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The degree distribution of fixed act-size collaboration networks

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Abstract. In this paper, we investigate a special evolving model of collaboration networks, where the act-size is fixed. Based on the first-passage probability of Markov chain theory, this paper provides a rigorous proof for the existence of a limiting degree distribution of this model and proves that the degree distribution obeys the power-law form with the exponent adjustable between 2 and 3.

Keywords. Fixed act-size collaboration networks; Markov chain; first-passage probability; scale-free; degree distribution.

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1. Introduction

The intensive study of complex networks is pervading all branches of science today, ranging from the physical to biological, to even social sciences. Typical complex networks include the World Wide Web [1], biological interacting networks [2–4], scientific cooperation networks [5–7], and so on. Research on fundamental properties and dynamical features of such complex networks has become overwhelming.

Barabasi and Albert discovered [8] that for many real-world complex networks, the fraction $P(k)$ of vertices with degree k has a power-law tail, $k^{-\gamma}$, where γ is a constant independent of the size of the network. To explain this phenomenon, they proposed a scale-free model, known as the BA model.

"starting with a small number (m_0) of vertices, at every time-step we add a new vertex with $m(\leq m_0)$ edges that link the new vertex to m different vertices already present in the system. To incorporate preferential attachment, we assume that the probability Π that a new vertex will be connected to a vertex i depends on the connectivity k_i of that vertex, so that $\Pi(k_i) = k_i/\sum_j k_j$. After t steps the model leads to a random network with $t + m_0$ vertices and mt edges."

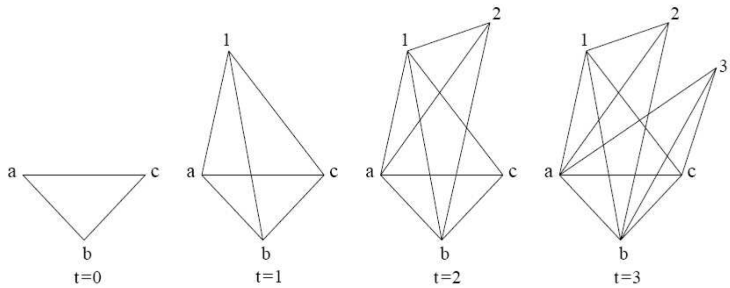

Figure 1. Illustration of fixed act-size network under consideration ($m = 3$).

Krapivsky *et al* [9] considered growing networks with $\Pi(k) \sim k^{\alpha}$, and found different behaviours arise for $\alpha < 1, \alpha > 1$, and $\alpha = 1$. For $\alpha < 1$, the number of vertices with k links, $P(k)$, varies as a stretched exponential. For $\alpha > 1$, a single vertex connects to nearly all other vertices. In the borderline case $\Pi(k) \sim k$, the power-law $P(k) \sim k^{-\gamma}$ is found.

Dorogovtsev *et al* [10] introduced a more general model of growing networks and allowed multiple edges between vertices, where each new vertex has an initial attractiveness A. Simultaneously, m new directed edges coming out from nonspecified vertices are introduced. The probability that a new link points to a given vertex is proportional to $k = A + q$ with q being in-degree of the vertex. They found that this has a power-law tail in the degree distribution.

There are a large number of fixed act-size collaboration networks [11]. For example, each football team has eleven players. In athletic sports or other items, the number of players is fixed, etc. In this paper, we propose a new approach to provide a rigorous proof for the existence of the degree distribution of this model, and we also prove that the degree distribution obeys power-law form.

2. Model description

In the model, at $t = 0$, we start with a complete graph which consists of m vertices. At each time-step, one picks at random one of the existing m-cliques, and adds a vertex by bonds to each member of the clique to form $(m + 1)$-cliques (see figure 1).

According to this rule, the bigger the degree of a vertex, the vertex will be a part of more m-cliques. In other words, the probability that a new vertex will be connected to vertex i depends on the degree $k_i(t)$ of vertex i at time t.

The number of m-cliques which contain vertex i increases $m - 1$ while the degree k_i of vertex i adds 1. Obviously, $k_i(i) = m$, and it is easy to obtain that the number of m-cliques which contain vertex i at time t is $(m - 1)k_i(t) - m(m - 2)$ and the number of m-cliques at time t is $mt + 1$. Thus, the probability $\Pi_m^{t+1}(k_i(t))$ that a new vertex will be connected to vertex i at time $t + 1$ is given by

$$\Pi_m^{t+1}(k_i(t)) = \frac{(m-1)k_i(t) - m(m-2)}{mt+1}. \tag{1}$$

3. Degree-distribution stability

$k_i(t)$ is a random variable for any fixed t and it is a nonhomogeneous Markov chain for the variable t [12,13], and let $P(k,i,t) = P\{k_i(t) = k\}$ be the probability of vertex i having degree k at time t.

From eq. (1), the state-transition probability of this Markov chain is given by

$$P\{k_i(t+1) = l | k_i(t) = k\} = \begin{cases} 1 - \frac{(m-1)k_i(t) - m(m-2)}{mt+1}, & l = k \\ \frac{(m-1)k_i(t) - m(m-2)}{mt+1}, & l = k+1, \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

where $k = 1, 2, \ldots, m + t - i$ and $i = 1, 2, \ldots$.

Consider the first-passage probability in Markov chain: $f(k,i,s) = P\{k_i(s) = k, k_i(l) \neq k, l = 1, 2, \ldots, s-1\}$. Then, we have the following lemma.

Lemma 3.1. When $k > m$

$$f(k,i,s) = P(k-1,i,s-1)\frac{(m-1)(k-1) - m(m-2)}{m(s-1)+1} \quad (3)$$

$$P(k,i,t) = \sum_{s=k+i-m}^{t} f(k,i,s) \times \prod_{j=s}^{t-1}\left[1 - \frac{(m-1)(k-1) - m(m-2)}{mj+1}\right]. \quad (4)$$

According to Dorogovtsev *et al* [10], we define $P(k,t) = \frac{1}{t}\sum_{i=1}^{t} P(k,i,t)$ and $P(k) = \lim_{t\to\infty} P(k,t)$.

Lemma 3.2. $P(m)$ exists, moreover,

$$P(m) = \frac{1}{2}.$$

Proof. From eq. (2), it follows that

$$P(m,i,t+1) = \left(1 - \frac{m}{mt+1}\right) P(m,i,t).$$

Since $P(m,t+1,t+1) = 1$, one has

$$P(m,t+1) = \frac{t}{t+1}\left(1 - \frac{m}{mt+1}\right) P(m,t) + \frac{1}{t+1}.$$

Then, by iteration and $P(m,1) = 1$, we have

$$P(m,t) = \frac{1}{t}\prod_{i=1}^{t-1}\left(1 - \frac{m}{mi+1}\right)\left[1 + \sum_{l=1}^{t-1}\prod_{j=1}^{l}\left(1 - \frac{m}{mj+1}\right)^{-1}\right].$$

Next, let

$$x_t = 1 + \sum_{l=1}^{t-1} \prod_{i=1}^{l} \left(1 - \frac{m}{mi+1}\right)^{-1}, \quad y_t = t \prod_{i=1}^{t-1} \left(1 - \frac{m}{mi+1}\right)^{-1}.$$

Using Stolz Theorem [14], we get

$$\begin{aligned} P(m) &= \lim_{t\to\infty} P(m,t) = \lim_{t\to\infty} \frac{x_t}{y_t} = \lim_{t\to\infty} \frac{x_{t+1} - x_t}{y_{t+1} - y_t} \\ &= \lim_{t\to\infty} \frac{1}{1 + \frac{mt}{mt+1}} = \frac{1}{2}. \end{aligned}$$

This completes the proof.

Lemma 3.3. For $k > m$, if $P(k-1)$ exists, then $P(k)$ also exists:

$$P(k) = \frac{(m-1)(k-1) - m(m-2)}{m + (m-1)k - m(m-2)} P(k-1). \tag{5}$$

Proof. By Lemma 3.1, the proof is similar to Lemma 3.2.

Theorem 3.4. The steady-state degree distribution of a fixed act-size collaboration network model exists, and is given by

$$P(k) = \frac{\Gamma(k - \frac{m(m-2)}{m-1})}{\Gamma(k + 1 + \frac{m-m(m-2)}{m-1})} \cdot \frac{\Gamma(m + 1 + \frac{m-m(m-2)}{m-1})}{\Gamma(m - \frac{m(m-2)}{m-1})} \cdot \frac{1}{2} \sim k^{-\frac{2m-1}{m-1}}.$$

Proof. By induction, from Lemmas 3.2 and 3.3, we get Theorem 3.4 easily.

Obviously, from Theorem 3.4, we found that the degree distribution of fixed act-size collaboration network follows a power-law with an exponent $\gamma = \frac{2m-1}{m-1} = 2 + \frac{1}{m-1}$ between 2 and 3. Furthermore, our analytic result agrees well with simulation result [11].

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