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Analysis of pulsed wire method for field integral measurements in undulators

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Abstract. Pulsed wire technique is a fast and accurate method for the measurement of first and second field integrals of undulators used in free-electron lasers and synchrotron light sources. In this paper, we present a theoretical analysis of this technique by finding out the analytic solution of the differential equation for the forced vibration of the wire taking dispersion due to stiffness into account. Method of images is used to extend these solutions to include reflections at the ends. For long undulators, the effect of dispersion of the acoustic wave in the wire could be significant and our analysis provides a method for the evaluation of the magnetic field profile even in such cases taking the effect due to dispersion into account in an exact way.

Keywords. Undulator; free-electron laser; synchrotron radiation source; magnetic characterization.

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1. Introduction

Undulators are important components of free-electron lasers and synchrotron radiation sources. The performance of free-electron lasers and synchrotron radiation sources depends critically on the quality of the magnetic field of the undulator. In an ideal case, the on-axis magnetic field in a planar undulator is transverse and varies sinusoidally along its axis such that the trajectory of a relativistic electron injected along its axis is sinusoidal in the plane of oscillation. Any deviation in the trajectory from the ideal sinusoidal trajectory produces phase error, which results in the reduction in the brightness of the radiation in the case of synchrotron radiation sources. In the case of free-electron lasers, this results in the reduction of laser gain that deteriorates the performance of the device. It is therefore important to characterize the field quality of undulators developed for free-electron lasers and synchrotron radiation sources.

One of the techniques for the characterization of field quality in the undulator is the pulsed wire method, which was first proposed by Warren [1] for the measurement of profiles of first and second field integrals of an undulator. The first and second integrals of the magnetic field in the undulator, denoted by $I_1(z)$ and $I_2(z)$ respectively, are defined as

$$I_1(z) = \int_0^z \mathrm{d}z\, B(z), \tag{1}$$

$$I_2(z) = \int_0^z \mathrm{d}z \int_0^z \mathrm{d}z'\, B(z'), \tag{2}$$

where $B(z)$ is the y-component of the magnetic field as a function of the location z along z-axis in the undulator. The method has since then been used and improved by several authors [2–8] for fast and accurate measurement of the magnetic field profile. In this method, a thin wire is stretched along the axis of the undulator and a current pulse is passed through the wire. The forced vibration is thus produced in the wire which propagates in both directions along the length of the wire. By observing the vibration of the wire as a function of time at any given point along the wire, one can obtain the information about the magnetic field profile in the undulator. It has been shown that for delta-function excitation of the current in the wire, the vibration in the wire at any location follows the profile of the first integral of the magnetic field and for the step-function excitation, the vibration follows the pattern of second integral of the magnetic field [1]. For a relativistic electron beam injected in the undulator, the x-component of the velocity v_x of the electron injected along z-axis at $z = 0$, as a function of z is given by

$$v_x(z) = \frac{e}{\gamma m} \int_0^z \mathrm{d}z\, B(z), \tag{3}$$

where e is the absolute value of the electronic charge, m is the electron's rest mass and γ is the electron energy in units of its rest mass energy. The displacement of electron along x-axis is given by

$$x(z) = \frac{e}{\gamma m v_z} \int_0^z \mathrm{d}z \int_0^z \mathrm{d}z'\, B(z'), \tag{4}$$

where v_z is the electron velocity along z-axis, which can be assumed to be constant for the relativistic case with $\gamma \gg 1$. From the above expressions, it is clear that the profile of the first integral of the magnetic field in the undulator is proportional to the velocity profile of the electron in the undulator. The second integral profile of the magnetic field in the undulator is a measure of the trajectory of the electron beam in the undulator. The error in the trajectory of the electron in the undulator critically affects the performance of the device. Measurement of these integrals therefore becomes an important part in the characterization of the field quality of an undulator.

Compared to the more conventional technique of performing these measurements using Hall probe, the pulsed wire technique has several advantages. First, it is much faster compared to the Hall probe technique since in the Hall probe technique, measurements are taken point by point and a sufficient settling time should be given at each point for the Hall probe reading to settle down. Second, it is more suitable for performing measurements with mini-gap undulators since the diameter of the wire can be very thin compared to a Hall probe. The measurement of the first field integral by the pulsed wire technique is typically as accurate as the Hall probe method, whereas the second integral measurement could be performed even more accurately using pulsed wire technique [2]. We would however like to clarify here that the point by point magnetic field data obtained using Hall probe have in general more information than the field integral measurements that we are discussing here. For example, the measurement of optical phase error, which is an important parameter to characterize the undulator field quality is more accurately done by point by point measurement of magnetic field [9]. Compared to other techniques of measuring field integrals, e.g., stretched wire method [10] and rotating coil method [11], the pulsed wire method has however the advantage that it gives the profile of first and second field integrals as a function of distance along the undulator axis. The stretched wire method gives a single number for each of these integrals, where the integration is performed over the entire length of the undulator.

The previous analyses of the pulsed wire technique available in the literature [1,3,4] are based on the qualitative arguments and not on the rigorous solution of the differential equation of the forced vibration of stretched wire with proper boundary conditions. This approach is needed for a complete analysis of the problem. A knowledge about the complete time evolution of the vibration of the wire for the most general case will help in the design of the pulsed wire set-up for measurements of field profile. In this paper, we present such an analysis based on the solution of the differential equation for the forced vibration of the stretched wire, including the boundary condition at the ends of the wire.

The pulsed wire technique has limitations due to several phenomena, e.g., sag in the wire due to self-weight, dispersion due to stiffness in the wire, attenuation in the wire, scattering due to inhomogeneity or impurity in the wire etc., as discussed by Warren [1]. For long undulators, the effect of dispersion of the acoustic wave propagating in the wire becomes important. The waves having different wave numbers travel with different speeds in the wire and the shape of the pulse propagating in the wire is not preserved and gets distorted. This gives rise to problem in the measurement of the magnetic field profile for the case of long undulators. One typically tries to avoid this problem by reducing the dispersion in the wire by increasing the tension in the wire up to the limit of yield strength and reducing the diameter of the wire. However, the reduced thickness of the wire gives rise to problem due to spurious signal arising due to inhomogeneity in the wire [3]. On the other hand, if one keeps the wire thickness large enough to average out the imperfections in the wire, the effect of dispersion becomes non-negligible. In this paper, we find out an analytical solution of the problem that allows us to calculate the magnetic field profile correctly from the vibration profile of the wire, even in the presence of significant dispersion, by taking the effect due to dispersion in an exact way.

In the next section, we discuss the basic theory where we give the most general solution of the fourth-order linear partial differential equation for the forced

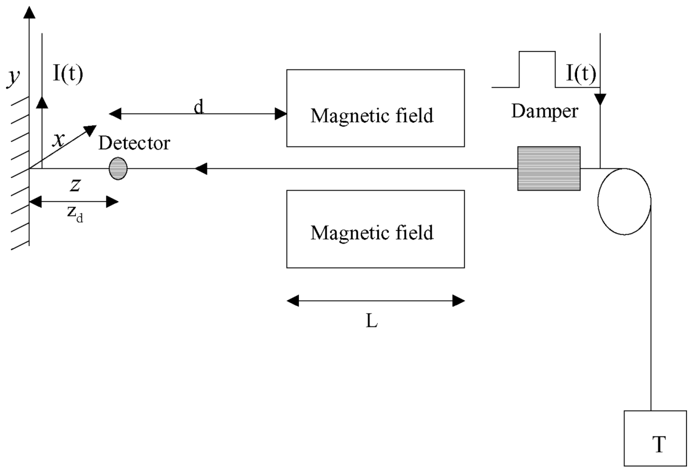

Figure 1. Schematic of the pulsed wire set-up. The wire is fixed at one end and at the other end, a damper is used such that there is no reflection from that end. The coordinate system used in the paper is shown. Current is flowing along the negative z-direction.

vibration of the stretched wire and show that the solution satisfies the differential equation. We use the method of images to include the boundary conditions at the ends of the wire. In §3, we discuss the results obtained for special cases and describe the time evolution for these cases. Finally, we present some conclusions.

2. Basic theory

The schematic of the pulsed wire set-up is shown in figure 1. The stretched wire is fixed at one end and the other end goes over a pulley. There is a damper on the pulley end to avoid reflections from this end. The fixed end and the pulley are typically assembled on a translation stage, which is not shown in the figure. The fixed end is at $z = 0$ and the detector is kept at $z = z_{\mathrm{d}}$. The magnetic field is assumed to be present only in the region $(z_{\mathrm{d}} + d) < z < (z_{\mathrm{d}} + d + L)$.

The differential equation for the forced vibration in the x-direction of a wire stretched along the z-axis is given by [12]

$$-EM\frac{\partial^4 x}{\partial z^4} + T\frac{\partial^2 x}{\partial z^2} - \mu\frac{\partial^2 x}{\partial t^2} = -f(z,t), \tag{5}$$

where E is the modulus of elasticity, M is the moment of inertia of the cross-sectional area with respect to the neutral axis, T is the tension and μ is the mass per unit length of the wire. Here, $f(z,t)$ is the force per unit length experienced by the wire in the x-direction at location z at time t and in this case is given by

$$f(z,t) = I(t)B(z), \tag{6}$$

where $I(t)$ is the current in the wire at time t and $B(z)$ is the magnetic field in the y-direction at z. Note that the first term in eq. (5) gives rise to dispersion of acoustic waves in the wire as first studied by Lord Rayleigh [3,13]. The solution of the above equation can be obtained by using Fourier transform technique. The solution for the initial condition $x(z,0) = 0$ and boundary condition $x(z,t) = 0$ for $z \to \pm\infty$ is given by

$$x(z,t) = \frac{1}{2\mu}\int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}k \frac{k}{\omega(k)} \int_{z-\frac{\omega(k)}{k}t'}^{z+\frac{\omega(k)}{k}t'} \mathrm{d}z' A(k) I(t-t') \mathrm{e}^{ikz'}, \tag{7}$$

where

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}z\, B(z) \mathrm{e}^{-ikz}, \tag{8}$$

and $\omega(k)$ is given by

$$\omega(k) = k\sqrt{\frac{T}{\mu}}\sqrt{1+\frac{EM}{T}k^2}. \tag{9}$$

As shown in Appendix A, the solution given by eq. (7) satisfies the differential equation given by eq. (5).

We now consider the case where the wire is fixed at the end $z = 0$. This will give rise to reflection. We can use the method of images to find the solution subjected to this boundary condition. Application of the method of images for solving the wave equation in the stretched wire is discussed in ref. [12]. Here, we apply this method for the analysis of the pulsed wire technique. In this problem, the actual magnetic field $B(z)$ is specified only for $z > 0$. We introduce a fictitious magnetic field given by

$$B(-z) = -B(z), \tag{10}$$

in the region $z < 0$, as shown in figure 2. One can show that this will give rise to the condition that $A(-k) = -A(k)$. Putting this condition in eq. (7), one can verify that the solution with the above fictitious magnetic field satisfies the boundary condition, $x(0,t) = 0$, as shown in Appendix A. The solution in the region $z > 0$ will therefore be given by eq. (7), where $A(k)$ corresponds to the total magnetic field, i.e., the actual field plus the fictitious field. This solution satisfies eq. (5) as well as the boundary condition.

For the situation when the wire is fixed at both the ends, the boundary condition will be $x(0,t) = 0$ and also $x(z_1,t) = 0$, where z_1 is the total length of the wire. Here, the actual magnetic field $B(z)$ is specified only for the region $0 < z < z_1$. We can define the fictitious magnetic field outside this region given by

$$B(-z) = -B(z), \quad B(z_1+z) = -B(z_1-z), \tag{11}$$

which is the inversion of mirror image of the magnetic field at $z = 0$ and z_1, as illustrated in figure 2. Note that in this case, the total magnetic field, including the fictitious magnetic field becomes periodic with a period $= 2z_1$. The allowed values of k are therefore restricted to $n\pi/z_1$, where n is an integer. With the

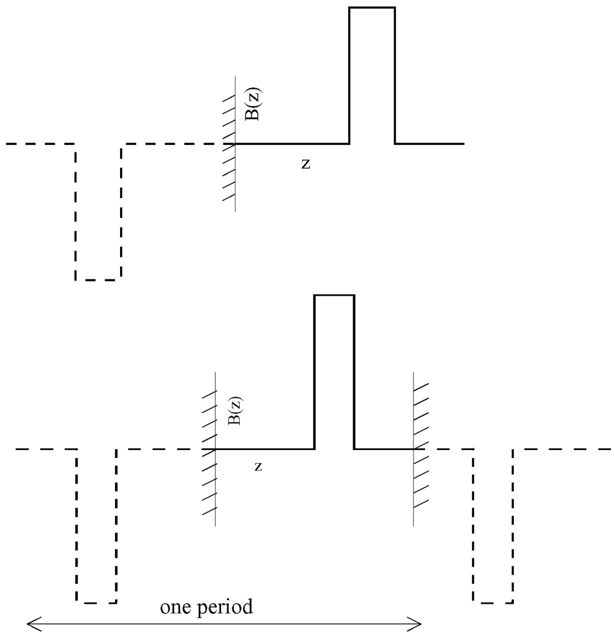

Figure 2. Illustration of fictitious magnetic field for two cases. Plots of the actual magnetic field and the fictitious magnetic field are shown by bold and dashed curves respectively. The upper figure shows the fictitious magnetic field for the case where the wire is fixed at one end only. The lower figure shows the fictitious magnetic field for the case where the wire is fixed at both ends. Note that in this case, the total magnetic field (including the fictitious magnetic field) becomes periodic.

above fictitious magnetic field, the solution given by eq. (7) satisfies the boundary conditions $x(0,t) = 0$ and $x(z_1,t) = 0$ as shown in Appendix A.

We have thus obtained the most general solution for the case when the wire is fixed either at one end or at both ends. Our solution takes the time dependence of the current and also the effect of dispersion due to stiffness in the wire. In the next section, we discuss the solutions for different cases.

3. Discussions

Let us discuss the case when the wire is fixed at $z = 0$, but is infinite in the positive z-direction and also the dispersion due to stiffness in the wire is negligible. In this case, the deflection in the wire at the detector located at $z = z_{\rm d}$ is given by

$$x(z_{\rm d},t) = \frac{v_0}{2T}\int_0^t {\rm d}t' \left\{ \int_{z_{\rm d}}^{z_{\rm d}+v_0 t'} {\rm d}z\, I(t-t')B(z) + \int_{z_{\rm d}-v_0 t'}^{z_{\rm d}} {\rm d}z\, I(t-t')B(z) \right\}, \qquad (12)$$

where $v_0 = \sqrt{T/\mu}$ is the phase velocity of the acoustic wave in the limit $k \to 0$. Here, the first term is due to the wave reaching z_d from the region $z > z_\mathrm{d}$ and the second term is due to the wave reaching from the region $z < z_\mathrm{d}$. We assume that the magnetic field is non-zero only in the region $(z_\mathrm{d} + d) < z < (z_\mathrm{d} + d + L)$. The second term in the above equation is then due to reflection. The first term is the main signal that we denote by $x^I(z, t)$ and the second term is the reflected signal that we denote by $x^R(z, t)$. Let us now consider the special case when the current pulse is a step function given by $I(t) = I_0$ for $0 < t < \tau$ and $I(t) = 0$ for other values of t. The first term of the solution for this case is given by

$$x^I(z_\mathrm{d}, t) = \frac{v_0 I_0}{2T} \int_0^t \mathrm{d}t' \int_{z_\mathrm{d}}^{z_\mathrm{d}+v_0 t'} \mathrm{d}z\, B(z), \tag{13}$$

for $t < \tau$ and

$$x^I(z_\mathrm{d}, t) = \frac{v_0 I_0}{2T} \int_{t-\tau}^t \mathrm{d}t' \int_{z_\mathrm{d}}^{z_\mathrm{d}+v_0 t'} \mathrm{d}z\, B(z), \tag{14}$$

for $t > \tau$. Note that eq. (13) has been obtained earlier in ref. [1] and eq. (14) has been earlier obtained in ref. [4]. We make some observations here. The second integrals of the magnetic field in eqs (13) and (14) are actually proportional to the transverse oscillation of the relativistic electron beam injected along the z-axis. In eq. (13), the second integral is proportional to the transverse displacement of the electron at $z = z_\mathrm{d} + v_0 t$, when injected along z-axis with $x = 0$ at $z = z_\mathrm{d}$. Similarly, in eq. (14), the second integral is proportional to the transverse displacement of the electron at $z = z_\mathrm{d} + v_0 t$, when injected along z-axis at $z = z_\mathrm{d}$ with a displacement along x-direction such that $x = 0$ at $z = z_\mathrm{d} + v_0(t - \tau)$. In the limit $\tau \to 0$, the displacement $x(z_\mathrm{d}, t)$ is given by

$$x^I(z_\mathrm{d}, t) = \frac{v_0 \tau I_0}{2T} \int_{z_\mathrm{d}}^{z_\mathrm{d}+v_0 t} \mathrm{d}z\, B(z), \tag{15}$$

which is proportional to the first integral of the magnetic field. Note that the above equation is valid as long as there is no significant variation in the magnetic field over the length $v_0\tau$. In that case, the above equation is valid for any arbitrary current pulse with charge $I_0\tau$ in a pulse.

We can draw some conclusions from the above results. If we want to perform the measurement of the second integral of the magnetic field that is non-zero only in the region $(z_\mathrm{d} + d) < z < (z_\mathrm{d} + d + L)$, we should keep the current pulse width $\tau = (d + L)/v_0$ such that we can differentiate x^I twice with respect to t in eq. (13) to get $B(z)$ for $t < \tau$. The contribution in the wire displacement due to the first term in eq. (12) will be during $d/v_0 < t < 2(d + L)/v_0$. During the period $0 < t < d/v_0$, there is no displacement in the wire at $z = z_\mathrm{d}$ since there is no magnetic field in the region $z_\mathrm{d} < z < z_\mathrm{d} + d$. The rise time of the current pulse τ_r should be such that there is no significant variation in the magnetic field over a length of $v_0\tau_\mathrm{r}$.

Note that in eqs (13)–(15), the contribution due to reflection from the fixed end is not included. The contribution from the reflection will be non-zero only for

$(2z_\mathrm{d}+d)/v_0 < t < (2z_\mathrm{d}+2d+2L)/v_0$. Therefore, if we satisfy $(2z_\mathrm{d}-d) > 2L$, we will make sure that the reflected signal does not interfere with the incident signal.

We will discuss here briefly the case when the wire is fixed at both ends. The first reflection from the left end will contribute during the period $(2z_\mathrm{d}+d)/v_0 < t < (2z_\mathrm{d}+2d+2L)/v_0$ and the reflection from the right end will contribute during the period $(2z_1-2z_\mathrm{d}-d-L)/v_0 < t < (2z_1-2z_\mathrm{d}+L)/v_0$. The timing of the further reflected signals can also be determined similarly. The design parameters can thus be carefully chosen such that main signal does not interfere with the reflected signals.

We will now discuss the effect of dispersion. The effect of dispersion on the wire vibration in the pulsed wire method using eq. (5) has been discussed qualitatively earlier in refs [3,7,8]. Here, we proceed for rigorous analysis by re-writing eq. (7) in the following form by making a change of variable $z'' = (z'-z)(kv_0/\omega)+z$ and then replacing z'' with z',

$$x(0,t) = \frac{1}{2v_0\mu}\int_0^t \mathrm{d}t' \int_{-v_0t}^{+v_0t} \mathrm{d}z' I(t-t') \int_{-\infty}^{\infty} \mathrm{d}k\, A(k)\mathrm{e}^{i\frac{\omega(k)}{v_0}z'}. \tag{16}$$

Here, we have also made a change in the order of integration. Note that we have changed the coordinate of the detector to $z=0$ from $z=z_\mathrm{d}$, for simplification. The fixed end of the wire is now at $z=-z_\mathrm{d}$ in this changed coordinate system. From the above equation, we find that if the undulator has a wave number k_u, the phase difference between the component having wave number k_u and the one having $k=0$ given by $[v_\mathrm{ph}(k_\mathrm{u})-v_0]k_\mathrm{u}L/v_0$ after travelling a distance L, where $v_\mathrm{ph}=\omega(k)/k$ is the phase velocity for the wave as a function of its wave number k. The distortion due to dispersion is less significant if this phase difference is much less than 2π. This imposes the following condition on the maximum length L_max of the wire

$$L_\mathrm{max} \ll \frac{4\pi T}{EMk_\mathrm{u}^3}. \tag{17}$$

The above expression has been derived assuming $EMk_\mathrm{u}^2/T \ll 1$. Note that this expression was derived earlier by Warren [1].

Next, we discuss the method for calculating the field integrals in the presence of effects due to dispersion, when eq. (17) is not valid. If we compare eqs (12) and (16), we observe that $B(z)$ in eq. (12) is replaced with $B_\mathrm{eff}(z)$ given by

$$B_\mathrm{eff}(z) = \int_{-\infty}^{+\infty} \mathrm{d}k\, A(k)\mathrm{e}^{i\frac{\omega(k)}{v_0}z}. \tag{18}$$

It is more useful to write the above integral in the following form of Fourier transform by making a change of variable $k'=\omega(k)/v_0$ and then replacing k' by k:

$$B_\mathrm{eff}(z) = \int_{-\infty}^{+\infty} \mathrm{d}k\, G(k)\mathrm{e}^{ikz}, \tag{19}$$

where

$$G(k) = A\left(k\sqrt{\frac{T}{2EMk^2}}\left[\sqrt{1+\frac{4EM}{T}k^2}-1\right]^{1/2}\right)$$
$$\times\frac{\sqrt{\frac{2EMk^2}{T}}}{\left[\sqrt{1+\frac{4EM}{T}k^2}-1\right]^{1/2}\sqrt{1+\frac{4EM}{T}k^2}}. \tag{20}$$

Note that for the limiting case $EM/T \to 0$, $G(k) \to A(k)$. From the above expressions, it is clear that if we include the effect of dispersion due to stiffness in the wire, the vibration in the wire is described by eqs (13) and (14) where $B(z)$ is replaced with $B_{\text{eff}}(z)$ mentioned above. Hence, knowing the vibration in the wire, we get the profile of $B_{\text{eff}}(z)$ by differentiating $x^I(0,t)$ twice with respect to t. Once $B_{\text{eff}}(z)$ is known, $G(k)$ can be calculated by taking inverse Fourier transform of $B_{\text{eff}}(z)$. Having calculated $G(k)$, $A(k)$ can be calculated by the following equation, which is obtained by inverting eq. (20):

$$A(k) = \frac{\left[\sqrt{1+\frac{4EM}{T}k^2+\frac{4E^2M^2}{T^2}k^4}-1\right]^{1/2}\sqrt{1+\frac{4EM}{T}k^2+\frac{4E^2M^2}{T^2}k^4}}{\sqrt{\frac{2EM}{T}k^2}\sqrt{1+\frac{EM}{T}k^2}}$$
$$\times G\left(k\sqrt{1+\frac{EM}{T}k^2}\right). \tag{21}$$

Having calculated $A(k)$, $B(z)$ can be obtained by taking its Fourier transform and the field integrals can then be calculated. It is important to note here that the absolute accuracy in the measurement of the profile of the magnetic field or the field integral will be dependent on the accuracy with which the data are available for E, M and T. The typical value of absolute accuracy possible using Hall probe is better than 0.05%. It may often not be possible to achieve similar absolute accuracy using the pulsed wire method due to limitation on the accuracy in the data for E, M and T. The data for E are available from the data sheet, but typically the absolute accuracy may not be better than 1%. Similarly, M and T can be measured in the laboratory, but the absolute accuracy may again not be better than 1%. Thus the absolute accuracy achieved in the pulsed wire method may be poor compared to the Hall probe technique and this will give rise to calibration error. However, this is not a drawback in this technique since one can obtain a comparable relative accuracy, which is more relevant for calculating the trajectory error arising due to field error.

4. Conclusions

In this paper, we have given a rigorous analysis of the pulsed wire technique based on the solution of the partial differential equation for the forced vibration of stretched wire. We have discussed the time evolution of the vibration in the wire at the location of the detector for various cases, taking reflection and dispersion into account.

In the absence of dispersion, by measuring the vibration in the wire at a given location for short and long current pulses, one can obtain the first and second field integrals directly. Our analytical treatment allows us to take into account the effect of dispersion in an exact manner, using which one can obtain the field integrals after processing the data on wire vibration as discussed in the previous section. Note that due to several sources of error, e.g., error in the measurement of wire vibration, background vibration already present in the wire, etc., the data on wire vibration can become noisy. Even in such cases, where the data on the wire vibration are not accurate enough for calculating the inverse Fourier transform and then the Fourier transform after the data processing, one can simply calculate the expected wire vibration for the theoretical field profile using eq. (16) and then compare this with the observed wire vibration to obtain a qualitative estimate of the error in the field profile. Our analysis will thus be useful in utilizing the pulsed wire technique for characterizing the magnetic field profile in long undulators.

Appendix A: Solution of differential equation for the forced vibration

In this appendix, we will show that the solution given by eq. (7) satisfies eq. (5). For this purpose, it is convenient to make a change of variable of integration in eq. (7) and re-write it in the following form:

$$x(z,t) = \frac{1}{2\mu}\int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}k\, \frac{k}{\omega(k)} \int_{z-\frac{\omega(k)}{k}(t-t')}^{z+\frac{\omega(k)}{k}(t-t')} \mathrm{d}z'\, A(k)I(t')\mathrm{e}^{ikz'}. \tag{A1}$$

By differentiating eq. (A1), we get

$$-EM\frac{\partial^4 x}{\partial z^4} + T\frac{\partial^2 x}{\partial z^2} = \frac{1}{2\mu}\int_0^t \mathrm{d}t' \times \int_{-\infty}^{+\infty} \mathrm{d}k\, \frac{A(k)I(t')}{i\omega(k)}(-EMk^4 - Tk^2)\Phi, \tag{A2}$$

$$-\mu\frac{\partial^2 x}{\partial t^2} = \frac{1}{2\mu}\int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}k\, \frac{A(k)I(t')}{i\omega(k)}\mu\omega^2(k)\Phi - \int_{-\infty}^{+\infty} \mathrm{d}k\, A(k)I(t)\mathrm{e}^{ikz}, \tag{A3}$$

where

$$\Phi(k,z,t,t') = \mathrm{e}^{i(kz+\omega(t-t'))} - \mathrm{e}^{i(kz-\omega(t-t'))}. \tag{A4}$$

Substituting eqs (A2) and (A3) in eq. (5), we can show that eq. (5) is satisfied.

Next, we show that if we satisfy $A(-k) = -A(k)$, the boundary condition is satisfied at $z = 0$. For this, we re-write eq. (7) for $z = 0$ as

$$x(0,t) = \frac{1}{\mu}\int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}k\, A(k)I(t-t')\frac{\sin\omega t'}{\omega}. \tag{A5}$$

Since the integrand in the above equation is an odd function of k, the integral will vanish. For the case, when the wire is fixed at both the ends, the displacement at $z = z_1$ is given by

$$x(z_1, t) = \frac{1}{\mu} \int_0^t \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}k\, A(k) I(t - t') \mathrm{e}^{ikz_1} \frac{\sin \omega t'}{\omega}. \tag{A6}$$

The integrand in the above integral is again an odd function of k since $A(-k) = -A(k)$ and only discrete values of k are allowed which are equal to $n\pi/z_1$, where n is an integer as discussed in §3. The integral therefore vanishes, proving that $x(z_1, t) = 0$.

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References

[1] R W Warren, *Nucl. Instrum. Methods in Phys.* **A272**, 257 (1988)
[2] P V Bousine, S V Tolmachev and A A Varfolomeev, *Nucl. Instrum. Methods in Phys.* **A393**, 414 (1997)
[3] T C Fan *et al*, *Rev. Sci. Instrum.* **73**, 1430 (2002)
[4] O Shahal and R Rohtagi, *Nucl. Instrum. Methods in Phys.* **A285**, 299 (1989)
[5] O Shahal, B V Elkonin and J S Sokolowski, *Nucl. Instrum. Methods in Phys.* **A296**, 588 (1990)
[6] A A Varfolomeev *et al*, *Nucl. Instrum. Methods in Phys.* **A359**, 93 (1995)
[7] T C Fan, C S Huang and F Y Lin, *Proceedings of EPAC 2004*, 428 (2004)
[8] T C Fan *et al*, *Proceedings of the 2001 Particle Accelerator Conference*, 2775 (2001)
[9] B L Bobbs *et al*, *Nucl. Instrum. Methods in Phys.* **A296**, 574 (1990)
[10] D Zangrando and R P Walker, *Nucl. Instrum. Methods in Phys.* **A376**, 574 (1996)
[11] W G Davies, *Nucl. Instrum. Methods in Phys.* **A311**, 399 (1992)
[12] Karl F Graff, *Wave motion in elastic solids* (Ohio State University Press, 1975)
[13] S Rayleigh, *Theory of sound* (Dover, New York, 1945) (reprint) 1877