PRAMANA
— journal of
physics

© Indian Academy of Sciences

Vol. 74, No. 6
June 2010
pp. 867–874

Understanding the spreading of a Gaussian wave packet using the Bohmian machinery

ALOK KUMAR PAN
CAPSS, Department of Physics, Bose Institute, Salt Lake, Kolkata 700 091, India
E-mail: apan@bosemain.boseinst.ac.in

MS received 19 September 2009; revised 15 January 2010; accepted 29 January 2010

Abstract. A freely propagating Gaussian wave packet naturally spreads with time. Exploiting the machinery of the Bohmian model of quantum mechanics, the way the wave packet spreads is re-examined.

Keywords. Bohmian model; Gaussian wave packet; turning point.

PACS No. 03.65.Ta

1. Introduction and motivation

The time evolution of a Gaussian wave packet is an undergraduate textbook issue. A wave packet naturally spreads during the free time evolution that can be calculated by solving the relevant Schrödinger equation. In this paper, exploiting the machinery of the Bohmian model of quantum mechanics, we present a physically transparent picture of this spreading phenomenon that may not be available in the standard formalism of quantum mechanics.

Developing the early ideas of de Broglie [1], in 1952, David Bohm [2] rigorously formulated a realistic causal description of quantum mechanics by including the actual position (the so-called hidden variable) of an individual particle as a part of the state description of a microsystem. In this paper, we refer it as the Bohmian model. This model is observationally equivalent to the standard quantum mechanics (i.e., Bohmian model is the different interpretation of the same experimental results) but introduces a radically different perception of the underlying physics of microsystem.

One of the fundamental aspects of the Bohmian interpretation of quantum mechanics is its alleged solution to the quantum measurement problem. Since all measurements of microphysical attributes ultimately reduce to observation of position variable of a macro-object (say, the centre-of-mass coordinate of a pointer), according to this approach, a definite outcome in an individual measurement is determined by the relevant ontological position variable associated with the apparatus

– this has an objective value at all instants. Interpreted in this way, the intrinsic inexactness of quantum theory seems to be eliminated by ensuring a correspondence between a definite outcome of an individual measurement and the position coordinate introduced in the theory at a fundamental level, irrespective of whether the outcome is observed. But, no clear consensus of this contentious solution has been achieved yet.

In this paper, keeping aside the subtle conceptual debates concerning the Bohmian model, we stress on the fact that this model possesses intrinsic heuristic value which helps one to understand the behaviour of microsystem.

Before proceeding further, we note the dramatic questions concerning the Bohmian model that have been asked by Bell [3]:

> "*Why is the pilot wave picture (Bohmian model) ignored in text books? Should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show us that vagueness, subjectivity, and indeterminism, are not forced on us by experimental facts, but by deliberate theoretical choice?*"

It is evident from the above statement that, although Bell was a proponent of the Bohmian model, he certainly did not regard Bohmian model as a complete solution to the subtle conceptual problems of the Copenhagen interpretation; rather he suggested to exploit the new elements of this model for realizing the quantum mechanics more deeply. This simply is the motivation for the present work.

In the standard framework of quantum mechanics, Born's interpretation of the squared modulus of the wave function (ψ) as the probability density of finding a particle within a specified region of space is a key ingredient. In the Bohmian model, $|\psi|^2$ is interpreted as the probability density of a particle being actually present within a specified region. Such actual particle follows a definite space-time trajectory that is determined by its wave function through an equation of motion in accordance with the initial position, formulated in a way consistent with the Schrödinger time evolution. Thus, the wave function plays a dual role in the Bohmian model; on the one hand, it determines the probability of the actual location of a particle, and on the other hand, it choreographs the motion of the particle. Similar to the standard formalism of quantum mechanics, in this model, the pre-measurement value of a dynamical variable is, in general, different from its post-measurement value depending upon the experimental context, except the position variable which has been given a special status whose pre- and post-measurement values are the same.

Although this model hinges on the notion of a definite space-time track used to provide a description of the objective motion of a single particle, such a trajectory is not directly measurable. Hence this trajectory can be essentially viewed as conceptual aids for understanding deeply the various features of quantum mechanics. Along this line, a number of studies have been reported by showing the applications of such trajectory as a useful computational tool [4,5]. In the present work, the machinery of the Bohmian model is used for understanding the spreading of a Gaussian wave packet – a realization that may not be available in the standard formalism of quantum mechanics, or in its other variants.

Using the notion of the Bohmian ontological trajectory of an individual particle, and, specifically, pointing out the feature that the 'particles' which lie in the front half of a freely evolving Gaussian ensemble accelerate and 'particles' which lie in the back half decelerate during the time evolution, we illustrate the way the spreading of a given wave packet occurs.

It is also interesting to note that because of the decelerated motion of the back half 'particles', an individual Bohmian particle lying in the left half of the wave packet may move in opposite direction to the motion of the peak of the wave packet, i.e., an individual Bohmian free particle of a given Gaussian ensemble can turn around from its initial direction of motion – a feature which has an important implication while calculating the arrival time distribution in quantum mechanics. We investigate the specific condition in which this turning point occurs.

The plan of the paper is as follows. In §2, we briefly summarize the essence of the Bohmian causal model of quantum mechanics and the related nonuniqueness of this model. The free time evolution of Gaussian wave packet is considered in §3. We calculate the Bohmian trajectories of individual particles and obtain the condition in which the turning point occurs. Finally, in §4 we present our concluding remarks.

2. The essence of the Bohmian model

Here we present the Bohmian description of quantum mechanics which is made easy for the people who are not familiar with this model. Bohm himself stressed on the 'quantum potential' as the important element in his model that is crucial for demonstrating the nonlocality in his model. But, the basic insights of this model can be introduced without using the notion of quantum potential following the spirit of Bell [3]. Anyway, we begin by briefly summarizing the basic tenets of the Bohmian model [2,6,7] as follows:

(a) A complete description of the state of an individual particle is provided by its position and the wave function, with the position changing with time (the Bohmian equation of motion) in a way that is specified by the Schrödinger time evolution of the wave function $\psi(x,t)$.

(b) Corresponding to a given wave function $\psi(x,t)$, positions of the particles are taken to be distributed over an ensemble according to the probability density $\rho(x,t) = |\psi(x,t)|^2$, with $|\psi(x,t)|^2\,\mathrm{d}x$ being interpreted as the probability of a particle to be actually present within a region of space between x and $x+\mathrm{d}x$ at an instant t. This is in contrast with the standard interpretation (*a la* Born) of $|\psi(x,t)|^2\,\mathrm{d}x$ as the probability of finding a particle within the specified region.

(c) The quantum mechanical equation of continuity, as derived from the Schrödinger equation, is interpreted as corresponding to an actual flow of particles with the velocity $v(x,t)$, and the Schrödinger probability current density $J(x,t)$ is regarded as the current density of actual particle flow given by $J(x,t) = \rho(x,t)v(x,t)$. Then the Bohmian equation of motion of a particle is given by the following 'guidance equation':

$$v(x,t) = \frac{1}{m}\frac{\partial S(x,t)}{\partial x}, \tag{1}$$

where we have used the Schrödinger expression for current density

$$J(x,t) = \mathrm{Re}[(-i\hbar/m)\psi^*(x,t)(\partial/\partial x)\psi(x,t)] \tag{2}$$

and the relevant wave function is written in the polar form $\psi(x,t) = R(x,t)\mathrm{e}^{iS(x,t)/\hbar}$ with $\rho(x,t) = R^2(x,t)$, R and S being real numbers which are functions of space and time.

Here it should be noted that the Schrödinger expression for the probability current density $J(\mathbf{x},t)$ and, consequently, $v(\mathbf{x},t)$ is inherently non-unique. This is seen from the feature that if one adds any divergence-free term to the above expression for $J(x,t)$, the new expression also satisfies the same equation of continuity derived from the Schrödinger equation. However, it has been discussed by Holland [8], that the probability current density derived from the Dirac equation for any spin-1/2 particle is unique, and that this uniqueness is preserved in the non-relativistic limit while containing a spin-dependent term in addition to the Schrödinger expression for $J(x,t)$. Interestingly, one can further argue that this property of the uniqueness of probability current density is not specific to the Dirac equation, but is a consequence of any relativistic quantum mechanical equation [9]. It has also been shown [10] that expression for probability current density for spin-1 and spin-1/2 are the same but for spin-0 particles the unique current is the Schrödinger current $J(\mathbf{x},t)$. Here we restrict our attention to spin-0 particles.

3. Bohmian trajectory of a freely evolving Gaussian wave packet

Let us consider a one-dimensional Gaussian wave function at $t=0$ given by

$$\psi(x,t=0) = \frac{1}{(2\pi\sigma_0^2)^{1/4}} \exp\left[-\frac{x^2}{4\sigma_0^2} + ikx\right], \tag{3}$$

where σ_0 is the initial width of the associated wave packet which is peaked at $x=0$ and moving freely along positive $\widehat{x}$ direction with the initial group velocity $u = \hbar k/m$. The Schrödinger time-evolved wave function for free motion at any instant t is given by

$$\psi(x,t) = \frac{1}{(2\pi A_t^2)^{1/4}} \exp\left[-\frac{(x-ut)^2}{4\sigma_0 A_t} + ik\left(x - \frac{ut}{2}\right)\right] \tag{4}$$

and hence the time-evolved position probability density is given by

$$\rho\left(x,t\right) = \left|\psi\left(x,t\right)\right|^2 = \frac{1}{\left(2\pi\sigma_t^2\right)^{1/2}} \exp\left[-\frac{\left(x-ut\right)^2}{2\sigma_t^2}\right], \tag{5}$$

where $A_t = \sigma_0(1 + \frac{i\hbar t}{2m\sigma_0^2})$ and $\sigma_t = |A_t| = \sigma_0(1 + \frac{\hbar^2 t^2}{4m^2\sigma_0^4})^{1/2}$; σ_t is the width of the wave packet at any instant t. Next, we note that eq. (4) can be written in the following form:

$$\psi\left(x,t\right) = R\left(x,t\right) \exp\left(-\frac{iS(x,t)}{\hbar}\right), \tag{6}$$

where the amplitude part

$$R(x,t) = \frac{1}{(2\pi\sigma_t^2)^{1/4}} \exp\left[-\frac{(x-ut)^2}{4\sigma_t^2}\right] \tag{7}$$

and the phase part

$$S(x,t) = -\frac{\hbar}{2}\tan^{-1}\left(\frac{\hbar t}{2m\sigma_0^2}\right) + mu\left(x - \frac{ut}{2}\right) + \frac{(x-ut)^2\hbar^2 t}{8m\sigma_0^2\sigma_t^2}. \tag{8}$$

Now, from eq. (1), the Bohmian ontological velocity of individual particle belonging to the Gaussian ensemble is calculated by using eq. (8)

$$v(x,t) = u + \frac{(x-ut)\,bt}{(1+bt^2)}, \tag{9}$$

where $b = \hbar^2/4m^2\sigma_0^4$.

Integrating eq. (9) and further simplifying, one can calculate the Bohmian trajectory equation of a freely evolving ith particle of the ensemble with the initial position x_{0i} given by

$$x_i(t) = ut + x_{0i}\sqrt{1+bt^2}, \tag{10}$$

where u, the initial group velocity of the wave packet, is the pre-measurement velocity of any individual particle of the ensemble and x_{0i} is the initial position of the ith individual particle corresponding to the wave function given by eq. (3).

To calculate the velocity of the ith individual particle we evaluate $\mathrm{d}x_i(t)/\mathrm{d}t$, then we obtain

$$v_i(x_{0i},t) = \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = u + \frac{x_{0i}bt}{\sqrt{1+bt^2}} \tag{11}$$

which is the function of initial position x_{0i} in contrast to eq. (9) that is not the function of initial positions. Equation (11) represents the velocity of the ith particle of the initial Gaussian ensemble at any instant t, and implies the following:

(i) If $x_{0i} = 0$ at which the wave packet associated with the wave function given by eq. (3) is peaked, then $v_i(x_{0i},t) = u$, i.e., the particle follows the Newtonian trajectory for free motion.

(ii) If x_{0i} is +ve, then $v_i(x_{0i},t) = u + \frac{x_{0i}bt}{\sqrt{1+bt^2}}$, i.e., the particles distributed in the right half of the initial ensemble are all accelerated. The velocity $v_i(x,t)$ can then never be zero for any x_{0i} since b is a positive constant.

(iii) If x_{0i} is −ve, then $v_i(x_{0i},t) = u - \frac{x_{0i}bt}{\sqrt{1+bt^2}}$, i.e., the particles distributed in the left half of the initial ensemble are all decelerated.

It is then evident from the above three cases that there is an asymmetry in the Bohmian velocities between any two symmetric particles; one lies in the left half (say, at $x_{0i} = -n\sigma_0$) and the other one belongs to the right half (at $x_{0i} = +n\sigma_0$) of the initial wave packet. Because of this asymmetry, after a certain time the distance in position space shifted by the particle initially lying at $x_{0i} = +n\sigma_0$ is greater than the particle lying at $x_{0i} = -n\sigma_0$. Thus, the position space distribution of the freely propagating Gaussian wave function at any given instant of time is broader than the initial wave packet envelope. Hence, from the Bohmian model of

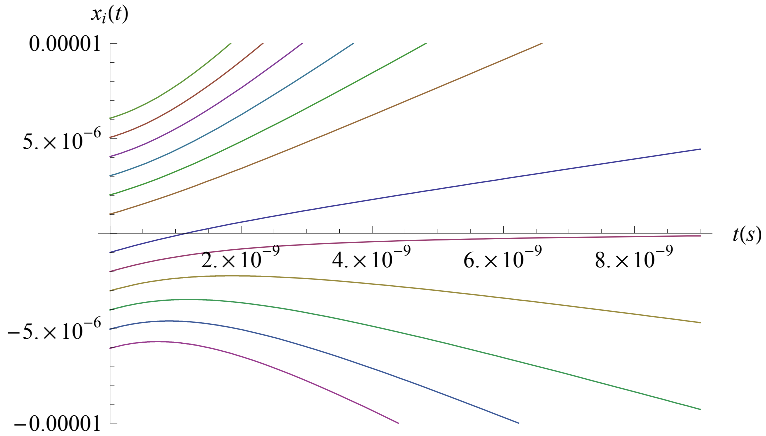

Figure 1. The Bohmian trajectories for the particles associated with a Gaussian wave packet with initial positions $\pm\sigma_0$, $\pm2\sigma_0$, $\pm3\sigma_0$, $\pm4\sigma_0$, $\pm5\sigma_0$ and $\pm6\sigma_0$ are plotted where the initial width $\sigma_0 = 10^{-6}$ cm and initial peak velocity $u = 10^4$ cm/s. The peak momentum is nearly two times the momentum uncertainty. It is seen that the particles that are distributed outside $-2\sigma_0$ along the $-\widehat{x}$-axis are all turned around and the particle at $-2\sigma_0$ gradually reaches initial peak position at $x = 0$.

quantum mechanics, a physically transparent account of the spreading of a Gaussian wave packet is obtained, that occurs due to the asymmetrical motions of the actual particles belonging to the left half and the right half. This trenchant argument of spreading may not be obtained from the standard formalism of quantum mechanics.

Now, since the particles lying in the left half of the initial Gaussian ensemble decelerate, then an individual particle in this half can turn around, if the following condition is satisfied:

$$u < \frac{x_{0i}bt}{\sqrt{1+bt^2}}. \tag{12}$$

Next, putting the value of b in eq. (12), and taking large time limit, one gets $p < x_{0i}\Delta p/\sigma_0$. Next, for an individual particle whose initial position $x_{0i} = -n\sigma_0$, one obtains the following condition:

$$p < n\Delta p, \tag{13}$$

where p and Δp are the initial peak momentum and the momentum uncertainty respectively.

Equation (13) implies that if the peak momentum is n times lower than momentum uncertainty, then the particle lying at the position $x_{0i} = -n\sigma_0$ of the initial wave packet turns around. As an example, if $p < 5\Delta p$ then particles that are distributed beyond $-5\sigma_0$ along $-\widehat{x}$-axis of the initial wave packet will turn around. For a Gaussian wave packet with initial width σ_0, an exceedingly small number (10^{-14}) of particles can be found outside the region $x = -10\sigma_0$ to $x = +10\sigma_0$. Thus, if no particle has the turning point, then the momentum uncertainty should not be less than at least one tenth times of the mean momentum. It is seen from figure 1, where $p \approx 2\Delta p$, that all the particles are distributed outside $-2\sigma_0$ along the $-\widehat{x}$-axis turn around, and, the particles at $-2\sigma_0$ gradually reach the initial peak position at $x = 0$.

Next, for large time limit the trajectory equation given by eq. (10) can be written as

$$x_i(t) = \frac{(p - n\Delta p)}{m} t. \tag{14}$$

From the above equation it is seen that for $p < n\Delta p$, $x_i(t)$ is negative, i.e., any particle which has a turning point cannot cross the initial peak position $x = 0$. When $p = n\Delta p$, $x_i(t) = 0$ which implies that the particle initially at $x_{0i} = -n\sigma_0$ will stop at the initial peak position of the wave packet.

4. Conclusions

Apart from the discussions of the merits of Bohmian model, in this paper, we have discussed that this model can help one to understand the quantum behaviour of the microsystem by re-examining the spreading of a freely evolving Gaussian wave packet from the Bohmian perspective. Exploiting the mathematical structure of the Bohmian model, it is shown that spreading is an effect that occurs due to the asymmetrical motions of the actual particles distributed in the left half and the right half of the given Gaussian ensemble. In the standard Bohr–Heisenberg version of quantum mechanics, this type of reasoning by taking into account the reality of the actual position of an individual particle is strictly not allowed. Thus, although the empirical discrimination between the Bohmian model and the standard interpretation of quantum mechanics is not possible, the standard formalism of quantum mechanics in conjunction with the machinery of the Bohmian model as a heuristic computational tool may provide a deeper understanding of the underlying physics of the microsystem.

Acknowledgements

The author is indebted to Dipankar Home for the insightful interactions. This work is supported by the Research Associateship of Bose Institute, Kolkata.

References

[1] L de Broglie, *Electrons et Photons* (Gauthier-Villars, Paris, 1928) p. 105
[2] D Bohm, *Phys. Rev.* **85**, 166 (1952); *Phys. Rev.* **85**, 180 (1952)
[3] J S Bell, *Speakable and unspeakable in quantum mechanics* (Cambridge University Press, Cambridge, 1987) p. 160
[4] P Holland, *Ann. Phys.* **315**, 505 (2005)
[5] G Bowman, *Am. J. Phys.* **70**, 313 (2002)
A Passon, *Eur. J. Phys.* **25**, 765 (2004)

[6] D Home, *Conceptual foundations of quantum physics* (Plenum, New York, 1997) Ch. 1
D Home and A Whitaker, *Einstein's struggles with quantum theory* (Springer Science, New York, 2007) Ch. 10
[7] P Holland, *The quantum theory of motion* (Cambridge University Press, Cambridge, 1993)
D Bohm and B Hiley, *The undivided Universe* (Routledge, London, 1993)
J T Cushing, *Quantum mechanics* (The University of Chicago Press, Chicago, 1994) Ch. 4
Bohmian mechanics and quantum theory: An appraisal edited by J T Cushing, A Fine and S Goldstein (Kluwer Academic Publishers, Dordrecht, 1996)
[8] P Holland, *Phys. Rev.* **A60**, 4326 (1999); *Ann. Phys. (Leipzig)* **12**, 446 (2003)
[9] A K Pan, M Ali and D Home, *Phys. Lett.* **A352**, 296 (2006)
[10] W Struyve, W De Baere, J De Neve and S De Weirdt, *Phys. Lett.* **A322**, 84 (2004)