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# Complexification of three potential models – II

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**Abstract.** A new kind of $\mathcal{PT}$ and non-$\mathcal{PT}$-symmetric complex potentials are constructed from a group theoretical viewpoint of the sl$(2, C)$ potential algebras. The real eigenvalues and the corresponding regular eigenfunctions are also obtained. The results are compared with the ones obtained before.

**Keywords.** Non-Hermitian Hamiltonians; Lie algebra, $\mathcal{PT}$ symmetry; real eigenvalues; regular eigenfunctions.

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## 1. Introduction

$\mathcal{PT}$-symmetric quantum mechanics have generated much interest in recent years [1–10]. Few years ago, Bender and others [1,2,4,9] have looked at several complex potentials with $\mathcal{PT}$ symmetry and have shown that the energy eigenvalues are real when $\mathcal{PT}$ symmetry is unbroken, whereas they come in complex conjugate pairs when $\mathcal{PT}$ symmetry is spontaneously broken. Recently, Mostafazadeh [10] in his very noteworthy work has introduced the concept of pseudo-Hermiticity and he has pointed out that all the $\mathcal{PT}$-symmetric Hamiltonians regarded so far are actually $\mathcal{P}$-pseudo-Hermitian, namely $\mathcal{P}H\mathcal{P}^{-1} = H^\dagger$. Again, it is claimed that generally, it is the $\eta$-pseudo-Hermiticity, i.e. $\eta H \eta^{-1} = H^\dagger$ [10] and not the $\mathcal{PT}$ symmetry, of a Hamiltonian which is the necessary condition for its real spectrum.

Bagchi and Quesne [11,12] have discussed the Lie algebra for hyperbolic potential. In this paper, we shall illustrate the Lie algebra for the deformed-type hyperbolic Scarf-II potential [13], Pöschl–Teller potential [13] and the Morse potential [14]. We shall show that our results are in good agreement with the results obtained by others.

The paper is organized as follows. In §2, we present a brief discussion of the sl$(2, C)$ potential algebra and its realization. In §3, we obtain general results for complex potential associated with the sl$(2, C)$ potential algebra. In §4, 5, 6 we discuss the solutions of the Scarf-II, the Pöschl–Teller and the Morse potentials respectively. Section 7 gives conclusion.

## 2. sl(2,$C$) potential algebras

The most general differential realization of the sl$(2, C)$ algebra is [11,12]

$$J_0 = -i\frac{\partial}{\partial\phi}, \quad J_\pm = \mathrm{e}^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} + \left(i\frac{\partial}{\partial\phi} \mp \frac{1}{2}\right) f(x) + g(x)\right], \tag{1}$$

where $o \le \phi < 2\pi$, $x \in R$ and the two functions $f(x), g(x) \in C$ satisfy

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 1 - f^2, \quad \frac{\mathrm{d}g}{\mathrm{d}x} = -fg \tag{2}$$

and the generators are connected by

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0. \tag{3}$$

The Casimir operator corresponding to the above generators is

$$J^2 = -J_\pm J_\mp + J_0^2 \mp J_0. \tag{4}$$

Using (1) and (4) one can obtain

$$J^2 = \frac{\partial^2}{\partial x^2} - \left(\frac{\partial^2}{\partial\phi^2} + \frac{1}{4}\right)\frac{\mathrm{d}f}{\mathrm{d}x} + 2i\frac{\partial}{\partial\phi}\frac{\mathrm{d}g}{\mathrm{d}x} - g^2 - \frac{1}{4}. \tag{5}$$

In the realization (1), the states are given by [11]

$$|jm\rangle = \Psi_{jm}(x,\phi) = \psi_{jm}\frac{\mathrm{e}^{im\phi}}{\sqrt{2\pi}} \tag{6}$$

with fixed $j$ for which

$$J_0|jm\rangle = m|jm\rangle, \quad m = j, j+1, ... \tag{7}$$

$$J^2|jm\rangle = j(j-1)|jm\rangle, \quad m = j, j+1, ... \tag{8}$$

and $j = j_1 + ij_2$, $m = m_1 + im_2$, $m_1 = j_1 + n$, $m_2 = j_2$, where $j_1, j_2, m_1, m_2 \in R$, $n \in N$. The states with $j = m$ (i.e., $n = 0$) satisfy the equation $J_-|jj\rangle = 0$, while those with higher values of $n$ can be obtained from them by repeated applications of $J_+$ and using the relation $J_+|jm\rangle \propto |jm+1\rangle$. Using eq. (8) it follows that the functions $\psi_{jm}(x)$ satisfies the Schrödinger equation

$$\begin{aligned} &\psi''_{jm} + V_m\psi_{jm} = -\left(j - \frac{1}{2}\right)^2\psi_{jm}, \\ &-\psi_n^{(m)''} + V_m\psi_n^{(m)} = -E_n^{(m)}\psi_n^{(m)}, \end{aligned} \tag{9}$$

where $\psi_{jm}(x) = \psi_n^{(m)}(x)$. The family of potentials $V_m(x)$ is represented by

$$V_m(x) = \left(\frac{1}{4} - m^2\right)\frac{\mathrm{d}f}{\mathrm{d}x} + 2m\frac{\mathrm{d}g}{\mathrm{d}x} + g^2 \tag{10}$$

and the energy eigenvalues are given by

$$E_n^{(m)} = -\left(m_1 + im_2 - n - \frac{1}{2}\right)^2. \tag{11}$$

Solving the differential equation $J_-\psi_0^{(m)}(x) = 0$, the eigenfunctions $\psi_0^{(m)}(x)$ are easily obtained. The remaining eigenfunctions are obtained by successive application of $J_+$ on $\psi_0^{(m)}(x)$. For bound states ($\psi_n^{(m)}(\pm\infty) \to 0$), $n$ is restricted to the range $n = 0, 1, 2, ..., n_{\max} < m_1 - \frac{1}{2}$.

## 3. General results

The solutions of eq. (2) are

$$\left\{\begin{array}{l} f(x) = \tanh_q(x - c - i\sigma) \\ g(x) = (d_1 + id_2)\,\mathrm{sech}_q(x - c - i\sigma) \end{array}\right\}, \tag{12}$$

$$\left\{\begin{array}{l} f(x) = \coth_q(x - c - i\sigma) \\ g(x) = (d_1 + id_2)\,\mathrm{cosech}_q(x - c - i\sigma) \end{array}\right\}, \tag{13}$$

$$\left\{\begin{array}{l} f(x) = \lambda \\ g(x) = (d_1 + id_2)\mathrm{e}^{-\lambda x} \end{array}\right\}, \tag{14}$$

where $q(>0), c, d_1, d_2(\neq 0)$ are real, $\lambda = \pm 1, -\frac{\pi}{4} \leq \sigma \leq \frac{\pi}{4}$, and the deformed hyperbolic functions are defined as

$$\sinh_q x = \frac{\mathrm{e}^x - q\mathrm{e}^{-x}}{2}, \quad \cosh_q x = \frac{\mathrm{e}^x + q\mathrm{e}^{-x}}{2}, \quad \tanh_q x = \frac{\sinh_q x}{\cosh_q x}$$

and we use the relations:

$$q\,\mathrm{sech}_q^2\,x + \tanh_q^2 x = 1, \quad \coth_q^2 x - q\,\mathrm{cosech}_q^2\,x = 1,$$

$$(\tanh_q x)' = q\,\mathrm{sech}_q^2\,x, \quad (\mathrm{cosech}_q\,x)' = -\mathrm{cosech}_q\,x\coth_q x,$$

$$(\coth_q x)' = -q\,\mathrm{cosech}_q^2\,x,$$

where the prime denotes the differentiation with respect to $x$. From eqs (10) and (12) we have the nonsingular Scarf-II [SF] potential, given by

$$
\begin{aligned}
V_m^{\rm SF}(x) &= \left((d_1+id_2)^2+\left(\frac{1}{4}-(m_1+im_2)^2\right)q\right)\operatorname{sech}_q^2(x-c-i\sigma)\\
&\quad -2(m_1+im_2)(d_1+id_2)\operatorname{sech}_q(x-c-i\sigma)\tanh_q(x-c-i\sigma)\\
&= \frac{2}{\left(\cosh_{q^2}(2x-2c)+q\cos 2\sigma\right)^2}\\
&\quad \times\Bigg\{\left(d_1^2-d_2^2+\left(\frac{1}{4}-m_1^2+m_2^2\right)q\right)\left(\cosh_{q^2}(2x-2c)\cos 2\sigma+q\right)\\
&\quad -2(d_1d_2-qm_1m_2)\sinh_{q^2}(2x-2c)\sin 2\sigma\\
&\quad -2(d_1m_1-d_2m_2)[\sinh_{q^2}(x-c)\cos\sigma\\
&\quad \times(\cosh_{q^2}(2x-2c)-q\cos 2\sigma+2q)]\\
&\quad +2(d_1m_2+d_2m_1)[\cosh_{q^2}(x-c)\sin\sigma\\
&\quad \times(\cosh_{q^2}(2x-2c)-q\cos 2\sigma-2q)]\Bigg\}\\
&\quad +\frac{2i}{\left(\cosh_{q^2}(2x-2c)+q\cos 2\sigma\right)^2}\\
&\quad \times\Bigg\{\left(d_1^2-d_2^2+\left(\frac{1}{4}-m_1^2+m_2^2\right)q\right)\sinh_{q^2}(2x-2c)\sin 2\sigma\\
&\quad +2(d_1d_2-qm_1m_2)\left(\cosh_{q^2}(2x-2c)\cos 2\sigma+q\right)\\
&\quad -2(d_1m_1-d_2m_2)[\cosh_{q^2}(x-c)\sin\sigma\\
&\quad \times(\cosh_{q^2}(2x-2c)-q\cos 2\sigma-2q)]\\
&\quad -2(d_1m_2+d_2m_1)[\sinh_{q^2}(x-c)\cos\sigma\\
&\quad \times(\cosh_{q^2}(2x-2c)-q\cos 2\sigma+2q)]\Bigg\}. \qquad (15)
\end{aligned}
$$

From eqs (10) and (13) we have the nonsingular Pöschl–Teller potential (PTL), given by

$$
\begin{aligned}
V_m^{\rm PTL}(x) &= \left((d_1+id_2)^2-\left(\frac{1}{4}-(m_1+im_2)^2\right)q\right)\operatorname{cosech}_q^2(x-c-i\sigma)\\
&\quad -2(m_1+im_2)(d_1+id_2)\operatorname{cosech}_q(x-c-i\sigma)\coth_q(x-c-i\sigma)\\
&= \frac{2}{\left(\cosh_{q^2}(2x-2c)-q\cos 2\sigma\right)^2}\\
&\quad \times\Bigg\{\left(d_1^2-d_2^2-\left(\frac{1}{4}-m_1^2+m_2^2\right)q\right)\\
&\quad \times\left(\cosh_{q^2}(2x-2c)\cos 2\sigma-q\right)\\
&\quad -2(d_1d_2+qm_1m_2)\sinh_{q^2}(2x-2c)\sin 2\sigma\\
&\quad -2(d_1m_1-d_2m_2)[\cosh_{q^2}(x-c)\cos\sigma\\
&\quad \times(\cosh_{q^2}(2x-2c)+q\cos 2\sigma-2q)]\\
&\quad +2(d_1m_2+d_2m_1)[\sinh_{q^2}(x-c)\sin\sigma
\end{aligned}
$$

$$
\begin{aligned}
&\left.\times(\cosh_{q^2}(2x-2c)+q\cos 2\sigma+2q)]\right\} \\
&+\frac{2i}{\left(\cosh_{q^2}(2x-2c)-q\cos 2\sigma\right)^2} \\
&\times\left\{\left(d_1^2-d_2^2-\left(\frac{1}{4}-m_1^2+m_2^2\right)q\right)\sinh_{q^2}(2x-2c)\sin 2\sigma\right. \\
&+2(d_1d_2+qm_1m_2)\left(\cosh_{q^2}(2x-2c)\cos 2\sigma-q\right) \\
&-2(d_1m_1-d_2m_2)[\sinh_{q^2}(x-c)\sin\sigma \\
&\times(\cosh_{q^2}(2x-2c)+q\cos 2\sigma+2q)] \\
&-2(d_1m_2+d_2m_1)[\cosh_{q^2}(x-c)\cos\sigma \\
&\left.\times(\cosh_{q^2}(2x-2c)+q\cos 2\sigma-2q)]\right\}
\end{aligned}
\tag{16}
$$

and from eqs (10) and (14) we have the nonsingular Morse potential (MP), given by

$$
\begin{aligned}
V_m^{\mathrm{MP}}(x) &= (d_1+id_2)^2\mathrm{e}^{-2x}-2(m_1+im_2)(d_1+id_2)\mathrm{e}^{-x} \\
&= \left(d_1^2-d_2^2\right)\mathrm{e}^{-2x}-2(d_1m_1-d_2m_2)\mathrm{e}^{-x} \\
&\quad +2i(d_1d_2\mathrm{e}^{-2x}-(d_1m_2+d_2m_1)\mathrm{e}^{-x}),
\end{aligned}
\tag{17}
$$

where eq. (17) corresponds to $\lambda = 1$ and to obtain eqs (15) and (16) we use the relations:

$$
\sinh_q(x+iy) = \sinh_q x\cos y + i\cosh_q x\sin y \tag{18}
$$

$$
\cosh_q(x+iy) = \cosh_q x\cos y + i\sinh_q x\sin y. \tag{19}
$$

The above potentials give a quite complete generalization of $\mathrm{sl}(2, C)$ algebra corresponding to the representation (1). In order to obtain the regular wave function $\psi_0^{(m)}(x)$, solve the differential equation $J_-\psi_{mm}(x,\phi) = 0$, we have

$$
\psi_0^{(m)}(x) \propto (\mathrm{sech}_q\, x')^{(m-\frac{1}{2})}\exp\left[\frac{(d_1+id_2)}{\sqrt{q}}\arctan\left(\frac{1}{\sqrt{q}}\sinh_q x'\right)\right], \tag{20}
$$

$$
\begin{aligned}
\psi_0^{(m)}(x) \propto &\left[\sinh_{\sqrt{q}}\left(\frac{x'}{2}\right)\right]^{(-m+\frac{1}{2}+\frac{d_1+id_2}{\sqrt{q}})} \\
&\times\left[\cosh_{\sqrt{q}}\left(\frac{x'}{2}\right)\right]^{(-m+\frac{1}{2}-\frac{d_1+id_2}{\sqrt{q}})},
\end{aligned}
\tag{21}
$$

$$
\psi_0^{(m)}(x) \propto \exp\left[-\left(m-\frac{1}{2}\right)x'-(d_1+id_2)\mathrm{e}^{-x'}\right], \tag{22}
$$

where $x' = x - c - i\sigma, m = m_1 + im_2$. Equations (20) and (21) are regular when $m_1 > \frac{1}{2}$, $d_1 > 0$ and eq. (22) is regular when $d_1 > 0$.

## 4. Complexification of the Scarf-II potential

The most general form of Scarf-II potential is

$$V(x) = -V_1 \operatorname{sech}_q^2 x - iV_2 \operatorname{sech}_q x \tanh_q x, \quad V_1 > 0, \quad V_2 \neq 0. \tag{23}$$

The potential (23) is $\mathcal{PT}$-symmetric under $\mathcal{P}$: $x \to \log q - x$, $\mathcal{T}$: $i \to -i$ and $\eta$-pseudo-Hermitian under $\eta x \eta^{-1} = x + i\pi$. Now for $c = \sigma = 0$, comparing eq. (23) with eq. (15) we have

$$d_1^2 - d_2^2 + \left(\frac{1}{4} - m_1^2 + m_2^2\right) q = -V_1 \tag{24}$$

$$d_1 d_2 - q m_1 m_2 = 0 \tag{25}$$

$$d_1 m_1 - d_2 m_2 = 0 \tag{26}$$

$$2(d_1 m_2 + d_2 m_1) = V_2. \tag{27}$$

Solving eqs (26) and (27) we have

$$m_1 = \frac{V_2 d_2}{2(d_1^2 + d_2^2)}, \quad m_2 = \frac{V_2 d_1}{2(d_1^2 + d_2^2)}. \tag{28}$$

Using eqs (24), (25) and (28) we have

$$(d_1^2 + d_2^2)\left[1 + \frac{q V_2^2}{4(d_1^2 + d_2^2)^2}\right] = -V_1 - \frac{q}{4} \tag{29}$$

$$d_1 d_2 \left[1 - \frac{q V_2^2}{4(d_1^2 + d_2^2)^2}\right] = 0. \tag{30}$$

Equation (30) implies that either $d_1 = 0$ or $d_1 \neq 0$ and $d_1^2 + d_2^2 = \frac{1}{2}\sqrt{q}|V_2|$. We shall now discuss two cases:

*Case* I. $d_1 = 0$. From eqs (28) and (29) we have

$$d_2^2 = \frac{1}{4}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\, V_2} + \lambda \sqrt{V_1 + \frac{q}{4} - \sqrt{q}\, V_2}\right)^2, \quad \lambda = \pm 1 \tag{31}$$

provided $|V_2| \leq \frac{1}{\sqrt{q}}\left(V_1 + \frac{q}{4}\right)$ and

$$m_1 = \frac{V_2}{2d_2}, \quad m_2 = 0. \tag{32}$$

From the regularity condition $m_1 > \frac{1}{2}$ of eq. (20), it then follows that $d_2$ must have the same sign as $V_2$, which we denote by $\eta$. In this case, the solutions of $d_1, d_2, m_1, m_2$ are

$$d_1 = 0, \quad d_2 = \frac{1}{2}\eta\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\,|V_2|} - \mu\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,|V_2|}\right), \quad (33)$$

$$m_1 = \frac{1}{2\sqrt{q}}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\,|V_2|} + \mu\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,|V_2|}\right), \quad m_2 = 0, \quad (34)$$

where $|V_2| \leq \frac{1}{\sqrt{q}}\left(V_1 + \frac{q}{4}\right)$, $\mu = \pm 1$, $\lambda + \mu = 0$ and

$$\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\,|V_2|} + \mu\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,|V_2|}\right) > \sqrt{q}. \quad (35)$$

So, from eqs (11) and (34) we get two series of real energy eigenvalues

$$\begin{aligned} E_n = &-\left(\frac{1}{2\sqrt{q}}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\,|V_2|}\right.\right.\\ &\left.\left.\pm\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,|V_2|}\right) - n - \frac{1}{2}\right)^2 \\ n = 0, 1, 2, \ldots < &\left(\frac{1}{2\sqrt{q}}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\,V_2}\right.\right.\\ &\left.\left.\pm\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,V_2}\right) - \frac{1}{2}\right). \end{aligned} \quad (36)$$

Let us take the potential parameter as $V_1 = (B^2 - A(A + \sqrt{q}))$, $V_2 = -\frac{B}{\sqrt{q}}\left(2A + \sqrt{q}\right), m_2 = 0$, the potential (23) is invariant under the transformation $A + \frac{\sqrt{q}}{2} \longleftrightarrow B$. Two values of $m$ are $\frac{A}{\sqrt{q}} + \frac{1}{2}$ and $\frac{B}{\sqrt{q}}$ and two series of real energy eigenvalues are

$$E_n^{(\frac{1}{\sqrt{q}}(A + \frac{\sqrt{q}}{2}))} = -\left(\frac{A}{\sqrt{q}} - n\right)^2, \quad n = 0, 1, 2, 3, \ldots < \frac{A}{\sqrt{q}} \quad (37)$$

$$E_n^{(\frac{B}{\sqrt{q}})} = -\left(\frac{B}{\sqrt{q}} - n - \frac{1}{2}\right)^2, \quad n = 0, 1, 2, 3, \ldots < \left(\frac{B}{\sqrt{q}} - \frac{1}{2}\right). \quad (38)$$

Now for the special choice $B = \sqrt{q}$, $A + \frac{\sqrt{q}}{2} = -\lambda\sqrt{q}$ ($\lambda < 0$), the energies (37) obtained from the first sl$(2, C)$ algebra become $E_n^{(-\lambda)} = -\left(\lambda + n + \frac{1}{2}\right)^2$, while the second sl$(2, C)$ algebra leads to a single energy level $E_0^{(1)} = -\frac{1}{4}$.

*Case* II. $d_1 \neq 0$ and $d_1^2 + d_2^2 = \frac{1}{2}\sqrt{q}\,|V_2|$. Applying regularity condition we must have

$$d_1 = \frac{1}{2}\eta\sqrt{\sqrt{q}\,|V_2| - V_1 - \frac{q}{4}}, \quad d_2 = \frac{1}{2}\eta\sqrt{\sqrt{q}\,|V_2| + V_1 + \frac{q}{4}}, \quad (39)$$

$$m_1 = \frac{1}{2\sqrt{q}}\sqrt{\sqrt{q}\,|V_2| + V_1 + \frac{q}{4}}, \quad m_2 = \frac{1}{2\sqrt{q}}\mu\sqrt{\sqrt{q}\,|V_2| - V_1 - \frac{q}{4}}, \quad (40)$$

where we assume $|V_2| > \frac{1}{\sqrt{q}}\left(V_1 + \frac{q}{4}\right)$ and $(\sqrt{q}\,|V_2| + V_1 + \frac{q}{4}) > q$. In this case, complex energy values are given by

$$E_n = -\left(\frac{1}{2\sqrt{q}}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}|V_2|} \pm i\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,|V_2|}\right) - n - \frac{1}{2}\right)^2$$
$$n = 0, 1, 2, ... < \frac{1}{2\sqrt{q}}\sqrt{\sqrt{q}\,|V_2| + V_1 + \frac{q}{4}}. \quad (41)$$

## 5. Complexification of the Pöschl–Teller potential

The generalized Pöschl–Teller potential is usually given in the form

$$V(x) = V_1 \operatorname{cosech}_q^2(x - c - i\sigma) - V_2 \operatorname{cosech}_q(x - c - i\sigma) \coth_q(x - c - i\sigma),$$
$$V_1 > -\frac{q}{4}, \quad V_2 \neq 0. \quad (42)$$

The potential (42) is $\mathcal{PT}$-symmetric under $\mathcal{P}$: $x \to \log q - x + 2c$, $\mathcal{T}$: $i \to -i$. Now for $c = \sigma = 0$, comparing eq. (42) with eq. (16) we have

$$d_1^2 - d_2^2 - \left(\frac{1}{4} - m_1^2 + m_2^2\right) q = V_1 \quad (43)$$

$$d_1 d_2 + q m_1 m_2 = 0 \quad (44)$$

$$2(d_1 m_1 - d_2 m_2) = V_2 \quad (45)$$

$$d_1 m_2 + d_2 m_1 = 0. \quad (46)$$

Using the same technique as in the previous section, we have for

*Case* I ($d_1 = 0$)

$$d_1 = 0, \quad d_2 = \frac{1}{2}\eta\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\,|V_2|} - \mu\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,|V_2|}\right), \quad (47)$$

$$m_1 = \frac{1}{2\sqrt{q}}\left(\sqrt{V_1 + \frac{q}{4} + \sqrt{q}\,|V_2|} + \mu\sqrt{V_1 + \frac{q}{4} - \sqrt{q}\,|V_2|}\right), \quad m_2 = 0, \quad (48)$$

where $|V_2| \leq \frac{1}{\sqrt{q}}\left(V_1+\frac{q}{4}\right)$, $\mu=\pm 1$, $\lambda+\mu=0$ and

$$\left(\sqrt{V_1+\frac{q}{4}+\sqrt{q}\,|V_2|}+\mu\sqrt{V_1+\frac{q}{4}-\sqrt{q}\,|V_2|}\right)>\sqrt{q}. \tag{49}$$

The two series of real energy eigenvalues are

$$E_n=-\left(\frac{1}{2\sqrt{q}}\left(\sqrt{V_1+\frac{q}{4}+\sqrt{q}\,|V_2|}\pm\sqrt{V_1+\frac{q}{4}-\sqrt{q}\,|V_2|}\right)-n-\frac{1}{2}\right)^2$$
$$n=0,1,2,\ldots<\left(\frac{1}{2\sqrt{q}}\left(\sqrt{V_1+\frac{q}{4}+\sqrt{q}\,V_2}\pm\sqrt{V_1+\frac{q}{4}-\sqrt{q}\,V_2}\right)-\frac{1}{2}\right) \tag{50}$$

and for

*Case* II ($d_1 \neq 0$ and $d_1^2+d_2^2=\frac{1}{2}\sqrt{q}|V_2|$)

$$d_1=\frac{1}{2}\eta\sqrt{\sqrt{q}\,|V_2|-V_1-\frac{q}{4}},\quad d_2=\frac{1}{2}\eta\sqrt{\sqrt{q}\,|V_2|+V_1+\frac{q}{4}}, \tag{51}$$

$$m_1=\frac{1}{2\sqrt{q}}\sqrt{\sqrt{q}\,|V_2|+V_1+\frac{q}{4}},\quad m_2=\frac{1}{2\sqrt{q}}\mu\sqrt{\sqrt{q}\,|V_2|-V_1-\frac{q}{4}}, \tag{52}$$

where we assume $|V_2| > \frac{1}{\sqrt{q}}\left(V_1+\frac{q}{4}\right)$ and $(\sqrt{q}\,|V_2|+V_1+\frac{q}{4})>q$. In this case, complex energy values are given by

$$E_n=-\left(\frac{1}{2\sqrt{q}}\left(\sqrt{V_1+\frac{q}{4}+\sqrt{q}\,|V_2|}\pm i\sqrt{V_1+\frac{q}{4}-\sqrt{q}\,|V_2|}\right)-n-\frac{1}{2}\right)^2$$
$$n=0,1,2,\ldots<\frac{1}{2\sqrt{q}}\sqrt{\sqrt{q}\,|V_2|+V_1+\frac{q}{4}}. \tag{53}$$

## 6. Complexification of the Morse potential

The generalized Morse potential is usually given by

$$V(x)=(V_1+iV_2)\mathrm{e}^{-2x}-(V_3+iV_4)\mathrm{e}^{-x},\quad V_1,V_2,V_3,V_4\in R. \tag{54}$$

The potential (54) is non-$\mathcal{PT}$-symmetric under $\mathcal{P}$: $x\to -x, \mathcal{T}: i\to -i$.

For $V_1+iV_2=(A+iB)^2, V_3+iV_4=K(A+iB)$ $(A,B,K\in R)$, potential is pseudo-Hermitian under the transformation $\eta x\eta^{-1}=x+i\theta$ with $\theta=2\tan^{-1}(\frac{B}{A})$. Comparing eqs (54) with (17) we have

$$d_1^2-d_2^2=V_1 \tag{55}$$

$$2d_1d_2=V_2 \tag{56}$$

$$2(d_1m_1 - d_2m_2) = V_3 \tag{57}$$

$$2(d_1m_2 + d_2m_1) = V_4. \tag{58}$$

Equations (55)–(58) have already been discussed in [11]. The complex eigenvalues [11] are

$$\begin{aligned} E_n = &-\frac{1}{2\sqrt{2(V_1^2+V_2^2)}} \\ &\times\left[\left(\sqrt{\sqrt{V_1^2+V_2^2}+V_1} - i\mu\sqrt{\sqrt{V_1^2+V_2^2}-V_1}\right)\right. \\ &\left.\times(V_3+iV_4) - n - \frac{1}{2}\right]^2, \end{aligned} \tag{59}$$

where

$$\begin{aligned} n = 0, 1, 2, ... \\ &< \left[\frac{1}{2\sqrt{2(V_1^2+V_2^2)}}\left(V_3\sqrt{\sqrt{V_1^2+V_2^2}-V_1}\right.\right. \\ &\left.\left.+\mu V_4\sqrt{\sqrt{V_1^2+V_2^2}+V_1}\right) - n - \frac{1}{2}\right]. \end{aligned}$$

The real energies correspond to $m_2 = 0$

$$E_n = -\left(\frac{V_3}{\sqrt{2}|V_2|}\sqrt{\sqrt{V_1^2+V_2^2}-V_1} - n - \frac{1}{2}\right)^2, \tag{60}$$

where

$$n = 0, 1, 2, ... < \frac{V_3}{\sqrt{2}|V_2|}\sqrt{\sqrt{V_1^2+V_2^2}-V_1} - n - \frac{1}{2}.$$

## 7. Conclusion

In this paper, the bound state eigenvalues of the Scarf-II, the Pöschl–Teller and the Morse potentials have been derived by $\mathrm{sl}(2, C)$ potential algebra. Our solution of eq. (2) which are given in eqs (12), (13) are more general than the solutions obtained by others [12]. For $q = 1, m_2 = 0, m_1 = m$, eqs (15) and (16) coincide with [12] and eq. (36) is consistent with [4] for $\alpha = 1$. For the case of the Scarf-II and the Pöschl–Teller potentials, we have calculated that symmetry breaking occurs when $|V_2| > \frac{1}{\sqrt{q}}\left(V_1 + \frac{q}{4}\right)$. We have also shown that for the Morse potential there is no symmetry breaking range.

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