PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 73, No. 6
December 2009
pp. 1105–1110

Magnons interaction of spinor Bose–Einstein condensates in an optical lattice

YONG-QING LIU
Department of Physics and Electronics, XinZhou Teachers University, XinZhou, 034000, China
E-mail: yqliu_phy@yahoo.com.cn

MS received 24 January 2009; revised 19 April 2009; accepted 30 July 2009

Abstract. We study the interaction of magnons in dipolar spinor Bose–Einstein condensates in an optical lattice. By means of Holstein–Primakoff and Fourier transformations the energy spectra of the ground and the excited states is obtained analytically. Our results show that the collision of magnons is elastic which is expressed by the conservation of wave numbers in the process of collision. At last, we found that the interaction of magnons is attractive which tends to self-localization to form spin waves, i.e., a cluster of a macroscopic number of coherent magnons. Because of the attraction, the instability of spin wave brings about the existence of solitary wave.

Keywords. Magnons; spin wave; dipolar spinor Bose–Einstein condensates.

PACS Nos 03.75.Lm; 05.30.Jp; 75.30.Ds

Of late, the dipolar spinor Bose–Einstein condensates (BECs) trapped in optical potentials [1–3] have been studied extensively. It offers new opportunity to confirm the dynamics of periodic structure in solid-state physics. In classical solid-state physics, the site-to-site interaction is caused mainly by the exchange interaction resulted from the direct Coulomb interaction among electrons and the Pauli exclusion principle. As a result, there are the phenomena of ferromagnetism or antiferromagnetism at temperatures below the Curie temperature. For the dipolar spinor BECs trapped in optical potentials, the internal degrees for the hyperfine spin of the atoms is freedom, which brings forth a rich variety of phenomena such as spin domains [4,5] and textures [6]. When the potential valley is very deep, the condensates at each lattice site would act as microscopic magnets and interact with each other through the long-range and anisotropic dipole–dipole interaction. These site-to-site dipolar interactions can cause the ferromagnetic phase transition [7,8] leading to a 'macroscopic' magnetization of the condensate array, the spin-wave-like excitation [7–10] and magnetic soliton [11,12] analogous to the spin-wave and magnetic soliton in a ferromagnetic spin chain. Therefore, the spinor BECs in an optical lattice offers a totally new environment to study spin dynamics in periodic structures. In this paper, we will study the magnons interaction of spinor BECs in an optical lattice.

The dynamics of $F = 1$ spinor BECs trapped in an optical lattice is primarily governed by two-body interactions: the isotropic short-range collision interaction and anisotropic long-range magnetic dipole–dipole interaction. While taking into account the Zeeman energy, the Hamiltonian takes the following form [3,4]:

$$\begin{aligned} H = &\sum_{\alpha} \int \mathrm{d}\mathbf{r} \hat{\psi}_{\alpha}^{\dagger}(\mathbf{r}) \left[-\frac{\hbar^2 \nabla^2}{2m} + U_{\mathrm{L}}(\mathbf{r}) \right] \hat{\psi}_{\alpha}(\mathbf{r}) \\ &+ \sum_{\alpha,\beta,\upsilon,\tau} \int \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r}' \hat{\psi}_{\alpha}^{\dagger}(\mathbf{r}) \hat{\psi}_{\beta}^{\dagger}(\mathbf{r}') [U_{\alpha\upsilon\beta\tau}^{\mathrm{coll}}(\mathbf{r}, \mathbf{r}') \\ &+ U_{\alpha\upsilon\beta\tau}^{d-d}(\mathbf{r}, \mathbf{r}')] \hat{\psi}_{\tau}(\mathbf{r}') \hat{\psi}_{\upsilon}(\mathbf{r}) + H_B, \end{aligned} \quad (1)$$

where $\hat{\psi}_{\alpha}(r)$ is the field annihilation operator for an atom in the hyperfine state $|f = 1, m_f = \alpha\rangle$, $U_{\mathrm{L}}(\mathbf{r})$ is the lattice potential, the indices $\alpha, \beta, \upsilon, \tau$ which run through the values $-1, 0, 1$ denote the Zeeman sublevels of the ground state. The parameter $U_{\alpha\upsilon\beta\tau}^{\mathrm{coll}}(\mathbf{r}, \mathbf{r}')$ describes the two-body ground-state collisions and $U_{\alpha\upsilon\beta\tau}^{d-d}(\mathbf{r}, \mathbf{r}')$ includes the magnetic dipole–dipole interaction.

When the optical lattice potential is deep enough, it is convenient to expand the spinor atomic field operator as $\hat{\psi}(\mathbf{r}) = \sum_i \sum_{\alpha=0,\pm 1} \hat{a}_{\alpha}(i) \phi_i(\mathbf{r})$, where i labels the lattice sites, $\phi_i(\mathbf{r})$ is the condensate wave function for the ith microtrap and the operators $\hat{a}_{\alpha}(i)$ satisfy the bosonic commutation relations $[\hat{a}_{\alpha}(i), \hat{a}_{\beta}^{\dagger}(j)] = \delta_{\alpha\beta}\delta_{ij}$. If the condensates at each lattice site contain the same number of atoms, the ground-state wave functions for different sites have the same form $\phi_i(\mathbf{r}) = \phi_i(\mathbf{r} - \mathbf{r}_i)$. Under the tight-banding approximation and ignoring the unimportant constant terms, the effective spin-Hamiltonian for the dipolar $F = 1$ condensates [7] trapped in a one-dimensional optical lattice along the z-direction can be constructed as

$$H = \sum_i [\lambda_a' \hat{S}_i^2 + \sum_{j(\neq i)} (J_{ij} \hat{S}_i \cdot \hat{S}_j - 3 J_{ij} \hat{S}_i^z \hat{S}_j^z) - \gamma \hat{S}_i \cdot \mathbf{B}], \quad (2)$$

where J_{ij} represents the magnetic dipole–dipole interaction. The direction of the magnetic field $\mathbf{B}$ is along the one-dimensional optical lattice, i.e., z-axis direction, and $\gamma = g_F \mu_B$ is the gyromagnetic ratio. The spin operators are defined as $\hat{S}_n = \hat{a}_{\alpha}^{\dagger}(n) \mathbf{F}_{\alpha\upsilon} \hat{a}_{\upsilon}(n)$, where $\mathbf{F}$ is the vector operator for the hyperfine spin of an atom, with components represented by 3×3 matrices in the $|f = 1, m_f = \alpha\rangle$ subspace. The first term in eq. (2) is resulted from the spin-dependent interatomic collisions at a given site, with $\lambda_a' = (1/2)\lambda_a \int \mathrm{d}^3 r\, |\phi_n(\mathbf{r})|^4$, where λ_a characterizes the spin-dependent s-wave collisions. The second and the third terms describe the site-to-site spin coupling induced by the static magnetic field dipolar interaction.

If one temporarily ignored the site-to-site coupling parameter J_{ij}, it is obvious that the ground state of the Hamiltonian (2) is $|GS\rangle = |N, -N\rangle$, where the spin at site i has the expectation value $\langle \hat{S}_i^z \rangle = -N_i \hbar$ and $N = \sum_i N_i$ denotes the total atomic number in the lattice. However, due to the large number of bosons at each site, the magnetic dipole–dipole interaction in the optical lattice was enhanced. As a result, the transfer of the transverse spin excitation from site to site is allowed, resulting in the distortion of the ground state spin structure. This distortion can propagate and hence generate spin waves along the atomic spin chain. The Holstein–Primakoff transformation [13] is a useful tool to study the properties of spin waves. In terms of this transformation the spin operators $\hat{S}$ can be expressed as

$$\begin{aligned} \hat{S}^+ &= (\sqrt{2S - \hat{a}^+\hat{a}})\hat{a}, \\ \hat{S}^- &= \hat{a}^+(\sqrt{2S - \hat{a}^+\hat{a}}), \\ \hat{S}_z &= (S - \hat{a}^+\hat{a}), \end{aligned} \tag{3}$$

where $\hat{S}^\pm \equiv \hat{S}_x \pm i\hat{S}_y$, and the creation and annihilation operators $\hat{a}$ and $\hat{a}^+$ satisfy the boson commutator relation $[\hat{a}, \hat{a}^+] = 1$, $[\hat{a}, \hat{a}] = [\hat{a}^+, \hat{a}^+] = 0$. Substituting eq. (3) into the Hamiltonian (2) and holding the sixth-order term, after a tedious calculation we obtain

$$\begin{aligned} H = {} & N\lambda'_a S(S+1) - \gamma NSB_z + \gamma B_z \sum_l \hat{a}_l^+\hat{a}_l \\ & + 2SJ \sum_{\langle lm \rangle} \hat{a}_l\hat{a}_m^+ - \frac{J}{2} \sum_{\langle lm \rangle} (\hat{a}_m^+\hat{a}_m^+\hat{a}_m\hat{a}_l + \hat{a}_l^\dagger\hat{a}_l\hat{a}_l\hat{a}_m^\dagger) \\ & - 2J \sum_{\langle lm \rangle} (S^2 - 2S\hat{a}_l^+\hat{a}_l + \hat{a}_l^\dagger\hat{a}_l\hat{a}_m^\dagger\hat{a}_m) \\ & - \frac{J}{16S} \sum_{\langle lm \rangle} (\hat{a}_m^+\hat{a}_m^+\hat{a}_m^+\hat{a}_m\hat{a}_m\hat{a}_l + \hat{a}_l^+\hat{a}_l^+\hat{a}_m^+\hat{a}_l\hat{a}_l\hat{a}_l \\ & - 2\hat{a}_m^+\hat{a}_m^+\hat{a}_l^+\hat{a}_m\hat{a}_l\hat{a}_l), \end{aligned} \tag{4}$$

where we assume that all the nearest-neighbour interactions are same, namely $J_{ij} = J$, which is a good approximation in the one-dimensional optical lattice for the large lattice constant [14]. The Hamiltonian in eq. (4) can be analysed in detail by the Fourier transformation which expresses the site space operator $\hat{a}_l$ in terms of the wave-vector-space operator $\hat{a}_{\mathbf{k}}$ as

$$\hat{a}_l = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{l}}\hat{a}_{\mathbf{k}}, \quad \hat{a}_l^+ = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{l}}\hat{a}_{\mathbf{k}}^+, \tag{5}$$

where the operator $\hat{a}_{\mathbf{k}}$ satisfies the commutator relation $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^+] = \delta_{\mathbf{k}\mathbf{k}'}$. Substituting eq. (5) into eq. (4) we get

$$H = E_0 + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\hat{a}_{\mathbf{k}}^+\hat{a}_{\mathbf{k}} - H_1 - H_2, \tag{6}$$

where

$$\begin{aligned} E_0 &= -\left(JN\chi - N\lambda'_a\right) S\left(S+1\right) - \gamma NSB_z, \\ \varepsilon_{\mathbf{k}} &= \gamma B_z + SJ\chi\gamma_{\mathbf{k}} + 4SJ, \end{aligned}$$

$$H_1 = \frac{J\chi}{4N} \sum_{\mathbf{kk'K}} \gamma_1 \hat{a}^+_{\mathbf{k}-\mathbf{K}} \hat{a}^+_{\mathbf{k'}+\mathbf{K}} \hat{a}_{\mathbf{k'}} \hat{a}_{\mathbf{k}},$$

$$H_2 = \frac{J\chi}{32SN^2} \sum_{\mathbf{kk'k''KK'}} \gamma_2 \hat{a}^+_{\mathbf{k}+\mathbf{K}} \hat{a}^+_{\mathbf{k'}+\mathbf{K'}} \hat{a}^+_{\mathbf{k''}-\mathbf{K}-\mathbf{K'}} \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k'}} \hat{a}_{\mathbf{k''}},$$

with the parameters

$$\gamma_{\mathbf{k}} = \frac{1}{\chi} \sum_{\boldsymbol{\rho}} \mathrm{e}^{+i\mathbf{k}\cdot\boldsymbol{\rho}},$$

$$\gamma_1 = \gamma_{\mathbf{k}} + \gamma_{\mathbf{k'}+\mathbf{K}} + 4\gamma_{\mathbf{k}-\mathbf{k'}-\mathbf{K}},$$

$$\gamma_2 = \gamma_{\mathbf{k''}} + \gamma_{\mathbf{k''}-\mathbf{K}-\mathbf{K'}} - 2\gamma_{\mathbf{k'}+\mathbf{K}+\mathbf{K'}},$$

where χ is the number of nearest neighbour, and $\boldsymbol{\rho} = \mathbf{l} - \mathbf{m}$ is the coordinate vector of the nearest neighbour.

Now we obtain the energy spectrum in eq. (6) where the first term E_0 presents the ground state and the second term denotes the excited state, i.e., spin waves with energy $\varepsilon_{\mathbf{k}}$ for each magnon. The third term H_1 presents obviously the interaction of two magnons with the wave numbers $\mathbf{k}$ and $\mathbf{k'}$ before collision, and the wave numbers $\mathbf{k} - \mathbf{K}$ and $\mathbf{k'} + \mathbf{K}$ after collision, respectively. The fourth term H_2 expresses the interaction of three magnons with the wave numbers $\mathbf{k}$, $\mathbf{k'}$ and $\mathbf{k''}$ before collision, and the wave numbers $\mathbf{k} + \mathbf{K}$, $\mathbf{k'}+\mathbf{K'}$ and $\mathbf{k''}-\mathbf{K''}-\mathbf{K''}$ after collision, respectively. These results show that the collision of magnons is elastic which is expressed by the conservation of wave numbers in the process of collision.

The interaction of magnons can also be discussed in another way as follows. In the presence of one magnon with wave vector $\mathbf{k}$, the energy of the system is given by

$$\begin{aligned} E_1 &= -\left(JN\chi - N\lambda'_a\right) S^2 + \varepsilon_{\mathbf{k}} \\ &\equiv -\left(JN\chi - N\lambda'_a\right) S^2 \left(1 - \frac{F_{\mathbf{k}}}{S}\right), \end{aligned} \tag{7}$$

where $F_{\mathbf{k}} = \varepsilon_{\mathbf{k}}/AN\chi S$, and we have ignored the external magnetic field for convenience. Equation (7) shows in fact that the interaction of magnons is zero since a single magnon has no interaction. But then the expression in eq. (7) can be seen as the energy of the system in the absence of a magnon while the spin S is reduced to an effective spin $S_1^* \equiv S\left(1 - F_{\mathbf{k}}/\left(2S\right)\right)$. Considering that $\varepsilon_{\mathbf{k}} \ll \left(JN\chi - N\lambda'_a\right) S^2$, we can present the energy in eq. (7) as

$$E_1 = -\left(JN\chi - N\lambda'_a\right) S_1^{*2}.$$

Repeating similar procedures as above we get the energy of the system where n or $n+1$ magnons are added

$$E_n = -(JN\chi - N\lambda'_a)(S_n^*)^2,$$

$$E_{n+1} = -(JN\chi - N\lambda'_a)(S_{n+1}^*)^2,$$

where $S_n^* = S(1 - \frac{F_n}{2S}), F_n = \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}} n_{\mathbf{k}}}{(JN\chi - N\lambda_a')S} = \sum_{\mathbf{k}} n_{\mathbf{k}} F_{\mathbf{k}}$. Therefore, the addition of energy for the system resulting from the nth magnon is obtained as

$$\hbar\omega_k^n \equiv E_{n+1} - E_n = \varepsilon_{\mathbf{k}} \left(1 - \frac{\sum_{\mathbf{k}'} \varepsilon_{\mathbf{k}'} n_{\mathbf{k}'}}{2\,|E_0|}\right),$$

which is smaller than the energy of one magnon $\varepsilon_{\mathbf{k}}$ which shows that the interaction of magnons is attractive. Owing to this attraction, a cluster of magnons tends to self-localization. In a certain sense, the attraction of magnons is critical for a one-dimensional atom spin chain for spinor BECs in an optical lattice because it produces a bound state of quasi-particles (magnons), i.e. self-localization. A spin wave may be regarded as a cluster of a macroscopic number of coherent magnons. Because of the attraction, the magnon cluster tends to be localized, and thus the spin wave becomes unstable. The developing instability causes magnetization localization and brings about a solitary wave [12]. These phenomena are analogous to the ferromagnetism in solid-state physics, but occur with bosons instead of fermions. For fermions, the site-to-site interaction is caused mainly by the exchange interaction; the dipole–dipole interaction is small and can be neglected. For the spinor BECs in the optical lattice, the exchange interaction is absent and the individual spin magnets are coupled by the magnetic dipole–dipole interaction.

References

[1] J Stenger, S Inouye, D M Stamper-Kurn, H J Miesner, A P Chikkatur and W Ketterle, *Nature (London)* **396**, 345 (1998)

[2] B P Anderson and M A Kasevich, *Science* **282**, 1686 (1998)
D M Stamper-Kurn, M R Andrews, A P Chikkatur, S Inouye, H J Miesner, J Stenger and W Ketterle, *Phys. Rev. Lett.* **80**, 2027 (1998)

[3] T L Ho, *Phys. Rev. Lett.* **81**, 742 (1998)
C K Law, T Ohmi and K Machida, *J. Phys. Soc. Jpn* **67**, 1822 (1998)
C K Law, H Pu and N P Bigelow, *Phys. Rev. Lett.* **81**, 5257 (1998)

[4] H J Miesner, D M Stamper-Kurn, J Stenger, S Inouye, A P Chikkatur and W Ketterle, *Phys. Rev. Lett.* **82**, 2228 (1999)
Jun'ichi Ieda, Takahiko Miyakawa and Miki Wadati, *Phys. Rev. Lett.* **93**, 194102 (2004)

[5] N Gemelke, E Sarajlic, Y Bidel, S Hong and S Chu, *Phys. Rev. Lett.* **95**, 170404 (2005)
Wenxian Zhang, D L Zhou, M S Chang, M S Chapman and L You, *Phys. Rev. Lett.* **95**, 180403 (2005)

[6] T Ohmi and K Machida, *J. Phys. Soc. Jpn* **67**, 1822 (1999)

[7] H Pu, W P Zhang and P Meystre, *Phys. Rev. Lett.* **87**, 140405 (2001)

[8] K Gross, C P Search, H Pu, W P Zhang and P Meystre, *Phys. Rev.* **A66**, 033603 (2002)

[9] W P Zhang, H Pu, C Search and P Meystre, *Phys. Rev. Lett.* **88**, 060401 (2002)
K Góral, L Santos and M Lewenstein, *Phys. Rev. Lett.* **88**, 170406 (2002)

[10] M Rigol, V Rousseau, R T Scalettar and R R P Singh, *Phys. Rev. Lett.* **95**, 110402 (2005)
J N Fuchs, D M Gangardt, T Keilmann and G V Shlyapnikov, *Phys. Rev. Lett.* **95**, 150402 (2005)

[11] Z W Xie, W Zhang, S T Chui and W M Liu, *Phys. Rev.* **A69**, 053609 (2004)
Z W Xie, Z X Cao, E I Kats and W M Liu, *Phys. Rev.* **A71**, 025601 (2005)
[12] Z D Li, Qiu-Yan Li, Xing-Hua Hu, Zhong-Xi Zheng and Yubao Sun, *Ann. Phys. (N.Y.)* **322**, 2545 (2007)
Z D Li and Qiu-Yan Li, *Ann. Phys. (N.Y.)* **322**, 1961 (2007)
[13] T Holstein and H Primakoff, *Phys. Rev.* **58**, 1098 (1940)
[14] V V Konotop, M Salerno and S Takeno, *Phys. Rev.* **E56**, 7240 (1997)