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Global chaos synchronization of coupled parametrically excited pendula

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Abstract. In this paper, we study the synchronization behaviour of two linearly coupled parametrically excited chaotic pendula. The stability of the synchronized state is examined using Lyapunov stability theory and linear matrix inequality (LMI); and some sufficient criteria for global asymptotic synchronization are derived from which an estimated critical coupling is determined. Numerical solutions are presented to verify the theoretical analysis. We also examined the transition to stable synchronous state and show that this corresponds to a boundary crisis of the chaotic attractor.

Keywords. Chaos; synchronization; boundary crisis; parametrically excited pendula; Lyapunov theory; linear matrix inequality.

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1. Introduction

At the beginning of the last decade, one of the discoveries that changed the viewpoints of the field of nonlinear dynamics and chaos is the fact that two or more chaotic systems evolving from different initial conditions can be synchronized. This was achieved by Pecora and Carroll [1]. Synchronization can be understood as a state in which two or more systems (with dynamics that can either be periodic or chaotic) adjust to each other giving rise to a common dynamical behaviour. This common behaviour can be induced either by coupling the systems (locally or globally) or by forcing them [2]. In view of its practical applications, the phenomenon of synchronization has been widely investigated theoretically [3,4], numerically (see ref. [5] and the references therein) and experimentally [6] for many sytems; thus many types of synchronization have been proposed. For a detailed study of different concepts of synchronization, the reader is referred to the fundamental book by Pikovsky *et al* [7]

Coupled dynamical systems are typically synthesized from simpler, low-dimensional systems to form new and more complex organizations. This is often done with the intent of realistically modelling spatially extended systems, with the belief that dominant features of the constituent systems will be retained. From an application point of view this building block approach can also be used to create a novel system whose behaviour is more complex than that of its constituents. These and several other motivations have led to intensive studies of coupled systems in a wide range of disciplines including optical systems [8,9], condensed matter [10,11], biological systems [12–14], neural networks [15–17] and physical systems [18,19] to mention only a few.

The phenomena of synchronized dynamics in coupled or driven nonlinear oscillators are of fundamental importance and possess wide practical applications ranging from secure communications to the monitoring of the dynamical systems and control [20,21]. The basic idea in secure communication is to mask the information-bearing signal to be transmitted with a chaotic signal that exhibits broadband features. This represents an alternative to mere classical noise-masking methods, wherein one uses a purely stochastic signal to mask the information to be transmitted.

Among the many interesting phenomena associated with synchronization in coupled or driven nonlinear oscillators that have been extensively studied are, intermittency [6,22], boundary crisis [23,24], interior crisis [25,26] and basin bifurcations leading to multistability [27]. For instance, in ref. [23], we examined unidirectionally coupled double-well Duffing oscillators (DDOs) and showed that synchronization was characterized by boundary crisis of the chaotic attractors. In our previous work [23,25], only numerical results were presented. In this paper, we extend our results to parametrically excited systems and in particular obtain sufficient criteria for global synchronization using Lyapunov stability theory and linear matrix inequality (LMI); from which we also show that an estimate critical coupling for complete synchronization to occur could be obtained. Oscillators with parametric modulations are common in real experimental practice and they have been widely used in electromechanical and electronic systems for communication purposes. For this reason, investigation of the synchronization behaviour of parametrically excited oscillators is relevant for a variety of applications. Till now, only a few works have been devoted to the study of synchronization of parametrically modulated systems [28–31].

The rest of this paper is organized as follows: In the following section, we give a brief description of the model and the master–slave synchronization scheme for non-autonomous parametrically excited pendulum. In §3, some sufficient criteria for global chaos synchronization are provided based on Lyapunov's direct method and LMI while §4 is devoted to the numerical simulations and in §5 we present the dynamical mechanism leading to complete synchronization in our model. Finally, a concluding remark is given in §6.

2. The model and synchronization scheme

Let us consider the following parametrically excited chaotic pendulum [32,33]:

$$\ddot{x} + b\dot{x} + (1 + \eta \cos \omega t) \sin x = 0, \tag{1}$$

where x is the displacement from the stable equilibrium position, overdots represent differentiation with respect to time t, b stands for the damping coefficient, ω and η represent the angular frequency and the amplitude of the parametric excitation of the system respectively.

A comparative study of the dynamics of this oscillator has been presented by Szemplinska and Tyrkiel [32]. They showed that tumbling chaotic motion consisting of an irregular combination of rotations and oscillations in this class of oscillators is preceded by two co-existing periodic attractors, which are simultaneously annihilated prior to period-doubling cascade scenario. Pendulum with parametric modulation are of practical importance in view of their applications in electrochemical and electronic systems for communication purposes.

When two such systems (1) interact with each other via a specific coupling scheme, the dynamics could be very rich and exciting. One such dynamical behaviour is the synchronization. In order to design the synchronization scheme, the oscillator (1) can be re-written in the autonomous form using the transformations $x_1 = x, x_2 = \dot{x}$ as follows:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -bx_2 - (1 + \eta \cos \omega t) \sin x. \tag{2}$$

In a compact form, we express eq. (2) as

$$\dot{\mathbf{X}} = \mathbf{ZX} + \alpha \mathbf{f}(\mathbf{x}), \tag{3}$$

where

$$\mathbf{X} = (x_1, x_2)^T \in \mathbf{R}^2, \quad \alpha(t) = 1 + \eta \cos \omega t,$$
$$\mathbf{Z} = \begin{pmatrix} 0 & 1 \\ 0 & -b \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} 0 \\ -\sin x \end{pmatrix}.$$

Next we construct drive-response synchronization scheme for two identical parametrically excited pendula by linear state error feedback controller in the following form:

$$\dot{\mathbf{X}} = \mathbf{ZX} + \alpha \mathbf{f}(\mathbf{x}), \tag{4}$$
$$\dot{\mathbf{Y}} = \mathbf{ZY} + \alpha \mathbf{f}(\mathbf{y}) + \mu, \tag{5}$$

where $\mathbf{Y} = (y_1, y_2)^T \in \mathbf{R}^2$ is the state variables of the response system, $\mu = \mathbf{C}(\mathbf{X} - \mathbf{Y})$ is the linear state feedback control input and $\mathbf{C} \in \mathbf{R}^{2\times 2}$ is a constant control matrix that determines the strength of the feedback into the response system.

In order to ascertain the error dynamics, we define the synchronization error as the difference between the relevant dynamical variable and it is given by

$$\mathbf{e} = \mathbf{X} - \mathbf{Y}. \tag{6}$$

By subtracting eq. (5) from eq. (4) and applying the definition of error system in eq. (6), one readily obtains

$$\dot{\mathbf{e}} = (\mathbf{Z} - \mathbf{C} + \mathbf{A}(x_1, y_1))\mathbf{e}, \tag{7}$$

where

$$\mathbf{A}(x_1, y_1) = \alpha \begin{pmatrix} 0 & 0 \\ g(x_1, y_1) & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

with

$$g(x_1, y_1) = -\frac{(\sin x_1 - \sin y_1)}{x_1 - y_1}. \tag{8}$$

In the absence of control matrix, $\mathbf{C}$, eq. (7) would have an equilibrium at $(0, 0)$. Obviously, $\mathbf{e} = 0$ is an equilibrium point of the error system (7).

For the error dynamics system (7), synchronization in a direct sense implies that the trajectories $x(t)$ and $y(t)$ of the drive and response systems, for any choice of the initial conditions $x(0)$ and $y(0)$ satisfy

$$\lim_{t\to\infty} ||\mathbf{e}|| = \lim_{t\to\infty} ||\mathbf{x}(t) - \mathbf{y}(t)|| = 0, \tag{9}$$

where $|| * ||$ represents the Euclidean norm of a vector.

3. Sufficient criteria for global synchronization

In this section, we shall employ the Lyapunov stability theory and linear matrix inequality (LMI) to obtain the main theorem of this paper and establish some criteria for global chaos synchronization in the sense of the error system (7). The Lyapunov stability theory employs Lyapunov functionals which could be used for the analysis and synthesis of synchronization dynamics; and has been employed by earlier authors (see for example refs [34,35]). However, attention has been paid mostly on the stability of the synchronization. Here, we also show that the method could be used to estimate the onset of synchronization. The basic idea is that a given system is stable, if there exist a continuous positive definite differentiable function (the Lyapunov function, V) defined along the system's trajectory, such that its time derivative $\dot{V} \leq 0$, as $t \to \infty$ [34–36]. To begin with, we shall apply the following lemma to prove the main theorem of this paper.

Lemma 1. For $g(x_1, y_1)$ defined by (8), the inequality

$$|g(x_1, y_1)| \leq 1 \tag{10}$$

holds.

Proof. By the differential mean-value theorem we have

$$\sin x_1 - \sin y_1 = (x_1 - y_1)\cos\phi, \quad \phi \in (x_1, y_1) \quad \text{or} \quad \phi \in (y_1, x_1). \tag{11}$$

So,

$$g(x_1, y_1) = -\frac{(\sin x_1 - \sin y_1)}{x_1 - y_1} = -\cos\phi \tag{12}$$

and hence the inequality (10) holds.

We proceed by utilizing the stability theory on time-varied systems to derive sufficient criteria for global chaos synchronization in the sense of the error system (7). The following theorem is related to the general control matrix:

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathbf{R}^{2\times 2}. \tag{13}$$

Theorem 1. *If there exists a symmetric positive definite matrix* $\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$ *and a coupling matrix* $\mathbf{C} \in \mathbf{R}^{2\times 2}$ *defined in* (13) *such that for any* $t > 0$

$$\Omega_1 = -p_{11}c_{11} - p_{12}c_{21} + |p_{12}|(1 + |\eta|) < 0, \tag{14}$$

$$\Omega_2 = p_{12}(1 - c_{12}) - p_{22}(c_{22} + b) < 0, \tag{15}$$

$$4\Omega_1\Omega_2 > [|p_{11}(1 - c_{12}) - p_{12}(c_{11} + c_{22} + b) - p_{22}c_{21}| + p_{22}(1 + |\eta|)]^2, \tag{16}$$

then the drive-response systems (4) *and* (5) *achieve global chaos synchronization.*

Proof. Let us assume a quadratic Lyapunov function of the form:

$$V(e) = \mathbf{e}^T\mathbf{P}\mathbf{e}, \tag{17}$$

where $\mathbf{P}$ is a positive definite symmetric matrix as defined earlier. The derivative of the Lyapunov function with respect to time, t, along the trajectory of the error system (7) is of the form

$$\dot{V}(e) = \dot{\mathbf{e}}^T\mathbf{P}\mathbf{e} + \mathbf{e}^T\mathbf{P}\dot{\mathbf{e}}. \tag{18}$$

Substituting eq. (7) into the system (18), we have

$$\dot{V}(\mathbf{e}) = \mathbf{e}^T\left[(\mathbf{Z}(t) + \mathbf{A}(t) - \mathbf{C})^T\mathbf{P} + \mathbf{P}(\mathbf{Z}(t) + \mathbf{A}(t) - \mathbf{C})\right]\mathbf{e} \tag{19}$$

$\dot{V}(e) < 0$ if

$$\gamma = \left[(\mathbf{Z}(t) + \mathbf{A}(t) - \mathbf{C})^T\mathbf{P} + \mathbf{P}(\mathbf{Z}(t) + \mathbf{A}(t) - \mathbf{C})\right] < 0 \ \forall t \geq 0. \tag{20}$$

According to Lyapunov stability theory [36], the inequality in (20) represents a sufficient condition for global asymptotic stability of the linear time-varied error system (7) at the equilibrium point. With $\mathbf{A}(t), \mathbf{C}, \mathbf{P}, \mathbf{Z}(t)$ as defined earlier, eq.

(20) thus becomes

$$\gamma = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix}, \tag{21}$$

where $\lambda_{11} = -2p_{11}c_{11} + 2p_{12}(\alpha g - c_{21})$, $\lambda_{12} = p_{11}(1 - c_{12}) - p_{12}(c_{11} + c_{22} + b) + p_{22}(\alpha g - c_{21})$ and $\lambda_{22} = 2p_{12}(1 - c_{12}) - 2p_{22}(c_{22} + b)$. Since γ is symmetric, γ is negative if and only if

$$-2p_{11}c_{11} + 2p_{12}(\alpha g - c_{21}) < 0 \tag{22}$$

$$2p_{12}(1 - c_{12}) - 2p_{22}(c_{22} + b) < 0 \tag{23}$$

$$\begin{aligned} &4[(\alpha g - c_{21})p_{12} - c_{11}p_{11}][(1 - c_{12})p_{12} - p_{22}(c_{22} + b)] \\ &\quad - [p_{11}(1 - c_{12}) - p_{12}(c_{11} + c_{22} + b) + p_{22}(\alpha g - c_{21})]^2 > 0. \end{aligned} \tag{24}$$

The trajectories of the parametrically excited pendulum describable by the model in eq. (1) is bounded. Thus for any $t \geq 0$

$$|\alpha(t)| = |1 + \eta \cos \omega t| \leq 1 + |\eta|. \tag{25}$$

Since the matrix $\mathbf{P}$ is positive definite, we have $p_{22} > 0$. It follows that

$$\begin{aligned} &-2p_{11}c_{11} + 2p_{12}(\alpha g - c_{21}) \leq -2p_{11}c_{11} - 2p_{12}c_{21} + |2p_{12}(\alpha g)| \leq 2\Omega_1, \\ &|p_{11}(1 - c_{12}) - p_{12}(c_{11} + c_{22} + b) + p_{22}(\alpha g - c_{21})| \leq |p_{11}(1 - c_{12}) \\ &\quad -p_{12}(c_{11} + c_{22} + b) - p_{22}c_{21}| + p_{22}(1 + |\eta|)). \end{aligned} \tag{26}$$

The inequalities in (22)–(24) hold if the inequalities in (14)–(16) are satisfied. Hence, it is evident that with the appropriate choice of the control matrix, $\mathbf{C}$, the drive-response systems (4) and (5) achieve global chaos synchronization. Thus, $\dot{V}$ vanishes identically only at the origin. Lyapunov theorem implies that $\mathbf{e} \to 0$ as $t \to \infty$.

COROLLARY 1

If a control matrix $\mathbf{C} = \mathrm{diag}\{c_1, c_2\}$ *and a symmetric positive definite matrix* $\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} > 0$ *are selected such that*

$$c_1 > \frac{|p_{12}|(1 + |\eta|)}{p_{11}} \tag{27}$$

$$c_2 > \frac{p_{12} - bp_{22}}{p_{22}} \tag{28}$$

$$\begin{aligned} &4[(1 + |\eta|)|p_{12}| - c_1p_{11}][p_{12} - p_{22}(c_2 + b)] \\ &\quad - [|p_{11} - p_{12}(c_1 + c_2 + b)| + p_{22}(1 + |\eta|)]^2 > 0 \end{aligned} \tag{29}$$

then the drive-response systems (4) *and* (5) *achieve global chaos synchronization.*

Proof. The inequalities (27)–(29) can be obtained from the inequalities (14)–(16) with $c_{11} = c_1, c_{22} = c_2$ and $c_{12} = c_{21} = 0$.

COROLLARY 2

If a control matrix $\mathbf{C} = \mathrm{diag}\{c, c\}$ *and a symmetric positive definite matrix* $\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} > 0$ *are selected such that*

$$c > \max\left(\frac{|p_{12}|(1+|\eta|)}{p_{11}}, \frac{p_{12} - bp_{22}}{p_{22}}\right) \tag{30}$$

$$\begin{aligned} &4(p_{11}p_{22} - p_{12}^2)c^2 - 4c\,[2p_{22}|p_{12}|(1+|\eta|) + p_{11}(p_{12} - bp_{22}) \\ &\quad + |p_{12}(p_{11} - bp_{12})|] + 4|p_{12}|(p_{12} - bp_{22})(1+|\eta|) \\ &\quad - [|p_{11} - bp_{12}| + p_{22}(1+|\eta|)]^2 > 0, \end{aligned} \tag{31}$$

then the drive-response systems (4) *and* (5) *achieve global chaos synchronization.*

Proof. Letting $c_{11} = c, c_{22} = c$ and $c_{12} = c_{21} = 0$, the inequalities (30) and (31) can be obtained according to the partial synchronization criteria (27)–(29).

Remark. If we take $p_{12} = 0$ and $p_{11} = p_{22}(1+|\eta|) > 0$, then the following synchronization criterion can be obtained by means of the inequalities (30) and (31)

$$\mathbf{C} = \mathrm{diag}\{c, c\}, \quad c > \frac{\sqrt{b^2 + 4(1+|\eta|)} - b}{2}. \tag{32}$$

4. Numerical simulation

Here, we employ numerical approach to validate the above theoretical analysis. The results to be presented in this section were computed with the following parameter settings: $\eta = 1.1$, $b = 0.1$ and $\omega = 1.65$. We set the initial conditions for the drive system (4) as $x_1(0) = 0.1, x_2(0) = 0.2$ and the initial conditions for the response system (6) as $y_1(0) = 0.4, y_2(0) = 0.8$. One could readily observe that the uncoupled parametrically excited pendulum exhibits persistent tumbling chaos as depicted by the chaotic attractor shown in figure 1; which corresponds to what was reported in refs [32,33].

Numerical solutions were obtained using fourth-order Runge–Kutta routine in double precision as well as the software *Dynamics* [37]. Firstly, in figure 2, we show the time evolution of the average error, $e_{\mathrm{ave}} = \sqrt{e_1^2 + e_2^2}$ for the case $\mathbf{C} = \mathrm{diag}\{0, 0\}$ which corresponds to the uncoupled system. Note that the error dynamics reveal irregular bursts that are comparable to the size of the chaotic attractor.

Next, we examine the variation of e_{av} as the coupling strength is progressively increased. Figure 3 shows that as the coupling strength c increases, and full synchronization is approached, $E_{av} \to 0$ asymptotically. It is clear from figure 3 that full synchrony is achieved at the critical $c = c_{\mathrm{cr}} \approx 1.38$. Then for all $c > c_{\mathrm{cr}}$, $E_{\mathrm{ave}} \approx 0$ and remains stable as $t \to \infty$, implying that the oscillators are completely

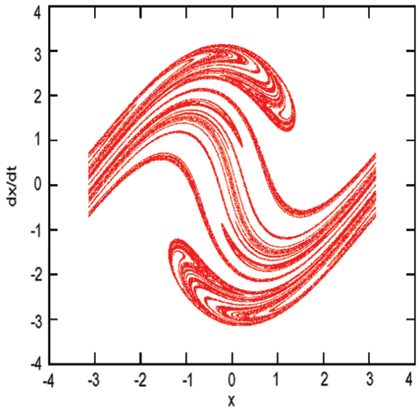

Figure 1. Tumbling chaotic attractor in the x–y ($y = \dot{x}$) plane in the parametrically excited pendulum with the following parameters: $b = 0.1$, $\eta = 1.1$, $\omega = 1.65$.

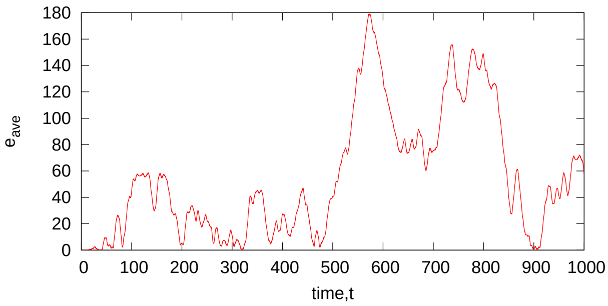

Figure 2. Time dependence of error dynamics e_{ave}, for $b = 0.1$, $\eta = 1.1$ and $\omega = 1.65$, for the uncoupled (i.e. $\mathbf{C} = \text{diag}\{0, 0\}$) parametrically excited pendulum.

synchronized. Remarkably, the numerical value of $c = c_{\text{cr}}$ is comparable to what was obtained using the synchronization criterion in eq. (32). By direct calculation of eq. (32), we find that for the control matrix $C = \text{diag}\{c, c\}$, $c > c_{\text{cr}} = 1.4$. Thus, our prediction is in good agreement with the numerical results.

Finally, we depict the simulation results for the second case in which we choose the constant control matrix, $\mathbf{C} = \text{diag}\{c, c\}$, such that $c = 1.45 > c_{\text{cr}}$, thus satisfying the condition (32). The simulation results shown in figure 4 confirm that complete synchronization is achieved for $c = 1.45 > c_{\text{cr}}$.

5. Transition to synchronization

The structural changes associated with the transition to stable synchronous behaviour is examined in this section. To illustrate this, we consider the strange attractor exhibited by the drive parametrically excited pendulum as shown in figure 1. When

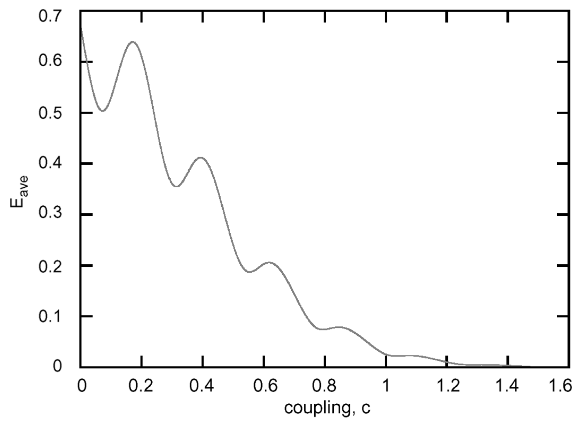

Figure 3. Average error dynamics $e_{\rm ave}$ as a function of the coupling strength, c. Here the parameters of the system are as in figure 1.

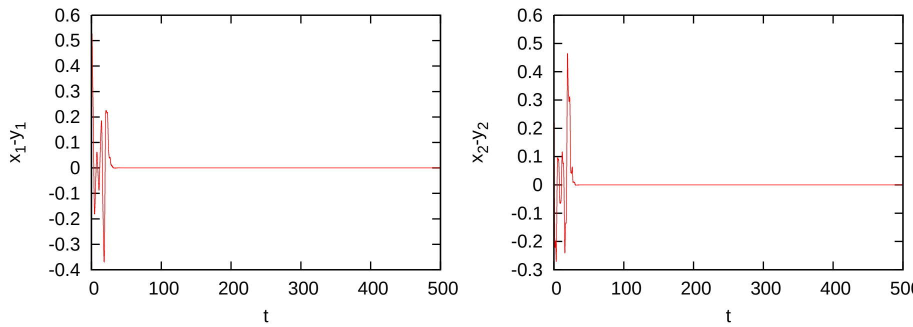

Figure 4. Chaos synchronization of two nonautonomous parametrically excited pendula by the controller $\mathbf{C} = {\rm diag}\{1.45, 1.45\}$.

the coupled oscillators become synchronized, the attractor for the response system would be precisely superimposed, point-to-point with that of the drive attractor. In the desynchronous regime, the Poincaré points move on the attractor in an uncorrelated manner.

To understand the structural changes that took place in the system, we obtain the Poincaré section of the phase portraits within and somewhere outside the synchronization region for the response system. By discarding the first 2000 iterates as a means of accounting for the initial transients, we display in figure 5 the Poincaré sections for the response system (5) for two different values of the coupling strength prior to $c_{\rm cr}$. The Poincaré sections were obtained by confining the dynamics in $x(t)$ between $-\pi$ and π using the software *Dynamics* [36]. This technique was employed to sample the phase portraits of the trajectory $x = x(t), \dot{x} = \dot{x}(t)$ in modulo$(2\pi/\omega)$. With this approach we reduce the three-dimensional state space $(x, \dot{x}, t)$ of each oscillator to two-dimensional phase plane $(x, \dot{x})$ with discrete time: $x \equiv x(nT), \dot{x} \equiv \dot{x}(nT), n = 0, 1, 2, \ldots$.

For $c = 0$, the tumbling chaotic attractor of the response system is idenitical to that of the drive depicted in figure 1. When the coupling is switched on, the

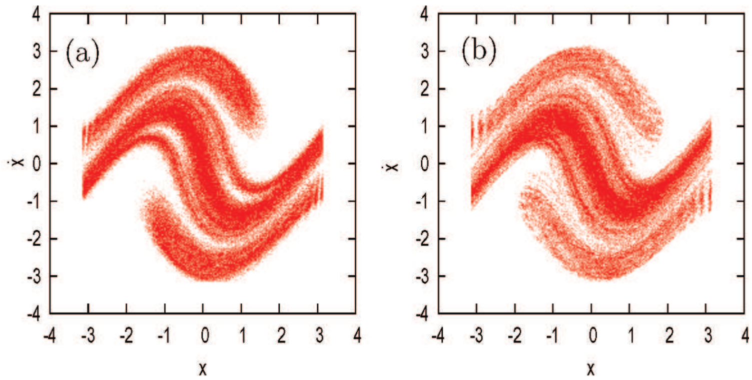

Figure 5. Poincaré sections for the response system of the parametrically excited pendulum for different values of the coupling strength. The velocity, $\dot{x}$, is plotted against the displacement, x. Other parameters of the system are as in figure 1: (**a**) $c = 0.15$; (**b**) $c = 0.28$.

Poincaré points move on the attractor in an uncorrelated fashion, gradually occupying the entire phase space. This is shown in figure 5a (for $c = 0.15$). With $c = 0.28$, prior to c_{cr}, the tumbling chaotic attractor is weakly registered initially, and then superposed by uncorrelated Poincaré points due to desynchronous points. We find that in the synchronization process, (i) the boundaries of the chaotic attractor is destroyed and (ii) the distance between it and the basin boundary approaches zero, so that the Poincaré points spread on the entire phase space. This is a more complex dynamics; and the phenomenon called boundary or exterior crisis of the chaotic attractor has been reported earlier as synchronization transition [23,31]. The concept of boundary or exterior crisis is characterized by the collision of the attractor by unstable periodic orbit on its basin boundary, or more equivalently, its stable manifold [38].

As the synchronous regime is however approached, the attractor of the response system is gradually re-built and in the full synchronous state, we find that the attractor of the response system is then registered point-for-point with the strange attractor of the drive shown in figure 1, and thus, the two strange attractors behave as one entity – indicating identical synchronization.

6. Conclusions

Conclusively, we have examined the synchronization behaviour of a drive-response system consisting of two coupled pendula with parametric excitation and established sufficient criteria for global and asymptotic chaos synchronization. By examining the stability of the synchronized state based on Lyapunov theory and linear matrix inequality (LMI), we obtained an estimate critical coupling for synchronization to occur in the drive-response parametrically excited pendula. The criteria obtained are in algebraic form and could be easily employed for designing the feedback control gains that would guarantee full synchronization. We present numerical simulations to verify these results. Prior to the obtained critical coupling, we also identify the boundary crisis event in which the main body of the chaotic attractor is destroyed and the Poincaré points filling the phase space.

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