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Degree distribution of a new model for evolving networks

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Abstract. We propose and study an evolving network model with both preferential and random attachments of new links, incorporating the addition of new nodes, new links, and the removal of links. We first show that the degree evolution of a node follows a nonhomogeneous Markov chain. Based on the concept of Markov chain, we provide the exact solution of the degree distribution of this model and show that the model can generate scale-free evolving network.

Keywords. Evolving networks; degree distribution; Markov chain; scale-free network.

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1. Introduction

A wide range of practical systems can be described as complex networks where the nodes represent elementary units and the links represent the interactions between pairs of units. Typical complex networks include the World Wide Web, the Internet, food webs, citation networks etc. (see [1–3]). Complex networks can be classified into two main categories, depending on their connectivity properties [4]. The first one is identified by the scale-free networks, whose degree distribution $P(k)$ (the probability that a node is connected to other k nodes) displays a power-law behaviour, $P(k) \sim k^{-\gamma}$, where γ represents the degree exponent. The second class is represented by exponential networks that exhibit an exponential degree distribution, $P(k) \sim \exp(-k/m)$, where m is a constant.

Empirical results demonstrate that many networks in nature appear to exhibit the scale-free feature. The origin of the power-law degree distribution observed in networks was first addressed by Barabási, Albert and Jeong (BA model) [5]. In this model, there are two main ingredients. First, the networks develop by the addition of new nodes. Second, the new node links to the old ones with preferential attachment rule. The two mechanisms, growth and preferential attachment, lead to the formulation of the power-law degree distribution with degree exponent $\gamma = 3$.

However, there are some differences between practical networks and the scale-free networks generated by BA model. First of all, the BA model incorporates only one mechanism for network growth: the addition of new nodes that connect to the nodes already in the system. In fact, many real networks are not purely growing; instead they are evolving networks with not only link and node additions but also link removals [1,6,7]. For example, in Internet, links are not only added but also break from time to time. In gene duplications and possible mutations, a protein network also is an evolving network with node and link additions and deletions. Secondly, some complex networks in nature should fall somewhere in between scale-free networks and exponential networks. There are many examples where the distribution is neither power-law nor exponential, such as the scientific collaboration networks [8]. Liu and Lai [9] have constructed a class of general growing networks based on intuitive but realistic consideration that nodes are added to the network with both preferential and random attachments. The degree distribution of the model is between a power-law and an exponential decay.

Motivated by the features of network evolution, we introduce a new model of evolving networks, incorporating the addition of new nodes, new links and the removal of links.

Starting with m_0 initial nodes, the model adopts the following two evolving rules independently.

(i) At each time step, a new node is added and $m(\leq m_0)$ new links from the new nodes are connected to m different nodes already present in the system. A node i with degree k_i will receive a connection from the new node with the probability

$$\Pi(k_i) = \frac{(1-p)(k_i+1)+p}{\sum_j[(1-p)(k_j+1)+p]}, \qquad (1)$$

where p $(0 \leq p \leq 1)$ is the probability that a new node is randomly connected to the existing node i and $(1-p)$ is the probability that the new node is preferentially attached to i.

(ii) At each time step, $c(< m_0)$ links between old nodes are removed with equal probability, i.e., we randomly choose two nodes in the network, if there exists a link between these two nodes, then the link is deleted; this process is repeated c times.

2. The degree exponent

We use the mean-field approach [5,10,11] to derive the scaling law for $P(k)$. Suppose that k_i is continuous, then k_i satisfies the following dynamic equation:

$$\frac{\partial k_i}{\partial t} = m\Pi(k_i) - \frac{2c}{t} \approx m\frac{(1-p)(k_i+1)+p}{(1-p)(2m+1)t+pt-2ct} - \frac{2c}{t}, \qquad (2)$$

where the approximation is based on $\sum_j[(1-p)(k_j+1)+p] \approx (1-p)(2m+1)t + pt - 2ct$ (note that we ignore the constant m_0 for large t).

Let the initial condition of eq. (2) be $k_i(t_i) = m$. Then we obtain

$$k_i(t) = [m+1+A(m,p,c)]\left(\frac{t}{t_i}\right)^{\beta} - 1 - A(m,p,c), \qquad (3)$$

where

$$A(m,p,c) = \frac{mp - 2c[(1-p)(2m+1) + p - 2c]}{m(1-p)},$$

$$\beta = \beta(m,p,c) = \frac{m(1-p)}{(1-p)(2m+1) + p - 2c}, \quad 0 \le p < 1.$$

Using the mean-field method, the degree distribution can be obtained as

$$P(k) = \frac{t}{m_0 + t}\frac{1}{\beta}[m + 1 + A(m,p,c)]^{1/\beta}(k + 1 + A(m,p,c))^{-\gamma}, \tag{4}$$

where the scaling exponent $\gamma = 1 + \frac{1}{\beta} = 3 + \frac{1-2c}{m(1-p)}$.

Thus, the degree distribution has a generalized power-law form as

$$P(k) \propto [k + 1 + A(m,p,c)]^{-\gamma}. \tag{5}$$

It should be stressed that eqs (4) and (5) are not valid when $m+1+A(m,p,c) = 0$. For example, when $p = 0$ and $m = 2c$ (or $2c - 1$), the continuum theory fails to predict the behaviour of the system. Therefore, we need an alternative approach to compute the exact solution of the degree distribution $P(k)$.

3. Degree distributions

Consider the degree $k_i(t)$ of node i at time t. The model indicates that the evolution of $k_i(t+1)$ only depends on $k_i(t)$. So $\{k_i(t), t = i, i+1, \ldots\}$ is a nonhomogeneous Markov chain [7,12,13] with the state space $\Omega = \{0, 1, 2, \ldots\}$.

From eq. (2), the probability that node i with degree k_i is connected to a new node at time t is $f_t^+(k_i) = (1 - 2c/t)m\Pi(k_i)$. The probability that node i's degree decreases by one is $f_t^-(k_i) = (2c/t)[1 - m\Pi(k_i)]$, while the probability that its degree decreases by more than one is $o(t)$ and will be ignored. Thus the probability that the degree of node i remains the same is $1 - f_t^+(k_i) - f_t^-(k_i)$. Hence, the state-transition probability of the Markov chain is given by

$$P\{k_i(t+1) = l | k_i(t) = k\} = \begin{cases} f_t^+(k), & l = k+1 \\ f_t^-(k), & l = k-1 \\ 1 - f_t^+(k) - f_t^-(k), & l = k \\ 0, & \text{otherwise} \end{cases}. \tag{6}$$

Let us introduce the probability, $p(k,i,t)$, that a node i has degree k at time t, namely, $p(k,i,t) = p\{k_i(t) = k\}$. Using the Markovian property, we obtain

$$p(0,i,t+1) = p(0,i,t)[1 - f_t^+(0)] + p(1,i,t)f_t^-(1) \tag{7}$$

$$\begin{aligned} p(k,i,t+1) = {} & p(k,i,t)[1 - f_t^+(k) - f_t^-(k)] \\ & + p(k+1,i,t)f_t^-(k+1) \\ & + p(k-1,i,t)f_t^+(k-1), \quad k \ge 1. \end{aligned} \tag{8}$$

The total degree distribution of the entire network $P(k,t) = \frac{1}{t}\sum_{i=1}^{t} p(k,i,t)$, and $P(k,t,t) = \delta_{k,m}$. Using this and applying $\sum_{i=1}^{t}$ to both sides of eqs (7) and (8), we get

$$P(0,t+1) = \frac{t}{t+1}P(0,t)[1-f_t^+(0)] + \frac{t}{t+1}P(1,t)f_t^-(1) \quad (9)$$

$$\begin{aligned} P(k,t+1) = &\frac{t}{t+1}P(k,t)[1-f_t^+(k)-f_t^-(k)] \\ &+\frac{t}{t+1}P(k+1,t)f_t^-(k+1) \\ &+\frac{t}{t+1}P(k-1,t)f_t^+(k-1)+\frac{1}{t+1}\delta_{k,m}, \quad k \geq 1. \end{aligned} \quad (10)$$

Under the assumption that the stationary degree distribution $P(k) \equiv P(k, t \to \infty)$ exists, the corresponding stationary equation is given by the following lemma.

Lemma 1

$$P(k) = \begin{cases} \dfrac{2c}{1+B}P(k+1), & k=0 \\ \dfrac{A(k-1)+B}{1+Ak+B+2c}P(k-1) + \dfrac{2c}{1+Ak+B+2c}P(k+1) & \\ \qquad +\dfrac{\delta_{k,m}}{1+Ak+B+2c}, & k \geq 1 \end{cases} \quad (11)$$

where

$$A = \frac{m(1-p)}{(1-p)(2m+1)+p-2c},$$

$$B = \frac{m}{(1-p)(2m+1)+p-2c}, \quad 0 \leq p < 1.$$

Proof. The difference equation (9) has the following recursive solution:

$$P(0,t) = \frac{1}{t}\prod_{i=1}^{t-1}[1-f_i^+(0)]\left\{P(0,1) + \prod_{l=1}^{t-1}\frac{lf_l^-(1)P(1,l)}{\prod_{j=1}^{l}[1-f_j^+(0)]}\right\}. \quad (12)$$

Let

$$x_t = P(0,1) + \prod_{l=1}^{t-1}\frac{lf_l^-(1)P(1,l)}{\prod_{j=1}^{l}[1-f_j^+(0)]}$$

and

$$y_t = t\prod_{i=1}^{t-1}[1-f_j^+(0)]^{-1}.$$

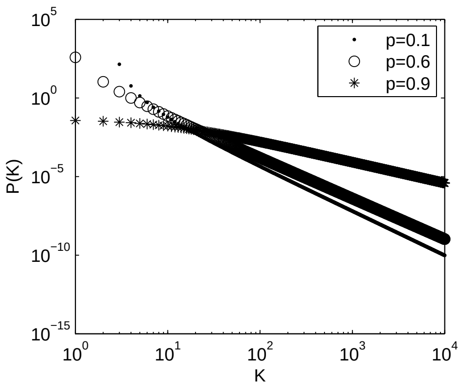

Figure 1. The degree distribution of the new model, with $m = 6, c = 1$ and $p = 0.1, 0.6$ and 0.9, respectively.

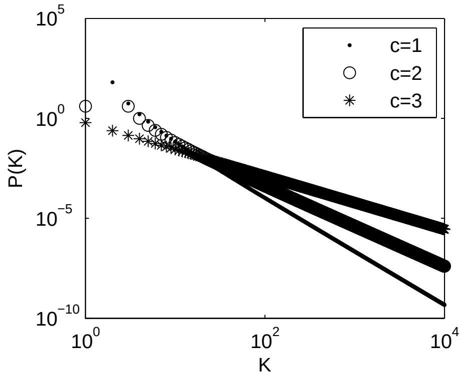

Figure 2. The degree distribution of the new model, with $m = 6, p = 0.5$ and $c = 1, 2$ and 3, respectively.

By the Stolz Theorem [14], we have

$$P(0) = \lim_{t\to\infty} \frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \lim_{t\to\infty} \frac{tf_t^-(1)P(1,t)}{1 + tf_t^+(0)}. \tag{13}$$

It is easy to check that $\lim_{t\to\infty} tf_t^+(0) = B$, $\lim_{t\to\infty} tf_t^-(1) = 2c$ and substituting this in (13), we get $P(0) = \frac{2c}{1+B}P(1)$.

Using similar arguments we have

$$P(k) = \frac{A(k-1)+B}{1+Ak+B+2c}P(k-1) + \frac{2c}{1+Ak+B+2c}P(k+1) + \frac{1}{1+Ak+B+2c}\delta_{k,m}, \quad k \geq 1.$$

Thus, the proof of Lemma 1 is completed.

By solving the above second-order difference eqs (11), one arrives at the following Lemma.

Lemma 2. Equation (11) has the following solution:

$$P(k) = C\int_0^1 z^{k+\frac{A}{B}-1}(1-z)^{\frac{1}{A}}\mathrm{e}^{-\frac{2c}{A}z}\mathrm{d}z, \tag{14}$$

where C is a constant to be determined by the normalization condition $\sum_{k=0}^{\infty} P(k) = 1$.

Theorem 1. The degree distribution $P(k)$ follows a power-law behaviour for large $k: P(k) \sim k^{-r}$, where the scaling exponent $r = 1 + \frac{1}{A} = 3 + \frac{1-2c}{m(1-p)}$.

Proof. From eq. (14), it follows that

$$\begin{aligned}\lim_{k\to\infty} P(k)\cdot k^{1+\frac{1}{A}} &= \lim_{k\to\infty} C\cdot k^{1+\frac{1}{A}} \int_0^1 z^{k+\frac{A}{B}-1}(1-z)^{\frac{1}{A}}\mathrm{e}^{-\frac{2c}{A}z}\mathrm{d}z\\ &= \lim_{k\to\infty} C\sum_{s=0}^{\infty}\frac{1}{s!}\left(-\frac{2c}{A}\right)^s k^{1+\frac{1}{A}}\int_0^1 z^{k+\frac{A}{B}+s-1}(1-z)^{\frac{1}{A}}\mathrm{d}z\\ &= C\sum_{s=0}^{\infty}\frac{1}{s!}\left(-\frac{2c}{A}\right)^s \lim_{k\to\infty} k^{1+\frac{1}{A}}\frac{\Gamma(k+\frac{A}{B}+s)\Gamma(1+\frac{1}{A})}{\Gamma(k+\frac{A}{B}+s+1+\frac{1}{A})}\\ &= C\,\Gamma(1+1/A)\mathrm{e}^{-\frac{2c}{A}},\end{aligned}$$

where $\Gamma(x)$ is the standard gamma function [15]. From here follows the expression for the degree distribution $P(k)\approx C\,\Gamma(1+1/A)\mathrm{e}^{-\frac{2c}{A}}k^{-(1+\frac{1}{A})}$ for large k.

Hence, for large k the degree distribution follows the asymptotic behaviour:

$$P(k)\sim k^{-\gamma},\quad \gamma = 3+\frac{1-2c}{m(1-p)}. \tag{15}$$

In order to check the theoretical prediction, we now turn to present numerical support for the scaling results obtained from eq. (15). Figure 1 depicts the degree distribution $P(k)$ obtained for different values of p with $m=6, c=1$, where the points, open circles and stars are for $p=0.1$ with $\gamma=2.81$, $p=0.6$ with $\gamma=2.59$, $p=0.9$ with $\gamma=1.33$, respectively. Figure 2 shows the degree distribution for different values of c with $m=6, p=0.5$, where the points, open circles and stars denote $c=1$ with $\gamma=2.67$, $c=2$ with $\gamma=2$, $c=3$ with $\gamma=1.33$, respectively. The results in the above figures agree well with the theoretical scaling results.

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