PRAMANA — journal of physics

Vol. 73, No. 6
December 2009
pp. 989–997

Random matrix theory for pseudo-Hermitian systems: Cyclic blocks

SUDHIR R JAIN* and SHASHI C L SRIVASTAVA
Nuclear Physics Division, Van de Graaff Building, Bhabha Atomic Research Centre, Trombay, Mumbai 400 085, India
*Corresponding author
E-mail: srjain@barc.gov.in (Sudhir R Jain); shashics@barc.gov.in (S C L Srivastava)

MS received 18 April 2009; revised 2 July 2009; accepted 30 July 2009

Abstract. We discuss the relevance of random matrix theory for pseudo-Hermitian systems, and, for Hamiltonians that break parity P and time-reversal invariance T. In an attempt to understand the random Ising model, we present the treatment of cyclic asymmetric matrices with blocks and show that the nearest-neighbour spacing distributions have the same form as obtained for the matrices with scalar entries. We also summarize the theory for random cyclic matrices with scalar entries. We have also found that for block matrices made of Hermitian and pseudo-Hermitian sub-blocks of the form appearing in Ising model depart from the known results for scalar entries. However, there is still similarity in trends even in log–log plots.

Keywords. Random matrices; circulants; quantum chaos; $\mathcal{PT}$ symmetry; pseudo-Hermiticity.

PACS Nos 05.30.-d; 05.45.Mt; 11.30.Er; 02.30.Hq

1. Introduction

Random matrix theory (RMT) was introduced by Wigner in 1954 to show that the local fluctuation properties of complex quantum systems have universal properties, independent of the details of the interaction among the constituents [1]. One considers a (say, Hamiltonian) matrix with elements that follow a distribution which results on imposing statistical independence among matrix elements, and, the symmetry of the underlying system. For instance, if a system is time-reversal invariant and rotationally invariant, the Hamiltonian matrix is real-symmetric and the system belongs to the Gaussian orthogonal ensemble of random matrices. If the system does not respect time-reversal invariance, then the Hamiltonian matrix can have complex elements – such a system will belong to the Gaussian unitary ensemble of random matrices. For a system with odd spin, time-reversal invariant and no rotational symmetry, the matrix will be a self-dual Hermitian matrix belonging to the Gaussian symplectic ensemble. The class of systems treated in the usual RMT preserve parity, P and time-reversal invariance, T. We discuss here those systems where both P and T are violated, and they are termed $\mathcal{PT}$-symmetric. Some years ago, we had introduced [2,3] the pseudo-unitary symmetry and developed RMT for pseudo-Hermitian operators which go hand-in-hand with $\mathcal{PT}$-symmetric systems. In this section, we present several important areas of physics where the study being carried out by us is of significance.

1.1 *Quantum chaos*

One of the first examples in the studies on 'quantum chaology and RMT' where $\mathcal{PT}$ symmetry appeared was the time-independent Schrödinger equation for a point particle in a rectangular box with an Aharonov–Bohm flux line [4]. The two-dimensional Lagrangian is given by

$$L(x,y,\dot{x},\dot{y}) = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \alpha\frac{x\dot{y}-y\dot{x}}{x^2+y^2}, \tag{1}$$

where α represents the coupling of the particle to the flux line. This system is pseudo-integrable [5,6] as the vector fields become singular at the origin. Hamiltonian for this system may be written as

$$H = \frac{1}{2}(\vec{p}+\vec{A})^2 = \frac{1}{2}(p_x^2+p_y^2) + \alpha\frac{yp_x - xp_y}{r^2} + \frac{\alpha^2}{2r^2}, \tag{2}$$

where $A_i = \alpha\epsilon^{ij}r_j/r^2$, $\epsilon_{12} = -\epsilon_{21} = 1$, and $r^2 = x^2+y^2$. As pointed out in [4], §III, paragraph 3, H is not invariant under either time reversal or parity but is invariant under a combined operation of both. Numerical studies showed that the nearest-neighbour level spacing distribution starts from an exponential form (Poisson) for $\alpha = 0$ and develops linear level repulsion as α becomes 0.5. To explain this behaviour, one needs to develop RMT for physical systems that are $\mathcal{PT}$-symmetric.

1.2 *Anyons*

The particles which assume $1/n$ fractional statistics are described by the real-time Lagrangian,

$$\mathcal{L} = \Psi^\dagger\left(i\partial_t - a_t - \frac{e}{c}A_t\right)\Psi - \frac{1}{2m}\Psi^\dagger\left(i\partial - \mathbf{a} + \frac{e}{c}\mathbf{A}\right)^2\Psi$$
$$+\frac{n}{4\pi}\epsilon^{\eta\nu\rho}(\partial_\mu a_\nu)a_\rho. \tag{3}$$

In this expression, Ψ is the fermionic field and a_μ is the Chern–Simons statistical gauge field; A_μ is the electromagnetic field. This system also breaks P and T and shows P, T-breaking properties of anyons [7,8].

In an attempt to obtain the momentum distribution of anyon gas [9,10], RMT for P, T-breaking systems become relevant.

1.3 *Random Ising model*

In the two-dimensional Ising model defined on a regular lattice, on each site a spin variable σ_i takes values $+1$ or -1. The interaction with the neighbouring spins σ_i and σ_j is $J_{ij}\sigma_i\sigma_j$ and between any other spins is zero. If $J_{ij} = J$ is fixed, we have the Ising model, whose partition function was found by Onsager [11,12]. However, as is well-known (p. 7 of [1]), if J_{ij} is a random variable, with a symmetric distribution around zero mean, we have the random Ising model. The calculation of the partition function is an open problem.

If we follow the beautiful treatment of Onsager and Kaufman, we arrive at the transfer matrix with elements containing the coupling. For the random Ising model, we will get a random matrix which is a cyclic asymmetric matrix. Each entry is a block. We have successfully developed the RMT for cyclic matrices with scalar entries recently [13]. The panorama of various results that we have found are discussed in §2. We have also observed that these matrices are pseudo-orthogonal matrices.

In §3, we present new results for block matrices just as they are needed to understand random Ising model. These are the first steps in this direction, not the final solution. But they are pointing to important universality class to which two-dimensional Ising model might belong.

2. Random cyclic matrices

Consider a cyclic, asymmetric matrix with real elements, $\{a_i\}$:

$$\mathsf{M} = \begin{bmatrix} a_1 & a_2 & \ldots & a_N \\ a_N & a_1 & \ldots & a_{N-1} \\ \vdots & & & \\ a_2 & a_3 & \ldots & a_1 \end{bmatrix}. \tag{4}$$

It is important to note that this matrix is pseudo-orthogonal with respect to $\boldsymbol{\eta}$:

$$\boldsymbol{\eta} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ \vdots & & & & & \\ 0 & 1 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{5}$$

that is,

$$\mathsf{M}^\dagger = \mathsf{M}^T = \boldsymbol{\eta}\mathsf{M}\boldsymbol{\eta}^{-1}. \tag{6}$$

Since $\boldsymbol{\eta}^2 =$ identity, I, $\boldsymbol{\eta}$ is introduced here as 'generalized parity'.

Owing to the commutativity of cyclic matrices, the eigenvalues of M are given as a linear combination of a's [14],

$$E_l = \sum_{p=1}^{N} a_p \exp\left[\frac{2\pi i}{N}(p-1)(l-1)\right], \tag{7}$$

($l = 1, 2, ..., N$), the maximum real eigenvalue being $\sum_i a_i$. The diagonalizing matrix is given by

$$U_{jl} = \frac{1}{\sqrt{N}} \exp\left[\frac{2\pi i}{N}(j-1)(l-1)\right]. \tag{8}$$

We consider a Gaussian ensemble of cyclic matrices with a distribution,

$$P(\mathbf{M}) \sim \exp[-A\ \mathrm{tr}(\mathbf{M}^\dagger \mathbf{M})], \tag{9}$$

where A sets the scale (of energy, for instance).

For the general case of $N \times N$ matrices, we need to invert (7), leading to the following relation:

$$a_i = \sum_l \mathbf{S}_{il} E_l, \tag{10}$$

where $\mathbf{S}_{il} = \omega^{(i-1)(N-(l-1))}$ and $\omega = \mathrm{e}^{2\pi i/N}$ is a root of unity. $\mathbf{S}$ is a symmetric matrix and $\mathbf{S}^2 = N\boldsymbol{\eta}$. Employing these relations, we can find $\sum_i a_i^2$, and hence the following result for the joint probability distribution function (JPDF) for even N:

$$P(\{E_i\}) = \left(\frac{A}{\pi}\right)^{N/2} \exp\left[-A\left(E_1^2 + E_{\frac{N}{2}+1}^2 + \sum_{i \neq 1, \frac{N}{2}+1}^{N} E_i E_{N+2-i}\right)\right], \tag{11}$$

where E_1 and $E_{\frac{N}{2}+1}$ are real and the rest of the eigenvalues may be complex. For odd N, the above result will hold except that there will be only one real eigenvalue, E_1, and the summation in the second term will extend over all i except 1. Note that (10) gives a unitary transformation and thus ensues the argument of the exponent in (11) as a sum of absolute squares of E's. Employing this general result on JPDF, we can now calculate the spacing distributions for the general case.

In general, there are three cases: (i) spacing among the complex conjugate pair of eigenvalues is found to be distributed again as a Gaussian:

$$p_{cc}(z) = \frac{2}{\pi}\mathrm{e}^{-z^2/\pi}. \tag{12}$$

where $z = S/\bar{S}$, and $\bar{S}$ is the average spacing among the complex conjugate pair.

(ii) Spacing between a real and a complex eigenvalue is distributed according to

$$p_{rc}(z) = \frac{3\sqrt{3}\pi}{16} c^2 z \exp\left(-\frac{3\pi}{16}c^2 z^2\right) I_0\left(\frac{3\pi}{32}c^2 z^2\right). \tag{13}$$

(iii) Two complex eigenvalues, $E_j = x_j + iy_j$ and $E_k = x_k + iy_k$ are spaced according to

$$p(s) = \frac{\int \prod_i \mathrm{d}\Re E_i \mathrm{d}\Im E_i P(\{E_i\}) \delta(|E_j - E_k| - s)}{\int \prod_i \mathrm{d}\Re E_i \mathrm{d}\Im E_i P(\{E_i\})}, \tag{14}$$

which reduces to the following integral on change of variables, $\xi(\eta)_\pm = x(y)_k \pm x(y)_j$

$$\begin{aligned} p(s) &= \frac{A}{\pi} \int \mathrm{d}\xi_- \mathrm{d}\eta_- \mathrm{e}^{-A(\xi_-^2 + \eta_-^2)} \delta(\sqrt{\xi_-^2 + \eta_-^2} - s) \\ &= \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right) \end{aligned} \tag{15}$$

which is exactly the Wigner distribution. Let us recall that Wigner's result holds exactly for 2×2 real symmetric matrices. It serves as an excellent approximation for $N \times N$ matrices though. We also know that the spacing distribution for a Poissonian random process in a plane is exactly the same as Wigner surmise. Thus our result proves that the complex eigenvalues of random cyclic matrices describe such a process. This is a very beautiful, non-intuitive result which brings out yet another characteristic of RCM.

We can now make the following observations: (i) the Gaussianity of $p_{cc}(z)$ implies that there is no level repulsion among the complex–conjugate pairs, at the same time there is no attraction, there is no tendency of clustering as in Poissonian spacing distribution; (ii) real and complex eigenvalues display linear level repulsion. These results are also borne out by the numerical simulations [13].

The eigenfunctions of $\mathbf{M}$ corresponding to the real eigenvalues (E_1 and $E_{\frac{N}{2}+1}$) are also simultaneously eigenfunctions of 'generalized parity' $\boldsymbol{\eta}$. However, the eigenfunctions of $\mathbf{M}$ corresponding to the complex conjugate pair of eigenvalues are not simultaneously eigenfunctions of $\boldsymbol{\eta}$. Thus, when these complex eigenvalues occur, 'generalized parity' is said to be spontaneously broken. Also, the eigenfunctions corresponding to the complex–conjugate pair of eigenvalues have zero $\mathcal{PT}$-norm. This is expected from the recent works [15,16] on $\mathcal{PT}$-symmetric quantum mechanics. This observation then fully embeds our findings into the new random matrix theory developed recently for pseudo-Hermitian Hamiltonians. However, we also note that the eigenvectors ψ_1 (ψ_2) corresponding to complex conjugate eigenvalues, λ (λ^*), satisfy orthogonality defined with respect to $\boldsymbol{\eta}$. Since these results are found for $N \times N$ matrices, we believe that this work extends the random matrix theory in a significant way. The findings on the spacing distributions have led us to a linear level repulsion among distinct complex eigenvalues, whereas the spacing between complex–conjugate pair is Gaussian-distributed.

3. Cyclic blocks

In this section, we present a special case of random matrices with block entries. Firstly, let us consider

$$\mathbf{B} = \begin{bmatrix} A_1 & A_2 & \dots & A_N \\ A_N & A_1 & \dots & A_{N-1} \\ \vdots & & & \\ A_2 & A_3 & \dots & A_1 \end{bmatrix}, \tag{16}$$

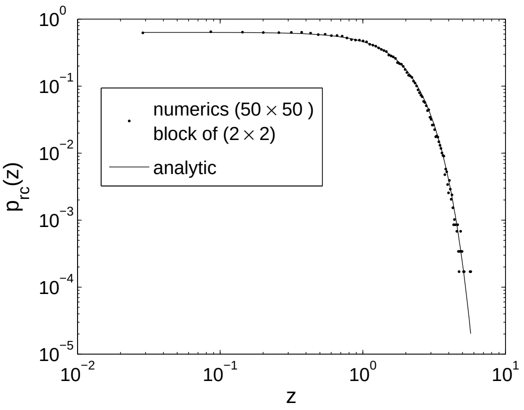

Figure 1. Log–log plot of the distribution of spacing among complex–conjugate pairs. The result is exactly in agreement with (12) for scalar entries (§2).

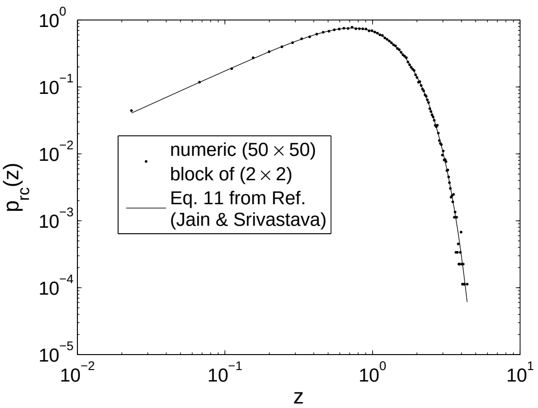

Figure 2. Log–log plot of the distribution of spacing among real and complex eigenvalues. The result is exactly in agreement with (13) for scalar entries (§2).

where each entry A_i is

$$A_i = \begin{bmatrix} a_i & -b_i \\ c_i & a_i \end{bmatrix}. \tag{17}$$

The realization of a random matrix is constructed by drawing real elements a_i, b_i, and c_i independently from a Gaussian distribution with zero mean and unit variance [2]. In this way, we will obtain a rather simple random matrix ensemble with real elements. However, seen as scalar entries, the resulting matrix is not a cyclic one. The matrix $\mathbf{B}$ is pseudo-orthogonal with respect to the 'generalized parity',

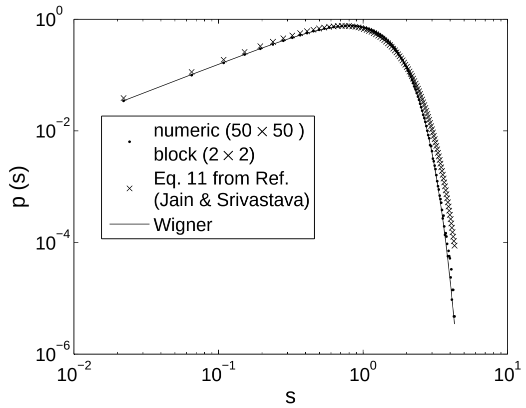

Figure 3. Log–log plot of the distribution of spacing among complex eigenvalues. The result is exactly in agreement with (15) for scalar entries (eq. (11) from [13]).

$$\Sigma = \begin{bmatrix} \sigma & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \sigma \\ \vdots & & & \\ 0 & \sigma & \ldots & 0 \end{bmatrix}, \tag{18}$$

where σ happens to be Pauli matrix,

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{19}$$

for this example. Consequently, $\mathbf{B}^{\dagger} = \Sigma \mathbf{B} \Sigma^{-1}$.

However, as we shall see below, the spectral fluctuations are just like the one for the scalar cyclic matrices. For instance, numerical investigations for 50×50 matrix, comprised of fifty 2×2 blocks per row reveal that the spacing distributions among complex–conjugate pairs, real–complex pair, complex–complex pair follow figures 1, 2 and 3 respectively.

These results are not at all obvious, considering the fact that the resulting matrix with scalar entries is not a cyclic matrix. This example encourages us to explore a possible universal class within the pseudo-orthogonal ensemble of random matrices.

Though analytical JPDF for cyclic blocks is still unknown to us, we have performed numerical calculation of spacing distribution for cyclic blocks of (2×2) matrices having complex entries. In particular, we have constructed cyclic matrices with the row $(A \;\; B \;\; B \;\; \ldots B^{\dagger})$ where

$$A = \begin{bmatrix} a_1 & ia_2 \\ -ia_2 & a_1 \end{bmatrix}, \tag{20}$$

$$B = \begin{bmatrix} -1/2 & ib_1 \\ ib_2 & -1/2 \end{bmatrix}. \tag{21}$$

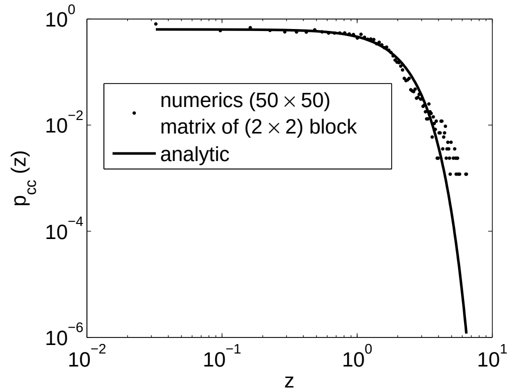

Figure 4. Log–log plot of the distribution of spacing among complex-conjugate pairs.

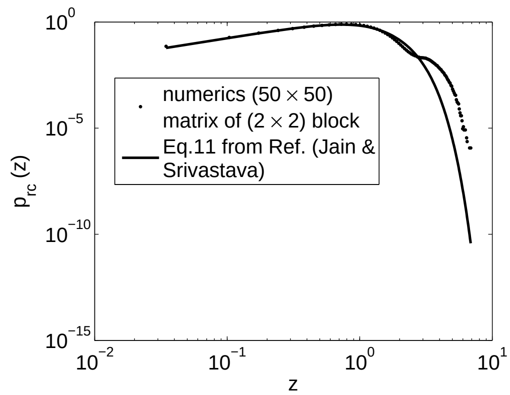

Figure 5. Log–log plot of the distribution of spacing among real and complex eigenvalues.

A and B are inspired from Ising model in two dimensions. The spacing distributions for different cases for random matrices thus constructed yield the following results shown in figures 4, 5 and 6 respectively. The complex-conjugate pairs are spaced as shown in figure 4 which is in reasonable agreement with (12).

The spacing distribution between real and complex eigenvalue shows a departure in the tail (figure 5). However we are yet to investigate the dependence of this deviation on the size of the matrix. Similar deviation of numerical result from Wigner distribution is seen in figure 6.

We notice deviations with the results obtained for scalar entries and for real blocks. It is known that cyclic matrices with blocks possess eigenvalues with same properties as those found for scalar entries. Although this is found to be the case when the entries of blocks were real, it is not true any more for complex entries. Nevertheless, as seen in the log–log plots, the results for two blocks – Hermitian and non-Hermitian – are not too far from the analytical results of §2. This points

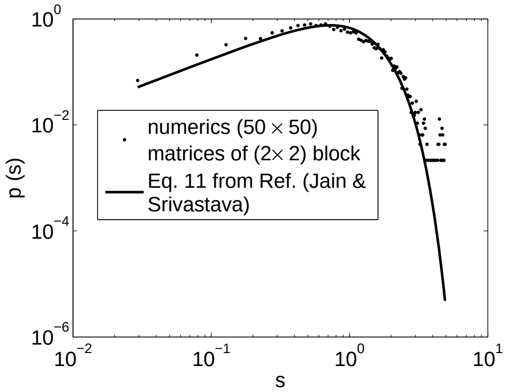

Figure 6. Log–log plot of the distribution of spacing among complex eigenvalues.

to the possibility of this system belonging to a universality class in the scaling limit, particularly because the behaviour near zero spacing is identical to that with scalar entries.

Having inspired by the random Ising model, clearly we have not solved the problem of calculating the partition function which is the same as JPDF. But we have taken the first step towards this by calculating the nearest-neighbour spacing distribution – a quantity that is connected to all the correlation functions. It would be very interesting if these results inspire analytical results.

References

[1] M L Mehta, *Random matrices* (Academic Press, London, 1991)
[2] Z Ahmed and S R Jain, *Phys. Rev.* **E67**, 045106(R) (2003)
[3] Z Ahmed and S R Jain, *J. Phys.* **A36**, 3349 (2003)
[4] G Date, S R Jain and M V N Murthy, *Phys. Rev.* **E51**, 198 (1995)
[5] A N Zemlyakov and A B Katok, *Math. Notes* **18**, 760 (1976)
P J Richens and M V Berry, *Physica* **D2**, 495 (1981)
[6] S R Jain and S V Lawande, *Proc. Indian Natl. Sci. Acad.* **A61**, 275 (1995)
[7] A Lerda, *Anyons*, Lecture notes in Physics, m14 (Springer, Berlin, 1992)
[8] Y Kitazawa, *Phys. Rev. Lett.* **65**, 1275 (1990)
[9] D Alonso and S R Jain, *Phys. Lett.* **B387**, 812 (1996)
[10] S R Jain and D Alonso, *J. Phys.* **A30**, 4993 (1997)
[11] L Onsager, *Phys. Rev.* **65**, 117 (1944)
[12] B Kaufman, *Phys. Rev.* **76**, 1232 (1949)
[13] S R Jain and Shashi C L Srivastava, *Phys. Rev.* **E78**, 036213 (2008)
[14] G Kowaleski, *Determinantentheorie* (Chelsea, New York, 1948) 3rd edn
[15] Z Ahmed and S R Jain, *Mod. Phys. Lett.* **21**, 331 (2006)
[16] C M Bender, D C Brody and H F Jones, *Phys. Rev. Lett.* **89**, 270401 (2002)