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Octonion wave equation and tachyon electrodynamics

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Abstract. The octonion wave equation is discussed to formulate the localization spaces for subluminal and superluminal particles. Accordingly, tachyon electrodynamics is established to obtain a consistent and manifestly covariant equation for superluminal electromagnetic fields. It is shown that the true localization space for bradyons (subluminal particles) is R^4- (three space and one time dimensions) space while that for the description of tachyons is T^4- (three time and one space dimensions) space.

Keywords. Octonion; bradyon; tachyon; electrodynamics.

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1. Definitions

An octonion x is expressed [1] as a set of eight real numbers

$$\begin{aligned} x &= e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3 + e_4x_4 + e_5x_5 + e_6x_6 + e_7x_7 \\ &= e_0x_0 + \sum_{A=1}^{7} e_Ax_A, \end{aligned} \tag{1}$$

where e_A $(A = 1, 2, ..., 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The octet $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is known as the octonion basis, and its elements satisfy the following multiplication rules:

$$\begin{aligned} &e_0 = 1, \quad e_0e_A = e_Ae_0 = e_A, \\ &e_Ae_B = -\delta_{AB}e_0 + f_{ABC}e_C, \quad (A, B, C = 1, 2, ..., 7). \end{aligned} \tag{2}$$

The structure constants f_{ABC} are completely antisymmetric and take the value 1 for the combinations, $f_{ABC} = +1$; $\forall(ABC)$ = (123), (471), (257), (165), (624), (543), (736). Here the octonion algebra $\mathcal{O}$ is described over the algebra of real numbers having the vector space dimension 8. We now get the following relations among octonion basis elements:

$$[e_A, e_B] = 2f_{ABC}e_C; \quad \{e_A, e_B\} = -2\delta_{AB}e_0; \quad e_A(e_Be_C) \neq (e_Ae_B)e_C, \tag{3}$$

where brackets [,] and { , } are used respectively for commutation and the anti-commutation relations while δ_{AB} is the usual Kronecker delta-Dirac symbol. The octonion conjugate is defined as

$$\begin{aligned}\bar{x} &= e_0x_0 - e_1x_1 - e_2x_2 - e_3x_3 - e_4x_4 - e_5x_5 - e_6x_6 - e_7x_7 \\ &= e_0x_0 - \sum_{A=1}^{7} e_Ax_A.\end{aligned} \tag{4}$$

The norm $N(x)$ and inverse x^{-1} (for a nonzero x) of an octonion are respectively defined as

$$\begin{aligned}N(x) &= x\bar{x} = \bar{x}x = \sum_{\alpha=0}^{7} x_\alpha^2 e_0, \\ x^{-1} &= \frac{\bar{x}}{N(x)} \Longrightarrow xx^{-1} = x^{-1}x = 1.\end{aligned} \tag{5}$$

The norm $N(x)$ of an octonion x is zero if $x = 0$, and is always positive otherwise. It also satisfies the following property of a normed algebra:

$$N(xy) = N(x)N(y) = N(y)\,N(x). \tag{6}$$

2. Octonion wave equation

A lot of literature [2–6] has already been available on octonion wave equation. Accordingly, let us define the octonion differential operator D as

$$D = \sum_{\mu=0}^{7} e_\mu D_\mu, \tag{7}$$

where D_μs are described as the components of a differential operator in an eight-dimensional representation. We describe a function of an octonion variable as

$$\mathcal{F}(X) = \sum_{\mu=0}^{7} e_\mu f_\mu(X) = f_0 + e_1f_1 + e_2f_2 + e_3f_3 + \cdots + e_7f_7, \tag{8}$$

where f_μs are scalar functions. Since octonions are neither commutative nor associative, one has to be very careful to multiply the octonion either from left or from right in terms of regularity conditions [3]. As such, a function $\mathcal{F}(X)$ of an octonion variable $X = \sum_{\mu=0}^{7} e_\mu X_\mu$ is left regular at X if and only if $\mathcal{F}(X)$ satisfies the condition

$$D\mathcal{F}(X) = 0. \tag{9}$$

Similarly, a function $G(X)$ is right regular if and only if

$$G(X)\overleftarrow{\overline{D}} = 0, \tag{10}$$

where $G(X) = g_0 + g_1 e_1 + g_2 e_2 + \cdots + g_7 e_7$. Then we get

$$D\mathcal{F} = I_0 + I_1 e_1 + I_2 e_2 + I_3 e_3 + I_4 e_4 + I_5 e_5 + I_6 e_6 + I_7 e_7, \tag{11}$$

where

$$\begin{aligned}
I_0 &= \partial_0 f_0 - \partial_1 f_1 - \partial_2 f_2 - \partial_3 f_3 - \partial_4 f_4 - \partial_5 f_5 - \partial_6 f_6 - \partial_7 f_7,\\
I_1 &= \partial_0 f_1 + \partial_1 f_0 + \partial_2 f_3 - \partial_3 f_2 + \partial_6 f_5 - \partial_5 f_6 - \partial_7 f_4 + \partial_4 f_7,\\
I_2 &= \partial_0 f_2 + \partial_2 f_0 + \partial_3 f_1 - \partial_1 f_3 + \partial_4 f_6 - \partial_6 f_4 - \partial_7 f_5 + \partial_5 f_7,\\
I_3 &= \partial_0 f_3 + \partial_3 f_0 + \partial_1 f_2 - \partial_2 f_1 + \partial_6 f_7 - \partial_7 f_6 + \partial_5 f_4 - \partial_4 f_5,\\
I_4 &= \partial_0 f_4 + \partial_4 f_0 + \partial_3 f_5 - \partial_5 f_3 - \partial_2 f_6 + \partial_6 f_2 - \partial_1 f_7 + \partial_7 f_1,\\
I_5 &= \partial_0 f_5 + \partial_5 f_0 + \partial_1 f_6 - \partial_6 f_1 + \partial_7 f_2 - \partial_2 f_7 - \partial_3 f_4 + \partial_4 f_3,\\
I_6 &= \partial_0 f_6 + \partial_6 f_0 - \partial_1 f_5 + \partial_5 f_1 + \partial_2 f_4 - \partial_4 f_2 - \partial_3 f_7 + \partial_7 f_3,\\
I_7 &= \partial_0 f_7 + \partial_7 f_0 + \partial_1 f_4 - \partial_4 f_1 + \partial_2 f_5 - \partial_5 f_2 - \partial_6 f_3 + \partial_3 f_6.
\end{aligned} \tag{12}$$

The regularity condition (9) may now be considered as a homogeneous octonion wave equation for octonion variables without sources. On the other hand, eq. (11) is considered as the inhomogeneous wave equation

$$D\mathcal{F} = I, \tag{13}$$

where I is again an octonion.

2.1 *Localization spaces for subluminal and superluminal objects*

Let us define an octonionic variable (respectively known as potential, differential operator and function variable) as the combination of two quaternion variables in the following manner [6]:

$$\emptyset = \emptyset_a + e_7\emptyset_b, \quad D = D_a + e_7 D_b, \quad F = F_a + e_7 F_b, \tag{14}$$

where $\emptyset_a$, $\emptyset_b$, D_a, D_b, F_a and F_b are quaternion variables (respectively known as potential, differential operator and function variable) given by

$$\begin{aligned}
\emptyset_a &= \emptyset_0 + e_1\emptyset_1 + e_2\emptyset_2 + e_3\emptyset_3,\\
\emptyset_b &= \emptyset_7 + e_1\emptyset_4 + e_2\emptyset_5 + e_3\emptyset_6,\\
D_a &= \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3,\\
D_b &= \partial_7 + e_1\partial_4 + e_2\partial_5 + e_3\partial_6,\\
F_a &= F_0 + e_1 F_1 + e_2 F_2 + e_3 F_3,\\
F_b &= F_7 + e_1 F_4 + e_2 F_5 + e_3 F_6.
\end{aligned} \tag{15}$$

Here e_1, e_2 and e_3 are three quaternion units which satisfy the multiplication rule $e_j e_k = -\delta_{jk} + \varepsilon_{jkl} e_l \quad \forall (j,k,l = 1,2,3)$. Accordingly, the octonion wave equation for a potential in terms of two quaternions is written as

$$\bar{D}\emptyset = (\bar{D}_a - e_7 D_b)(\emptyset_a + e_7\emptyset_b) = (\phi + e_7\varphi) = F, \tag{16}$$

where

$$\begin{aligned}
\phi &= \bar{D}_a\emptyset_a + \emptyset_b\widetilde{D}_b \\
&= (D_0 - D_1e_1 - D_2e_2 - D_3e_3)\emptyset_a + \emptyset_b(\overleftarrow{D_7} - \overleftarrow{D_4}e_1 - \overleftarrow{D_5}e_2 - \overleftarrow{D_6}e_3), \\
\bar{D}_a &= D_0 - D_1e_1 - D_2e_2 - D_3e_3, \\
\widetilde{D}_b &= D_7 - D_4e_1 - D_5e_2 - D_6e_3, \\
\varphi &= \bar{\bar{D}}_a\emptyset_b - \emptyset_a D_b = D_a\emptyset_b - \emptyset_a D_b.
\end{aligned} \tag{17}$$

As such, we may express the current equation in terms of two quaternions as

$$DF = (D_a + e_7D_b)(g + e_7h) = S = s_a + e_7s_b, \tag{18}$$

where

$$\begin{aligned}
s_a &= D_a g - hD_b = D_a\bar{D}_b\emptyset_a + \emptyset_b D_b\bar{D}_b, \\
s_b &= \widetilde{D}_a h + gD_b = \bar{D}_a D_a\emptyset_b + \emptyset_b\bar{D}_b D_b.
\end{aligned} \tag{19}$$

Thus $S = s_a + e_7s_b$ is the octonion expression for a generalized current density.

The octonion (eight-dimensional) space is considered in terms of two quaternion (namely the external and internal four-dimensional) spaces. Let us assume the external four spaces as the usual Minkowski (or Euclidean) $R^4 \Rightarrow M(1,3)$-space (consisting of one time and three space coordinates), i.e. the localization space of bradyons [7] (particles travelling slower than light $\rightarrow$ subluminal particles). Accordingly, we denote the internal four-dimensional space by $T^4 \Rightarrow M(3,1)$-space (consisting of three time and one space coordinates). This space has been [7] considered as the localization space of tachyons (particles travelling faster than light $\rightarrow$ superluminal particles). Hence an eight-dimensional octonion space is described as the unified space containing both external $R^4 \Rightarrow M(1,3)$ and internal $T^4 \Rightarrow M(3,1)$ four-dimensional spaces, i.e. the unified localization space for bradyons and tachyons expressed as $R^8 = R^4 \bigcup T^4$. First quaternion variable of eq. (18) whose components are (x_0, x_1, x_2, x_3), thus maps to four-dimensional space for bradyons $R^4 \simeq (t, \vec{r})$ with its coordinates (t, x, y, z). Accordingly, the second quaternion variable whose components are (x_7, x_4, x_5, x_6) hence maps to four-dimensional space for tachyons $T^4 \simeq (r, \vec{t})$ with its coordinates (r, t_x, t_y, t_z). So, in the absence of the second quaternion, eq. (19) reduces to

$$s_a = D_a\bar{D}_a\emptyset_a \Rightarrow \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\phi = \Box(R)\phi, \tag{20}$$

which resembles the Cauchy–Feuter equation [8] of quaternion variables in $R^4 \simeq (t, \vec{r})$ space and thus describes the usual Maxwell equations of electromagnetic fields in R^4-space. On the other hand, in the absence of the first quaternion, eq. (19) reduces to

$$s_b = \emptyset_b \bar{D}_b D_b \Rightarrow \varphi \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t_x^2} - \frac{\partial^2}{\partial t_y^2} - \frac{\partial^2}{\partial t_z^2} \right) = \varphi \overleftarrow{\Box}(T), \tag{21}$$

which also describes a Cauchy–Feuter equation of quaternion variables in $T^4 \simeq (r, \vec{t})$ space and coincides with the Maxwell equations of superluminal photons [7] in T^4-space.

3. Split octonion wave equation

The split octonions [9] are a nonassociative extension of quaternions (or the split quaternions). They differ from octonions in the signature of the quadratic form. The split octonions [9] have signature $(4,4)$ whereas the octonions have positive signature $(8,0)$. The Cayley algebra of octonions over the field of complex numbers $\mathbb{C}_{\mathbb{C}} = \mathbb{C} \otimes \mathbb{C}$ is visualized as the algebra of split octonions with basis elements $u_0 = \frac{1}{2}(e_0 + ie_7)$, $u_0^\star = \frac{1}{2}(e_0 - ie_7)$, $u_j = \frac{1}{2}(e_j + ie_{j+3})$, $u_j^\star = \frac{1}{2}(e_j - ie_{j+3})$ $(j = 1, 2, 3;\ i = \sqrt{-1})$ as bi-valued (or bi-dimensional) representations of quaternion units e_0, e_1, e_2, e_3. An arbitrary split octonion A in terms of the following 2×2 Zorn vector matrix realization [9] is

$$A = au_0^\star + bu_0 + x_i u_i^\star + y_i u_i = \begin{pmatrix} a & -\vec{x} \\ \vec{y} & b \end{pmatrix}, \tag{22}$$

where $(\star)$ is used for complex conjugation. The split octonion conjugation of eq. (22) is then described as

$$\bar{A} = au_0 + bu_0^\star - x_i u_i^\star - y_i u_i = \begin{pmatrix} b & \vec{x} \\ -\vec{y} & a \end{pmatrix}. \tag{23}$$

The norm of A is defined as $\bar{A}A = A\bar{A} = (ab + \vec{x} \cdot \vec{y})\hat{1}$ with $\hat{1}$ as the unit matrix of order 2×2. Any four-vector A_μ (complex or real) can equivalently be written in terms of the following Zorn matrix realization as

$$Z(A) = \begin{pmatrix} x_4 & -\vec{x} \\ \vec{y} & y_4 \end{pmatrix}, \quad Z(\bar{A}) = \begin{pmatrix} x_4 & \vec{x} \\ -\vec{y} & y_4 \end{pmatrix}. \tag{24}$$

The split octonion differential operator D and its conjugate $\bar{D}$ in terms of the 2×2 Zorn matrix realization are,

$$D = \begin{pmatrix} \partial_4 & -\vec{\nabla} \\ \vec{\nabla}' & \partial_4' \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} \partial_4 & \vec{\nabla} \\ -\vec{\nabla}' & \partial_4' \end{pmatrix} \tag{25}$$

with the assumption that the primed variables are represented in internal T^4-space (i.e. $(\partial_4'; \vec{\nabla}')$ denote (x_7, x_4, x_5, x_6)) whereas the unprimed variables are defined in external four-dimensional R^4-space (i.e. $(\partial_4; \vec{\nabla})$ denote (x_0, x_1, x_2, x_3)) parts of an octonion.

3.1 *Potential equation*

Let us now write the the inhomogeneous octonion wave equation in its split form as

$$\bar{D}\emptyset = \begin{pmatrix} \partial_4 A_4 + \vec{\nabla}\cdot\vec{A} & -(\partial_4\vec{B} - \vec{\nabla}B_4 - \vec{\nabla}\times\vec{A}) \\ \partial_4'\vec{A} - \vec{\nabla}'A_4 - \vec{\nabla}'\times\vec{B} & \partial_4' B_4 + \vec{\nabla}'\cdot\vec{B} \end{pmatrix} = F, \tag{26}$$

where

$$F = \begin{pmatrix} f_4 & -\vec{f} \\ \vec{f}' & f_4' \end{pmatrix}, \quad \emptyset = \begin{pmatrix} A_4 & -\vec{B} \\ \vec{A} & B_4 \end{pmatrix}$$

and

$$f_4 = \partial_4 A_4 + \vec{\nabla}\cdot\vec{A} = 0, \quad f_4' = \partial_4' B_4 + \vec{\nabla}'\cdot\vec{B} = 0,$$
$$\vec{f} = \partial_4\vec{B} - \vec{\nabla}B_4 - \vec{\nabla}\times\vec{A}, \quad \vec{f}' = \partial_4'\vec{A} - \vec{\nabla}'A_4 - \vec{\nabla}'\times\vec{B}, \tag{27}$$

with the assumption that $A_\mu \equiv (A_4, \vec{A})$ and $B_\mu \equiv (B_4, \vec{B})$ are being used respectively in internal and external spaces.

3.2 *Bradyonic case*

Let us substitute $B_4 \longmapsto A_4$, $\vec{B} \longmapsto \vec{A}$ in eqs (25)–(27) and correspondingly $\partial_4 \longmapsto \partial_4'$, $\vec{\nabla} \longmapsto \vec{\nabla}'$ and we get

$$\bar{D}\emptyset = F = \begin{pmatrix} \partial_4 A_4 + \vec{\nabla}\cdot\vec{A} & -(\partial_4\vec{A} - \vec{\nabla}A_4 - \vec{\nabla}\times\vec{A}) \\ \partial_4\vec{A} - \vec{\nabla}A_4 - \vec{\nabla}\times\vec{A} & \partial_4 A_4 + \vec{\nabla}\cdot\vec{A} \end{pmatrix}. \tag{28}$$

By imposing Lorentz gauge condition $\partial_4 A_4 + \vec{\nabla}\cdot\vec{A} = \frac{\partial\emptyset_e}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0,$ and using $\vec{\psi} = \vec{E} - i\vec{\mathcal{H}}(i = \sqrt{-1})$, we get

$$-\partial_4\vec{A} + \vec{\nabla}A_4 + \vec{\nabla}\times\vec{A} = -i\vec{E} + \vec{\mathcal{H}} = -i\,\overrightarrow{\psi^\star}, \tag{29}$$

and

$$\partial_4\vec{A} - \vec{\nabla}A_4 - \vec{\nabla}\times\vec{A} = i\vec{E} - \vec{\mathcal{H}} = i\,\overrightarrow{\psi^\star}, \tag{30}$$

where we have used $A_\mu \equiv (A_4, \vec{A}) = (\emptyset_e, \vec{A})$, $\vec{E} = -\frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\emptyset_e$ and $\vec{\mathcal{H}} = \vec{\nabla}\times\vec{A}$ along with the natural units $c = \hbar = 1$. So, we get the simple and compact forms of split octonion wave equations in the following manner:

$$\bar{D}\emptyset = F = \begin{pmatrix} 0 & -i\,\overrightarrow{\psi^\star} \\ i\,\overrightarrow{\psi^\star} & 0 \end{pmatrix} \tag{31}$$

and

$$D\bar{\emptyset} = F' = \begin{pmatrix} 0 & i\,\overrightarrow{\psi^\star} \\ -i\,\overrightarrow{\psi^\star} & 0 \end{pmatrix}. \tag{32}$$

3.3 *Tachyonic case*

By substituting $A_4 \longmapsto B_4$, $\vec{A} \longmapsto \vec{B}$ in eqs (25)–(27), we get the representation of T^4-space. Thus, we obtain

$$\bar{D}\emptyset = F' = \begin{pmatrix} \partial_4' B_4 + \vec{\nabla}\cdot\vec{B} & -(\partial_4'\vec{B} - \vec{\nabla}' B_4 - \vec{\nabla}'\times\vec{B}) \\ \partial_4'\vec{B} - \vec{\nabla} B_4 - \vec{\nabla}'\times\vec{B} & \partial_4' B_4 + \vec{\nabla}'\cdot\vec{B} \end{pmatrix}. \tag{33}$$

We get

$$-\partial_4'\vec{B} + \vec{\nabla}' B_4 + \vec{\nabla}'\times\vec{B} = -i\vec{E}_t + \vec{H}_t = -i\overrightarrow{\psi_{\rm T}^{\star}} \tag{34}$$

and

$$\partial_4'\vec{B} - \vec{\nabla} B_4 - \vec{\nabla}'\times\vec{B} = i\vec{E}_t - \vec{H}_t = i\overrightarrow{\psi_{\rm T}^{\star}}. \tag{35}$$

Here $\overrightarrow{\psi_{\rm T}} = \overrightarrow{E_{\rm T}} - i\overrightarrow{H_{\rm T}}$ (T denotes tachyonic representations). Then eq. (33) reduces to

$$F' = \begin{pmatrix} 0 & -i\overrightarrow{\psi_{\rm T}^{\star}} \\ i\overrightarrow{\psi_{\rm T}^{\star}} & 0 \end{pmatrix}. \tag{36}$$

Here also we have used the analogous Lorentz gauge condition in T^4-space in order to maintain the structural symmetry between internal and external spaces. Equation (33) is the split octonion form of the field tensor $F'_{\mu\nu} = \partial'_\mu B'_\nu - \partial'_\nu B'_\mu$ components which describe the electric and magnetic fields in T^4-space. Similarly, we obtain

$$D\bar{\emptyset} = F'_{\rm T}, \tag{37}$$

where

$$F'_{\rm T} = \begin{pmatrix} 0 & i\overrightarrow{\psi_{\rm T}^{\star}} \\ -i\overrightarrow{\psi_{\rm T}^{\star}} & 0 \end{pmatrix}. \tag{38}$$

4. Field equation

Let us write the split octonion wave equation as

$$DF = J, \tag{39}$$

where J is the split octonion form of the current source density given by

$$J = \begin{pmatrix} J_4 & -\vec{J} \\ \vec{J}' & J_4' \end{pmatrix}. \tag{40}$$

So, we get

$$DF = \begin{pmatrix} \partial_4 f_4 - \vec{\nabla} \cdot \vec{f}\,' & -(\partial_4 \vec{f} + \vec{\nabla} f_4' + \vec{\nabla}' \times f') \\ \partial_4' \vec{f}\,' + \vec{\nabla}' f_4 + \vec{\nabla} \times \vec{f} & \partial_4' f_4' - \vec{\nabla}' \cdot \vec{f} \end{pmatrix}. \tag{41}$$

Comparing eqs (39)–(41), we get

$$\begin{aligned} J_4 &= \partial_4 f_4 - \vec{\nabla} \cdot \vec{f}\,', \\ J_4' &= \partial_4' f_4' - \vec{\nabla}' \cdot \vec{f}, \\ \vec{J} &= \partial_4 \vec{f} + \vec{\nabla} f_4' + \vec{\nabla}' \times \vec{f}\,', \\ \vec{J}' &= \partial_4' \vec{f}\,' + \vec{\nabla}' f_4 + \vec{\nabla} \times \vec{f}. \end{aligned} \tag{42}$$

Let us discuss the following different cases of generalized field equation.

4.1 *Bradyonic* (*subluminal*) *case*

For the description of bradyons, let us substitute $B_4 \longmapsto A_4, \vec{B} \longmapsto \vec{A}, \partial_4' \longmapsto \partial_4, \vec{\nabla}' \longmapsto \vec{\nabla}$, $\vec{f'} \longmapsto \vec{f}$, $f_4' \longmapsto f_4$ and $J_4' \longmapsto J_4$, $\vec{J'} \longmapsto \vec{J}$. Then we get

$$DF = \begin{pmatrix} \vec{\nabla} \cdot \vec{\mathcal{H}} + i\vec{\nabla} \cdot \vec{E} & i\frac{\partial \vec{\mathcal{H}}}{\partial t} + i\vec{\nabla} \times \vec{E} - \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{\mathcal{H}} \\ -i\frac{\partial \vec{\mathcal{H}}}{\partial t} - i\vec{\nabla} \times \vec{E} + \frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \times \vec{\mathcal{H}} & \vec{\nabla} \cdot \vec{\mathcal{H}} + i\vec{\nabla} \cdot \vec{E} \end{pmatrix} = J. \tag{43}$$

4.2 *Tachyonic (superluminal) case*

In this, we may substitute $A_4 \longmapsto B_4$, $\vec{A} \longmapsto \vec{B}$, $\partial_4 \longmapsto \partial_4'$, $\vec{\nabla} \longmapsto \vec{\nabla}'$,$\vec{f} \longmapsto \vec{f'}$, $f_4 \longmapsto f_4'$ and $J_4 \longmapsto J_4'$, $\vec{J} \longmapsto \vec{J}\,'$ and accordingly, we get

$$DF' = \begin{pmatrix} J_4' & -\vec{J}\,' \\ \vec{J} & J_4 \end{pmatrix} = J'. \tag{44}$$

This equation is the split octonionic form of the Maxwell equation $F_{\mu\nu} = J_\mu$ in T^4-space.

5. Conclusion

In order to interpret octonion wave equations in eight-dimensional space-time, here, we have made an attempt to discuss the compact and simpler forms of octonion field equations as the unified picture of equations of motion for particles carrying simultaneously subluminal (bradyonic) and superluminal (tachyonic) objects. Visualizing the external four spaces as the localization space for bradyons and the internal space as the localization space for tachyons, it is shown that the octonion wave equation reduces to the Maxwell's equation (field equation) for bradyons in R^4-space as well as tachyons in T^4-space. Then we have made an attempt to investigate the split octonion wave in terms of Zorn vector matrix realization by describing electrodynamics potential, current and other dynamical quantities as octonion variables. It has been shown that split octonion Zorn vector realization reproduces two different spaces to demonstrate the wave equations for electromagnetic fields associated with bradyons and tachyons. Visualizing the external (internal) four-dimensional space as the localization space for bradyons (tachyons), we have concluded that the entire octonion wave equation when expressed in terms of split octonions, reduces to the Maxwell (field) equation for bradyons (tachyons) in R^4-space (T^4-space) in the absence of other. As such the eight-dimensional octonion wave equation describes well the representation of localization of subluminal (bradyonic) and superluminal (tachyonic) objects (particles) consistently.

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