd} - 2R_{a[c}g_{d]b} + 2R_{b[c}g_{d]a} + Rg_{a[c}g_{d]b}. \tag{12}$$

It is easy to see that when it is substituted in eq. (10), we obtain the Gauss–Bonnet $\mathcal{R}_{abcd}$ as given in eq. (4) with $\alpha = 4, \beta = 1$. We could thus write $\mathcal{R}_{abcd}$ by using eqs (11) and (10) for any term in the Lovelock polynomial and the corresponding H_{ab} for the nth polynomial will be given by

$$H_{ab} = n\mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}g_{ab} \tag{13}$$

which will for the linear case $n = 1$ be the Einstein tensor, G_{ab}. In this case, the trace of the Bianchi derivative vanishes because the Riemann tensor satisfies the Bianchi identity. However, H_{ab} as defined above is the analogue of the Einstein tensor for the nth-order polynomial and hence the generalized Einstein tensor is divergence-free. As trace of R_{abcd} is the Einstein–Hilbert Lagrangian, trace of $\mathcal{R}_{abcd}$ gives the Gauss–Bonnet and higher-order Lovelock Lagrangian. Note that eq. (5) holds good in general for any order n of the polynomial $\mathcal{R}_{abcd}$. We can thus write in general for eq. (7) as

$$\mathcal{R}^{cd}{}_{[cd;e]} - \frac{n-1}{n}(\mathcal{R}\delta_e{}^c)_{;c} = -\frac{2}{n}H_{e;c}^{\ \ c} = 0, \tag{14}$$

where $\mathcal{R}$ is the corresponding nth-order Lagrangian. For $n = 1$, the second term on the left vanishes indicating the Bianchi identity and $H_{ab} = G_{ab}$. For $n > 1$, the trace of Bianchi derivative does not vanish unless the trace $\mathcal{R}$ is also included as indicated by the second term which then leads to the required divergence-free tensor, H_{ab}. Apart from the quadratic Gauss–Bonnet case, we have also verified the above relation for $n = 3$ cubic polynomial. It is interesting that the order of the polynomial appears in the definition of the corresponding curvature polynomial tensor yielding the second-rank symmetric divergence-free tensor.

We now turn to redefining the curvature polynomial which also includes its trace such that the trace of the Bianchi derivative vanishes directly giving the required differential operator in terms of divergence-free H_{ab}. We thus write

$$\mathcal{F}_{abcd} = \mathcal{R}_{abcd} - \frac{n-1}{n(d-1)(d-2)}\mathcal{R}(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{15}$$

where n is the order of the polynomial and d is the dimension of space-time under consideration. Then

$$-\frac{n}{2}\mathcal{F}^{cd}{}_{[cd;e]} = H_{e;c}^{\ \ c} = 0 \tag{16}$$

which verifies eq. (13). In terms of $\mathcal{F}_{abcd}$, H_{ab} is given by

$$n\left(\mathcal{F}_{ab}-\frac{1}{2}\mathcal{F}g_{ab}\right)=H_{ab}, \tag{17}$$

where $\mathcal{F}_{ab}=g^{cd}\mathcal{F}_{acbd}$ and $\mathcal{F}=g^{ab}\mathcal{F}_{ab}$. It is interesting to note that $\mathcal{F}_{ab}$ is the analogue of the Ricci, R_{ab} and H_{ab} of the Einstein, G_{ab} with the order n of the polynomial $\mathcal{R}_{abcd}$ which is the analogue of the Riemann, R_{abcd}. Thus, H_{ab} is obtained from $\mathcal{F}_{abcd}$ in the same manner as G_{ab} is from R_{abcd}, and for $n=1$, $\mathcal{F}_{abcd}=\mathcal{R}_{abcd}=R_{abcd}$. All these however happen only for the specific Lovelock coefficients in the polynomial thereby determining the Lovelock polynomial.

Note that H_{ab} defines the general conserved (Einstein) tensor for any order n of the homogeneous curvature polynomial and let us take its trace which would give

$$\mathcal{F}=\frac{d-2n}{n(d-2)}\mathcal{R}. \tag{18}$$

For the usual Einstein gravity, $n=1$, it is $\mathcal{F}=\mathcal{R}$ for $d>2$ and the Einstein tensor vanishes for $d=2$. Thus, we have the general result that $d>2n$ for nth-order polynomial and H_{ab} vanishes for $d=2n$. It is the variation of $\mathcal{R}$ (and so also $\mathcal{F}$) which gives H_{ab} which vanishes for $d=2n$ but not $\mathcal{R}$. If we take $\mathcal{F}$ which is proportional to the Lovelock Lagrangian $\mathcal{R}$, then the Lagrangian itself vanishes for $d=2n$. It is remarkable that the existence of an analogue of the Einstein tensor in general – a conserved tensor which is non-zero only for $d>2n$, uniquely determines the Lovelock polynomial.

By redefining the curvature polynomial we have been able to derive the divergence-free differential operator for the gravitational dynamics. This is what we had set out to do and it could be stated as follows:

Theorem

The second-order quasilinear differential operator as a second-rank divergence-free tensor in the equation of motion for gravitation could always be derived from the trace of the Bianchi derivative of the fourth-rank tensor, $\mathcal{F}_{abcd}$, which is a homogeneous polynomial in curvatures. The trace of the curvature polynomial is proportional to the corresponding term in the Lovelock action and corresponding to each term in the Lovelock Lagrangian, there exists a fourth-rank tensor which is a new characterization of the Lovelock Lagrangian.

It is the requirement of quasilinearity (linear in second derivative) of the equation of motion which singled out the Lovelock polynomial with specific coefficients. In our case it is replaced by the requirement that the fourth-rank tensor which is homogeneous in curvatures yields a divergence-free second-rank tensor through the trace of its Bianchi derivative. This automatically ensures the quasilinearity of the equation. For non-Lovelock action, there would not exist a fourth-rank tensor with this property.

In a sense, the Riemann curvature could be looked upon as a Bianchi potential giving the Einstein tensor for the Einstein gravity while $\mathcal{F}_{abcd}$ is the Bianchi potential giving H_{ab} for the Lovelock dynamics. Thus, each term in the Lovelock polynomial has a potential tensor $\mathcal{F}_{abcd}$ given by eqs (10), (11) and (15). It

is remarkable that there exists Bianchi potential for each term in the Lovelock Lagrangian. Non-existence of potential characterizes all other actions like $f(R)$ gravity. The existence of Bianchi potential thus becomes a distinguishing feature for the Lovelock action. In this context it may be noted that the requirement that both Palitini and metric action give the same equation of motion also picks up the Lovelock action [6]. We have thus three distinct properties (namely quasilinearity of equation of motion, equivalence of Palitini and metric formulation and existence of Bianchi potential) which characterize the Lovelock action. I believe that there should exist a thread knitting them and it would be interesting to probe that.

However, the relevance of the order in the Lagrangian for the gravitational dynamics depends upon the space-time dimension. For instance, for $d < 5$, the quadratic Gauss–Bonnet term makes no contribution to the equation of motion and similarly the cubic term becomes relevant only for $d > 6$. We would however like to emphasize that for $d > 4$, H_{ab} must be included along with G_{ab} for the classical dynamics of gravitation. In the ultraviolet limit of the theory signifying superstrong gravitational field, it is pertinent to include higher-order curvature effects. If we continue to have a well-defined evolution of the field, the equation must be quasilinear. This will inevitably and uniquely lead to Lovelock polynomial and higher dimension. Further, one loop correction in string theory also generates the Gauss–Bonnet term [7]. That is, strong field limit of classical gravity and one-loop quantum correction seem to share the same Gauss–Bonnet ground. It could therefore be envisioned that the classical limit to quantum gravity is perhaps via the Lovelock gravity and the relevant order in the polynomial (loop correction) being given by the space-time dimension under consideration. The Gauss–Bonnet and higher orders may therefore represent an intermediary state between the classical Einstein gravity and quantum gravity [8].

Gravity is an inherently self-interactive force and the self-interaction could only be evaluated by successive iterations. The Einstein gravity is self-interactive but it contains only the first iteration through the square of first derivative in Riemann curvature. The question is, how do we stop at the first iteration? The second iteration would ask for a quadratic polynomial in Riemann curvature which should give the corresponding term in the equation of motion. Thus, the quadratic tensor, $\mathcal{R}_{abcd}$ as given in eq. (4) with specific coefficients, will alone meet the requirement (quasilinearity of the differential operator) for inclusion of the second iteration. However, its effect in the equation of motion can be felt only for $d > 4$, and hence we have to go to higher dimension for the physical realization of the second iteration of self-interaction [8,9]. It is remarkable that even classical dynamics of gravity asks for dimension > 4. As two and three dimensions were not big enough for free propagation of gravity, similarly four dimension is not big enough to fully accommodate self-interaction dynamics of gravity. Then the most pertinent question is, where does this chain end? Let us envision the AdS/CFT-like scenario where the 3-brane ($(3+1)$-space-time) forms the boundary enclosing the higher-dimensional bulk space-time. If matter fields remain confined to 3-brane, the bulk would then be free of matter and hence it would be maximally symmetric (homogeneous and isotropic in space and homogeneous in time). It is then a space-time of constant curvature, dS/AdS, with vanishing Weyl curvature. There is no free gravity to propagate any further and hence the chain stops at the second level at least in

this particular construction. Whether this is a generic setting or not is however an open question. It may be noted here that gravitational equation with the inclusion of both G_{ab} and H_{ab} for empty space in higher dimension admits dS/AdS as solution [8].

In what dimension should matter live is to be entirely determined by the matter dynamics. If the matter fields are gauge vector fields described by 2-form, the conformal invariance (universal scale change, $g_{ab} \to f^2 g_{ab}$) of their dynamics will dictate that they live in four dimension. This general principle is always obeyed by the matter field dynamics unless a scale in terms of mass etc. is introduced by spontaneous symmetry breaking. The symmetry breaking is indicative of theory being incomplete and it is hoped that the complete theory would restore the conformal invariance. It is therefore reasonable to take that matter remains confined to 3-brane. This view is also supported by the string theory where open strings have their endpoints on the brane indicating residence of matter there.

The most fundamental question is, how do we know there exist higher dimensions and if so why do we not see them? The existence of dimension can only be probed by a physical interaction. All our probes are matter field forces like electromagnetic, which remain, as argued above, confined to 3-brane and hence they cannot fathom higher dimension. Since gravity is universal, it cannot be confined entirely to the brane and can propagate in higher dimension. Above we have argued quite convincingly that there is a strong case for higher dimension for physical realization of the second iteration of self-interaction. It is a purely classical motivation (we have also elsewhere [8,9] given a couple of more classical arguments) while higher dimension is a natural arena for string theory. The only way higher dimension could thus be probed is by a very high energy pure gravitational experiment. This is what we have not yet been able to do and hence the question remains open.

We had set out to establish the general principle that gravitational dynamics resides in space-time curvature and hence it should always and entirely be driven by space-time geometry. We have shown this by deriving the quasilinear differential operator for the equation of motion for the Lovelock gravity. In particular, we have found a new geometric characterization of the Lovelock gravity in existence of Bianchi potential for each term in the polynomial. This is indeed an interesting general property.

Acknowledgement

The author wishes to thank T Padmanabhan and Dawood Kothawala for useful discussion and the latter also for eq. (13).

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[1a] For a vector field, vanishing of Bianchi derivative means the field is a curl of gauge potential, i.e. $F_{ik} = A_{i,k} - A_{k,i} = -F_{ki}$. This trivially implies $F^{ik}_{;ik} = 0$ and consequently we can write $F^{ik}_{;k} = J^i$ with $J^i_{;i} = 0$. This is indeed the equation of motion for the gauge field with J^i being identified as the current vector.

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