PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 73, No. 6
December 2009
pp. 961–968

Note on the natural system of units

LUIS J BOYA[1,*], CRISTIAN RIVERA[1] and E C G SUDARSHAN[2]
[1]Departamento de Física Teórica, Universidad de Zaragoza, E-50009 Zaragoza, Spain
[2]Physics Department, University of Texas, Austin 78712, USA
[*]Corresponding author
E-mail: luisjo@unizar.es (Louis J Boya); cristian_elfisico@hotmail.com (Cristian Rivera)

MS received 25 February 2009; revised 9 June 2009; accepted 22 June 2009

Abstract. We propose to substitute Newton's constant $G_{\rm N}$ for another constant G_2, as if the gravitational force would fall off with the $1/r$ law, instead of the $1/r^2$; so we describe a system of natural units with G_2, c and $\hbar$. We adjust the value of G_2 so that the fundamental length $L = L_{\rm Pl}$ is still the Planck's length and so $G_{\rm N} = L \times G_2$. We argue for this system as (1) it would express longitude, time and mass without square roots; (2) G_2 is in principle disentangled from gravitation, as in $(2+1)$ dimensions there is no field outside the sources. So G_2 would be truly universal; (3) modern physics is not necessarily tied up to $(3+1)$-dim. scenarios and (4) extended objects with $p = 2$ (membranes) play an important role both in M-theory and in F-theory, which distinguishes three $(2,1)$ dimensions.

As an alternative we consider also the clash between gravitation and quantum theory; the suggestion is that non-commutative geometry $[x_i, x_j] = \Lambda^2\theta_{ij}$ would cure some infinities and improve black hole evaporation. Then the new length Λ shall determine, among other things, the gravitational constant $G_{\rm N}$.

Keywords. Fundamental constants; gravitation.

PACS Nos 01.10.-m; 01.30.Rr; 01.90.+g

1. Introduction

Conventional physics uses length, time and mass as fundamental magnitudes, from which all others like e.g. acceleration, electric charge etc. can be derived. The old CGS system, which seems to stem from Gauss, has been substituted by SI (Système Internationale), with the three fundamental magnitudes as L (meter), K (kilogram) and T (second) (MKS is the acronym) and four derived ones. A typical derived magnitude is temperature: with Planck's equation (not due to Boltzmann!) $E = kT$, temperature becomes energy, and the conversion factor k is really a convention, given the traditional units of energy and temperature.

It is a natural desire of the scientists to find primary constants of magnitudes selected by natural phenomena, so that they would serve as units, and then go to express L, M and T and therefore all other physical magnitudes in terms of the primary ones. In this Note we take stand with the conventional choice of natural units, making a single change for $G_{\rm N}$ (Newton's constant) so as to avoid the square roots in expressing length, mass and time. We analyse the new proposal in the light of some evidence in current physics; our program is not completed, though, in the sense that both extra dimensions and extended objects, which we invoke, are at the moment speculative at best. However, we feel that the weight of our arguments is strong enough and so attention should be paid to the proposal.

At the end of the paper we contemplate briefly another idea: to consider the (new) fundamental length as coming from non-commutativity of space-time, in the sense of A Connes.

2. The natural system of units of Planck

The natural system of units was first established by Max Planck in 1899; he introduced the constant h (for *Hilfsmittel*) to express Wien's radiation law; h has dimension of action, $[h] = ML^2T^{-1}$. This he did before he found, in October 1900, his (correct) radiation law [1]; he admitted at once the universal character of h, as the black body radiation is supposed to be a universal phenomenon. Then h with Newton's gravitational constant $G \equiv G_{\rm N}$ and the velocity of light c (which already played a decisive role in Maxwell's equations, although we were six years before special relativity), the three then make up a system of natural units; Planck was exultant about this natural system and even went so far as to say that extraterrestrials would be using the same system, an outstanding statement for a conservative person like him [1].

It is important to remark that even today the status of the three constants, chronologically $G_{\rm N}$, c and h, is rather different. c is the most universally admitted, and in modern parlance it is justified because there is a maximum velocity in any physical phenomenon, and geometrically speaking because in the manifold for space-time there is signature, with space-like dimensions (apparently 3) and time-like ones (apparently one); if one wishes, one can think of c as changing the scale from space-like to time-like directions (however, there are recent discussions on possible violations of Lorentz invariance; we do not enter in it). This holds also for higher dimensions, and even includes the possibility of several times, like in the $(2, 2)$ membrane of Hewson and Perry [2] and in the F-theory $(2, 10)$ of Vafa [3]. As space-like and time-like directions are physically distinguishable, the existence of c is not merely a 'convention'.

On the other hand, the meaning of h today is rather elaborated since its inception by Planck 110 years ago. It has dimensions of action, as said, and also of angular momentum. Action in classical mechanics is already a distinguished magnitude, function of the path, $S = S[\gamma]$ where γ is a possible path for the system. However, classically the scale of the action is irrelevant, as the extremals are not sensible to an overall scale. In the modern conception of quantum mechanics, as epitomized by Feynman's path integrals (1942), there is a phase (or angle) for the action of a system, and so the contribution of path γ to the amplitude is just $\exp({\rm i}S[\gamma]/\hbar)$.

In physical terms h measures the quantum aspects of a system. In a geometrical sense it weights the contribution of any path to the full interference amplitude, with the associated uncertainty interpretation: in a way Planck's constant and Feynman's path integral are the quantitative expressions of Heisenberg's uncertainty principle. This undoubtedly endows the constant h (or rather in Dirac's form $\hbar$) with a universal character, as long as we believe quantization is universal. However, the fact that even today we are still at odds with quantizing gravity, puts the constant $\hbar$ in a lesser universal character than, say, c, and one should keep an open mind whether quantum mechanics may be reformulated in the future, as some people think (e.g. 't Hooft): whereas c fixes the kinematics, $\hbar$ refers to dynamics, although not a particular one, so it seems to us less universal. As $\hbar$ has also the dimension of an angular momentum, in our mundane $(1, 3)$ space there is a crucial distinction between integer (bosons) and half-integer (fermions) angular momentum, with the attendant coherent states for the first, and the exclusion (Pauli) principle for the latter. However, the spin–statistics connection does not follow in arbitrary dimension [4], so that the consequence of the quantum of action is tied up to special dimensions only. In supersymmetry, much theoretized in physics since around 1976, the Bose–Fermi symmetry is best realized in eight space dimensions, as the vector and the two spinor representations of the spin(8) group are isomorphic. It is possible that remnants of this (super-)symmetry 'flows down' to the actual $(1 + 3)$ space, and in fact there is a minimal extension ($\mathcal{N} = 1$) of the standard model, the MSSM, which would be eagerly tested soon at the LHC machine.

As a last comment on quantization, the recent work of Gukov and Witten [5] makes it clear that the conventional quantization procedure is too linear, that is, too much tied up to quadratic Lagrangians. There is no universal, intrinsic quantization procedure applicable to any symplectic manifold, the arena of classical mechanics, supposed to be quantizable. All these considerations warrant a less universal character of h as compared to c.

The 'universal' character of the gravitational constant is even much weaker, as many authors have already signalled, e.g. [6]. For one thing, G_{N} is tied up to a particular interaction, gravitation, however its universality. For another, it depends on the space dimension being three (through Gauss' law: arbitrarily far away spheres collect the same flux, hence the force should decay as the sphere surfaces increases). But considerations of extra dimensions are the current parlance in physics since 1980.

The difference between G_{N} and the others c, h, makes it possible to consider the system in which velocities respectively actions are measured in multiples of c respectively $\hbar$: this is the 'natural system' used e.g. in high energy physics: all magnitudes can then be expressed in terms of length L_1 or mass M_1. For example, the gravitation constant $[G_{\mathrm{N}}] = L^2$, and the fine structure constant is dimensionless, $\alpha = e^2/\hbar c \approx 1/137$; $[\alpha] = 1$. The (primitive) weak interaction theory included Fermi's constant, with $[G_{\mathrm{F}}] = L^2$. Heisenberg was the first, in 1938 [7], to remark that there was an essential difference in the dimensions of the coupling constants: those like α or the Yukawa coupling which were dimensionless belong to theories well-behaved at high momenta, whereas gravitation and the four-fermion

Fermi theory behaved much worse: in modern parlance the first were renormalizable, the second were not.

The three constants $G_{\rm N}, c$ and $\hbar$, make it possible to draw the so-called Okun cube [8], namely according to the limits in which $G_{\rm N}, 1/c$ and $\hbar$ go to zero:

- $(0,0,0)$: Classical mechanics in the most primitive form
- $(G_{\rm N},0,0)$: Newton theory of gravitation
- $(0,c,0)$: Classical relativistic mechanics
- $(0,0,\hbar)$: Quantum mechanics of finite systems
- $(G_{\rm N},c,0)$: General relativity
- $(0,c,\hbar)$: Quantum field theory
- $(G_{\rm N},0,\hbar)$: Quantum effects of static gravitation (e.g. black hole evaporation)
- $(G_{\rm N},c,\hbar)$: Theory of everything (TOE) (for the future!)

To be sure, black body evaporation is static only in reference to gravitation: pair creation of course is a quantum-field theory process. It is amusing to compare the above partition in eight cases with the division algebra of octonions, in particular with the Fano plane [9]: there three independent imaginary units, i, j, k combine to give the eight units of octonions: $1; i, j, k; ij, jk, kj; (ij)k$, or $8 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$.

The dimensions of the magnitudes of the new units in terms of the conventional L, M and T are

$$[G_{\rm N}] = L^3 M^{-1} T^{-2}, \quad [c] = LT^{-1}, \quad [h] = ML^2 T^{-1} \tag{1}$$

with the approximate values in the MKS system

$$G_{\rm N} \approx 6.67 \times 10^{11}, \quad c \approx 3 \cdot 10^8, \quad \hbar \approx 10^{-34}. \tag{2}$$

The error margins in these values reflect the different character already: for c we have an exact value, by definition, so the second is defined in terms of the meter, itself defined by spectroscopic standards; for h the error bar is in the 10^{-8} range, whereas $G_{\rm N}$ is known only up to 10^{-4}.

So the program of expressing all physical magnitudes in terms of $G_{\rm N}$, c and $\hbar$ starts by inverting (1), defining fundamental length, time and mass as

$$\begin{aligned} &L_0 = \sqrt{\hbar G_{\rm N}/c^3} \approx 1.6 \times 10^{-35}\ \text{m}, \quad T_0 = \sqrt{\hbar G_{\rm N}/c^5} \approx 5.4 \times 10^{-44}\ \text{s}, \\ &M_0 = \sqrt{\hbar c/G_{\rm N}} \approx 2.2 \times 10^{-8}\ \text{kg} \end{aligned} \tag{3}$$

which are called *Planck's units* for the respective magnitudes. Notice the peculiarity for the mass: the fundamental length and time are, in a sense, the smallest conceivable ones, whereas the Planck mass is of the order of a bacterium! This is of course a consequence of $G_{\rm N}$ appearing in the case of the mass in the denominator, together with the intrinsic weakness of gravitation. Indeed in the microscopic scale this mass unit is not small but very big! ($\approx 10^{19}$ GeV in modern high-energy units). As an example, the smallest massive particles, the neutrinos, are believed to have masses in the 10^{-2} eV range, a factor of 10^{-30}!

3. Changing G_N

Is it unavoidable to accept Planck's choice? This has been generally assumed, with some resilient die-hards (e.g. Veneziano *et al* [6]; they discussed how one can, in string theory for example, argue for only two fundamental constants); many people are aware, of course, of the different status of the three universal constants; an interesting discussion of this is in ref. [6].

It is objectively clear that the status of G_N is rather different from that of c and $\hbar$; the latter are more universal, independent of forces and space dimensions; there is also an aesthetic reason to change G_N (that was the original motivation for this Note): in eq. (3), the fundamental anthropocentric magnitudes L, M and T are under the square root, what clearly hinges on the 'square' in Newton's gravitational force equation (or, more explicitly, on the above remarked fact that $[G_N] = L^2$, in the high-energy units '$\hbar = c = 1$').

So the program is open now: we aim to find another system of natural units, changing at least G_N for something else, more universal if possible and trying at the same time to get rid of the nasty square roots in (3). That should be the more conservative program; of course, one should keep an open eye for the future, in particular for the status of quantum mechanics hundred years from now!

We endeavour to find arguments for substituting (G_N/r^2) in Newton's gravitation formula for something like (G_2/r), for example, because then $[G_2] = L$ in the 'partial' natural system. The answer is easy and natural, that would be the case if the space dimension would be two, not three, and accepting naively that the original force law would apply.

So let us adopt this idea and pursue where it leads us. When G_2 is defined by the fictitious law of force $F = G_2 mM/r$, without fixing its magnitude for the moment, the fundamental units of length, time and mass are

$$L = c^{-3} h G_2, \quad T = c^{-4} h G_2 \quad \text{and} \quad M = c^2/G_2, \tag{4}$$

that is, rational functions: our first objective has been fulfilled. Note also that the fundamental mass does not depend on h, as it would be the case for the independence of quantization from gravitation, which is so sensitive to mass. When we try to apply quantization to gravity, we know irreducible infinities remain.

Now we are able to express all constants in nature in terms of c, $\hbar$ and G_2, in particular the electric charge e and also the true Newton's constant G_N!

G_2 appears somewhat fictitious, so what about its magnitude? Gravitation in two space dimensions is not Newtonian gravitation [10]: one should formulate Einstein theory of gravitation directly in three dimensions; but then we find a happy surprise: in vacuum the Einstein tensor equation $G_{\mu\nu} = 0$ is equivalent to $\mathrm{Ric}_{\mu\nu} = 0$, and in three $(2 + 1)$ dimensions the Riemann curvature tensor and its Ricci (first) contraction coincide (the liberties are six in both cases): there is no 'curvature' outside the sources in two spatial dimensions; we turn this to an advantage for us: it means that G_2 is not really connected to gravitation! (We do not imply, however, that $(2+1)$ gravitation in its full glory could prescind of a specific coupling constant). We also know that in $2 + 1$ space, gravitation is really 'conic', so the effect of matter is a kind of angle [11]. So we are free to fix G_2 by other means.

Of course, the physics to fix G_2 should be the acceptable physics! So we propose the simplest hypothesis: to accept the values of Planck for length, time and mass, and invert G_2 in (4) to define

$$G_2 := c^2/M_{\mathrm{Pl}} \approx 4.1 \times 10^{24} \text{ MKS } \approx 4.1 \times 10^{25} \text{ CGS}. \tag{5}$$

There are many equivalent definitions, of course, e.g. $G_2 = G_{\mathrm{N}}/L_{\mathrm{Pl}}$, etc.

As rationale for our hypothesis we cite the existence of Dirac's law of big numbers [12]. Namely defining the pure (dimensionless) approximate number $\Omega \approx 10^{10}$, we find it a factor in powers of which many experimental relations arise, in some of which the elementary units are present. For example

1. Ω is about the ratio: number of photons to number of baryons in the present Universe.
2. Ω^3 is of the order: Planck mass to neutrino masses (10^{19} GeV vs. 10^{-2} eV).
3. Ω^4 is the ratio: electric to gravity forces for the proton–electron pair.
4. Ω^8 is close to the number of protons in the Universe.
5. The recently measured cosmological constant is of the order of the neutrino masses.
6. Ω^{12} is near the naive 'expected' value for the cosmological constant vs. the actual value.
7. Ω^6 is near the ratio: time elapsed since the Big Bang to Planck's time.

To be sure, the point 7 would indicate a time variation of the constants, as obviously the Universe evolves in time; in fact, that was part of Dirac's argument; and, although the question of varying universal constants is actually an active one (see again [6]), we do not want to pursue this issue here.

Why do we advocate Dirac's argument? Only because Planck's length (or time) enters into the large number scheme of Dirac (see the last item above).

4. Conclusions

Our choice selects 'something' in two spatial dimensions. What about that? We appeal now here to another concept of contemporary physics, albeit speculative, namely p-branes. $p = 0, 1, 2$ corresponds to particles, strings and membranes. Is there anything special about $p = 2$ membranes? They do play a role both in $11 = (10+1)$ dimensions, which includes maximal supergravity and M-theory, which embraces five consistent superstring theories, and also in F-theory [3,14], where a $(2 + 2)$ membrane is seen floating, the Perry membrane [2]; in the hands of Vafa and coll. F-theory with decoupled gravity seems to be a promising approach to the physics of the microworld (even moderately successful already [14]), through the minimal supersymmetric standard model (MSSM) (with $\mathcal{N} = 1$ supersymmetry) and grand unified theories (GUT). How this putative membrane could fix the value of a certain G_2 is left for the future; in any case, $p = 2$ membranes seem to be more promising than strings, and in M-theory for example, the underlying membrane generates the fundamental IIA string.

In mathematical terms, 3-manifolds now join curves and surfaces as being fully classified (the Thurston geometrization program, very much completed after Perelman's solution of the Poincare conjecture [15]). On the other hand, there are good reasons why manifolds with $d > 3$ cannot be enumerated; so membranes are the last extended objects well understood mathematically. Quantization of membranes is a tall order, however, perhaps connected with our actual inability to quantize gravitation.

This study is very preliminary and we do not seriously advocate the use of G_2 for practical applications. Also, we have not really found a reason to select a particular scale, with dimensions of length or mass, to define G_2; a possible one would be the scale of supersymmetry breaking, but this is again something for the future; another idea is to use the length of non-commutative geometry (see below). We think one should really resort to $G_\mathrm{N} = G_2 \times L_\mathrm{Pl}$ whenever realistic values of the constants are to be taken: the visible space is three-dimensional, after all. Indeed, there are 'anthropic' reasons for us to be in three open (or at least very large) dimensions: In quantum mechanics for example, the hydrogen atom gives a sensible spectrum, bound and continuous, only in 3d: in 1d and also in 2d the 'Coulomb' attractive forces yield confinement, whereas for $d > 3$ the particle 'falls into the center' [16]. Does quantum mechanics force space to have three open dimensions?

Another possible suggestion to define the fundamental length would be A Connes' introduction of non-commutative geometry: for two space coordinates one has

$$[x_i, x_j] = \Lambda^2 \theta_{ij} \tag{6}$$

and the consequent cell discretization of space, much as $\hbar$ determines elementary volumes in phase space. This alternative would improve the physics of black-hole evaporation (see e.g. [17]), and therefore might be a step in the right direction to tame quantum gravitation. If Λ, c and $\hbar$ are the fundamental magnitudes for all physics, mass is a derived one, as $[M] = \hbar/c\Lambda$ and $[G_\mathrm{N}] = \Lambda^2 c^3/\hbar$.

Acknowledgements

LJB acknowledges support from CICYT (Grant FPA2006-02315). The authors recognize illustrative discussions with J L Cortés; and also thank Prof. 't Hooft for encouragement.

References

[1] T S Kuhn, *Black body theory and quantum discontinuity* (Oxford University Press, Oxford, UK, 1978) Ch. 3
[2] S Hewson and M Perry, *Nucl. Phys.* **B492**, 249 (1997)
[3] C Vafa, *Nucl. Phys.* **B469**, 403 (1996)
[4] L J Boya and E C G Sudarshan, *Int. J. Theor. Phys.* **46**, 3285 (2007)
[5] S Gukov and E Witten, *Branes and quantization*, arXiv:0809.0305 [hep-th]
[6] M Duff, L Okun and G Veneziano, *J. High Energy Phys.* **3**, 20 (2002). Our position is closer to Okun's. As pointed out by A Rivero (Zaragoza Univ.), the title of the

paper, *Trialogue on the number of fundamental constants*, is a misnomer, as 'dia-' in dialogue means 'through', not two, as e.g. adiabatic means impermeable, isolated
[7] W Heisenberg, *Ann. Phys.* **32**, 20 (1938)
[8] L Okun, *Sov. Phys. Usp.* **34**, 818 (1991)
[9] Compare J Baez, *Bull. Am. Math. Soc.* **39**, 145 (2001)
[10] LJB acknowledges an interesting conversation over this point with R Jackiw in Montreal in 1985
[11] S Deser, R Jackiw and G 't Hooft, *Ann. Phys.* **152**, 220 (1984)
[12] P A M Dirac, *Nature (London)* **139**, 323 (1937)
[13] L J Boya, *Mod. Phys. Lett.* **A21**, 287 (2006)
[14] See e.g. the recent review by J Heckman and C Vafa, *From F-Theory GUTs to the LHC*, arXiv:0809.3452 [hep-th]
[15] *Proc. Int. Congr. Math.*, Madrid 2006,Vol. I
Eur. Math. Soc., talks by J Lott (p. 66) and J W Morgan (p. 713)
[16] L Landau and E Lifshitz, *Quantum mechanics* (Pergamon, London, 1959) p. 118
[17] P Nicolini *et al*, *Phys. Lett.* **B632**, 547 (2006)