

## Random matrix ensembles with random interactions: Results for EGUE(2)- $SU(4)$

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**Abstract.** We introduce in this paper embedded Gaussian unitary ensemble of random matrices, for  $m$  fermions in  $\Omega$  number of single particle orbits, generated by random two-body interactions that are  $SU(4)$  scalar, called EGUE(2)- $SU(4)$ . Here the  $SU(4)$  algebra corresponds to Wigner's supermultiplet  $SU(4)$  symmetry in nuclei. Formulation based on Wigner–Racah algebra of the embedding algebra  $U(4\Omega) \supset U(\Omega) \otimes SU(4)$  allows for analytical treatment of this ensemble and using this analytical formulas are derived for the covariances in energy centroids and spectral variances. It is found that these covariances increase in magnitude as we go from EGUE(2) to EGUE(2)-s to EGUE(2)- $SU(4)$  implying that symmetries may be responsible for chaos in finite interacting quantum systems.

**Keywords.** Embedded ensembles; random interactions; EGUE(2); EGUE(2)-s; EGUE(2)- $SU(4)$ ; Wigner–Racah algebra; covariances; chaos.

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### 1. Introduction

Hamiltonians for finite quantum systems such as nuclei, atoms, quantum dots, small metallic grains, interacting spin systems modelling quantum computing core, Bose condensates and so on consist of interactions of low body rank and therefore embedded Gaussian ensembles (EGE) of random matrices generated by random interactions, first introduced in 1970 in the context of nuclear shell model and explored to some extent in the 1970s and 1980s, are appropriate for these systems. Note that EGEs that correspond to the classical ensembles GOE, GUE and GSE are EGOE, EGUE and EGSE respectively. With the interest in many-body chaos, EGEs received new emphasis from 1996 and since then a wide variety of EGEs have been introduced in literature, both for fermion and boson systems [1–4] (see [1,2,5,6] and references therein for recent applications of EGEs).

EGEs generated by two-body interactions [EGE(2)] for spinless fermion systems are the simplest of these ensembles. For  $m$  fermions in  $N$  single-particle (sp) states, the embedding algebra is  $SU(N)$ . It is well established that  $SU(N)$  Wigner–Racah algebra solves EGUE(2) and also the more general EGUE( $k$ ) as well as EGOE( $k$ )

[7,8]. Realistic systems carry good quantum numbers (for example spin  $S$  for quantum dots, angular momentum  $J$  for nuclei) in addition to particle number  $m$ . Therefore EGEs with good symmetries should be studied. EGUE(2)- $\mathbf{s}$  and EGOE(2)- $\mathbf{s}$ , for fermions with spin  $\mathbf{s} = \frac{1}{2}$  degree of freedom, are the simplest non-trivial EGEs with immediate physical applications. For  $m$  fermions occupying  $\Omega$  number of orbits with total spin  $S$  a good quantum number, the embedding algebra for EGUE(2)- $\mathbf{s}$  and also for EGOE(2)- $\mathbf{s}$  is  $U(2\Omega) \supset U(\Omega) \otimes SU(2)$  [5,9]. In particular the EGOE(2)- $\mathbf{s}$  with its extension including mean-field one body part has been extensively used in the study of quantum dots, small metallic grains and atomic nuclei [6,10,11].

Wigner introduced in 1937 [12] the spin-isospin  $SU(4)$  supermultiplet scheme for nuclei. There is good evidence for the goodness of this symmetry in some parts of the periodic table [13] and also more recently there is a new interest in  $SU(4)$  symmetry for heavy  $N \sim Z$  nuclei [14]. Therefore, it is clearly of importance to define and study embedded Gaussian unitary ensemble of random matrices generated by random two-body interactions with  $SU(4)$  symmetry, hereafter called EGUE(2)- $SU(4)$ . Given  $m$  fermions (nucleons) in  $\Omega$  number of sp levels with spin and isospin degrees of freedom, for  $SU(4)$  scalar Hamiltonians, the symmetry algebra is  $U(4\Omega) \supset U(\Omega) \otimes SU(4)$  and all the states within an  $SU(4)$  irrep will be degenerate in energy. Our purpose in this paper is to define EGUE(2)- $SU(4)$ , develop analytical formulation for solving the ensemble and report the first results for lower-order cross-correlations generated by this ensemble. Now we will give a preview.

Section 2 gives a brief discussion of  $SU(4)$  algebra. EGUE(2)- $SU(4)$  ensemble is defined in §3. Also given here is the mathematical formulation based on Wigner-Racah algebra of the embedding  $U(4\Omega) \supset U(\Omega) \otimes SU(4)$  algebra for solving the ensemble. In §4, analytical formulas for  $m$  fermion  $U(\Omega)$  irreps  $f_m = \{4^r, p\}$  are given for the covariances in energy centroids and spectral variances generated by this ensemble. Section 5 gives discussion of some numerical results. Finally §6 gives summary and future outlook.

## 2. Preliminaries of $U(4\Omega) \supset U(\Omega) \otimes SU(4)$ algebra

Let us begin with  $m$  nucleons distributed in  $\Omega$  number of orbits each with spin ( $\mathbf{s} = \frac{1}{2}$ ) and isospin ( $\mathbf{t} = \frac{1}{2}$ ) degrees of freedom. Then the total number of sp states  $N = 4\Omega$  and the spectrum generating algebra is  $U(4\Omega)$ . The sp states in uncoupled representation are  $a_{i,\alpha}^\dagger |0\rangle = |i, \alpha\rangle$  with  $i = 1, 2, \dots, \Omega$  denoting the spatial orbits and  $\alpha = 1, 2, 3, 4$  are the four spin-isospin states  $|m_{\mathbf{s}}, m_{\mathbf{t}}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle, |-\frac{1}{2}, \frac{1}{2}\rangle$  and  $|-\frac{1}{2}, -\frac{1}{2}\rangle$  respectively. The  $(4\Omega)^2$  number of operators  $C_{i\alpha;j\beta}$  generate  $U(4\Omega)$  algebra. For  $m$  fermions, all states belong to the  $U(4\Omega)$  irrep  $\{1^m\}$ . In uncoupled notation,  $C_{i\alpha;j\beta} = a_{i,\alpha}^\dagger a_{j,\beta}$ . Similarly,  $U(\Omega)$  and  $U(4)$  algebras are generated by  $A_{ij}$  and  $B_{\alpha\beta}$  respectively, where  $A_{ij} = \sum_{\alpha=1}^4 C_{i\alpha;j\alpha}$  and  $B_{\alpha\beta} = \sum_{i=1}^{\Omega} C_{i\alpha;i\beta}$ . The number operator  $\hat{n}$ , the spin operator  $\hat{S} = S_{\mu}^1$ , the isospin operator  $\hat{T} = T_{\mu}^1$  and the Gamow-Teller operator  $\sigma\tau = (\sigma\tau)_{\mu,\mu'}^{1,1}$  of  $U(4)$  in spin-isospin coupled notation are [15],

$$\begin{aligned} \hat{n} &= 2 \sum_i \mathcal{A}_{ii;0,0}^{0,0}, \quad S_\mu^1 = \sum_i \mathcal{A}_{ii;\mu,0}^{1,0}, \quad T_\mu^1 = \sum_i \mathcal{A}_{ii;0,\mu}^{0,1}, \\ (\sigma\tau)_{\mu,\mu'}^{1,1} &= \sum_i \mathcal{A}_{ii;\mu,\mu'}^{1,1}; \quad \mathcal{A}_{ij;\mu_s,\mu_t}^{s,t} = \left( a_i^\dagger \tilde{a}_j \right)_{\mu_s,\mu_t}^{s,t}. \end{aligned} \quad (1)$$

Note that  $\tilde{a}_{j;\mu_s,\mu_t} = (-1)^{1+\mu_s+\mu_t} a_{j;-\mu_s,-\mu_t}$ . These 16 operators form  $U(4)$  algebra. Dropping the number operator, we have  $SU(4)$  algebra.

For the  $U(4)$  algebra, the irreps are characterized by the partitions  $\{F\} = \{F_1, F_2, F_3, F_4\}$  with  $F_1 \geq F_2 \geq F_3 \geq F_4 \geq 0$  and  $m = \sum_{i=1}^4 F_i$ . Note that  $F_\alpha$  are the eigenvalues of  $B_{\alpha\alpha}$ . Due to the antisymmetry constraint on the total wave function, the orbital space  $U(\Omega)$  irreps  $\{f\}$  are given by  $\{f\} = \{\tilde{F}\}$  which is obtained by changing rows to columns in  $\{F\}$ . It is important to note that, due to this symmetry constraint, for the irrep  $\{F\}$  each  $F_j \leq \Omega$  where  $j = 1, 2, 3, 4$  and for the irrep  $\{f\}$  each  $f_i \leq 4$  with  $i = 1, 2, \dots, \Omega$ . The irreps for the  $SU(4)$  group are characterized by three rowed Young shapes  $\{F'\} = \{F'_1, F'_2, F'_3\} = \{F_1 - F_4, F_2 - F_4, F_3 - F_4\}$ . Also they can be mapped to  $SO(6)$  irreps  $[P_1, P_2, P_3]$  as the  $SU(4)$  and  $SO(6)$  algebras are isomorphic to each other,  $[P_1, P_2, P_3] = [(F_1 + F_2 - F_3 - F_4)/2, (F_1 - F_2 + F_3 - F_4)/2, (F_1 - F_2 - F_3 + F_4)/2]$ . Before proceeding further, let us examine the quadratic Casimir invariants of  $U(\Omega)$ ,  $U(4)$ ,  $SU(4)$  and  $SO(6)$  algebras. For example,

$$\begin{aligned} C_2[U(\Omega)] &= \sum_{i,j} A_{ij} A_{ji} = \hat{n}\Omega - \sum_{i,j,\alpha,\beta} a_{i,\alpha}^\dagger a_{j,\beta}^\dagger a_{j,\alpha} a_{i,\beta}, \\ C_2[U(4)] &= \sum_{\alpha,\beta} B_{\alpha,\beta} B_{\beta,\alpha} \Rightarrow C_2[U(\Omega)] + C_2[U(4)] = \hat{n}(\Omega + 4). \end{aligned} \quad (2)$$

Also, in terms of spin, isospin and Gamow-Teller operators,  $C_2[SU(4)] = C_2[SO(6)] = S^2 + T^2 + (\sigma\tau) \cdot (\sigma\tau)$ . Now we have the general results,

$$\begin{aligned} \langle C_2[U(4)] \rangle^{\{F\}} &= \sum_{i=1}^4 F_i (F_i + 5 - 2i) = \left\langle C_2[SU(4)] + \frac{\hat{n}^2}{4} \right\rangle^{\{F'\}}, \\ \langle C_2[SO(6)] \rangle^{[P]} &= \langle C_2[SU(4)] \rangle^{\{F'\}} = P_1(P_1 + 4) + P_2(P_2 + 2) + P_3^2. \end{aligned} \quad (3)$$

In order to understand the significance of  $SU(4)$  symmetry, let us consider the space exchange or the Majorana operator  $M$  that exchanges the spatial coordinates of the particles and leaves the spin-isospin quantum numbers unchanged,

$$M|i, \alpha, \alpha'; j, \beta, \beta'\rangle = |j, \alpha, \alpha'; i, \beta, \beta'\rangle, \quad (4)$$

where  $\alpha, \beta$  are labels for spin and  $\alpha', \beta'$  are labels for isospin. As  $|i, \alpha, \alpha'; j, \beta, \beta'\rangle = a_{i,\alpha,\alpha'}^\dagger a_{j,\beta,\beta'}^\dagger |0\rangle$ , eqs (4), (2) and (3) in that order will give

$$\begin{aligned} 2M &= \sum_{i,j,\alpha,\beta,\alpha',\beta'} (a_{j,\alpha,\alpha'}^\dagger a_{i,\beta,\beta'}^\dagger) (a_{i,\alpha,\alpha'}^\dagger a_{j,\beta,\beta'}^\dagger)^\dagger \\ &= C_2[U(\Omega)] - \Omega\hat{n} = 4\hat{n} - C_2[U(4)] \\ &\Rightarrow \alpha M = \alpha \left\{ 2\hat{n} \left( 1 + \frac{\hat{n}}{16} \right) - \frac{1}{2} C_2[SU(4)] \right\}. \end{aligned} \quad (5)$$

The preferred  $U(\Omega)$  irrep for the ground state of a  $m$  nucleon system is the most symmetric one. Therefore,  $\langle C_2[U(\Omega)] \rangle$  should be maximum for the ground state irrep. This implies, as seen from eq. (5), that the strength  $\alpha$  of  $M$  must be negative. As a consequence, as follows from the last equality in eq. (5), the ground states are labelled by  $SU(4)$  irreps with the smallest eigenvalue for the quadratic Casimir invariant consistent with a given  $(m, T_z)$ ,  $T = |T_z|$ .

For even–even nuclei, the ground state  $SO(6)$  irreps are  $T = |N - Z|/2$ . For odd–odd  $N = Z$  nuclei, the ground state is  $[1]$  and for  $N \neq Z$  nuclei, it is  $[T, 1]$ . For odd-A nuclei, the irreps are  $[T, \frac{1}{2}, \pm \frac{1}{2}]$ . Therefore, for  $N = Z$  even–even,  $N = Z$  odd–odd and  $N = Z \pm 1$  odd-A nuclei the  $U(\Omega)$  irreps for the ground states are  $\{4^r\}$ ,  $\{4^r, 2\}$ ,  $\{4^r, 1\}$  and  $\{4^r, 3\}$  with spin–isospin structure being  $(0,0)$ ,  $(1, 0) \oplus (0, 1)$ ,  $(\frac{1}{2}, \frac{1}{2})$  and  $(\frac{1}{2}, \frac{1}{2})$  respectively. For simplicity, in this paper we will present final results only for these  $U(\Omega)$  irreps. Other irreps will be considered elsewhere. Now we will define EGUE(2)- $SU(4)$  ensemble and derive some of its properties.

### 3. EGUE(2)- $SU(4)$ ensemble: Definition and $U(4\Omega) \supset U(\Omega) \otimes SU(4)$ Wigner–Racah algebra for covariances

Here we follow closely the approach used for EGUE(2)-s recently [9]. Let us begin with normalized two-particle states  $|f_2 F_2; v_2 \beta_2\rangle$  where the  $U(4)$  irreps  $F_2 = \{1^2\}$  and  $\{2\}$  and the corresponding  $U(\Omega)$  irreps  $f_2$  are  $\{2\}$  (symmetric) and  $\{1^2\}$  (antisymmetric) respectively. Similarly,  $v_2$  are additional quantum numbers that belong to  $f_2$  and  $\beta_2$  belong to  $F_2$ . As  $f_2$  uniquely defines  $F_2$ , from now on we will drop  $F_2$  unless it is explicitly needed and also we will use the  $f_2 \leftrightarrow F_2$  equivalence whenever needed. With  $A^\dagger(f_2 v_2 \beta_2)$  and  $A(f_2 v_2 \beta_2)$  denoting creation and annihilation operators for the normalized two-particle states, a general two-body Hamiltonian  $H$  preserving  $SU(4)$  symmetry can be written as

$$H = \sum_{f_2, v_2^i, v_2^f, \beta_2} V_{f_2 v_2^i v_2^f}(2) A^\dagger(f_2 v_2^f \beta_2) A(f_2 v_2^i \beta_2). \quad (6)$$

In eq. (6),  $V_{f_2 v_2^i v_2^f}(2) = \langle f_2 v_2^f \beta_2 | H | f_2 v_2^i \beta_2 \rangle$  independent of the  $\beta_2$ s. For EGUE(2)- $SU(4)$  the  $V_{f_2 v_2^i v_2^f}$  are independent Gaussian variables with zero centre and variance given by (with bar representing ensemble average),

$$\overline{V_{f_2 v_2^1 v_2^2}(2) V_{f_2' v_2^3 v_2^4}(2)} = \delta_{f_2 f_2'} \delta_{v_2^1 v_2^3} \delta_{v_2^2 v_2^4} (\lambda_{f_2})^2. \quad (7)$$

Thus  $V(2)$  is a direct sum of GUE matrices for  $F_2 = \{2\}$  and  $F_2 = \{1^2\}$  with variances  $(\lambda_{f_2})^2$  for the diagonal matrix elements and  $(\lambda_{f_2})^2/2$  for the real and imaginary parts of the off-diagonal matrix elements. As discussed before for EGUE( $k$ ) [7] and EGUE(2)-s [9], tensorial decomposition of  $H$  with respect to the embedding algebra  $U(\Omega) \otimes SU(4)$  plays a crucial role in generating analytical results; as in [7], the  $U(\Omega) \leftrightarrow SU(\Omega)$  correspondence is used throughout and therefore we use  $U(\Omega)$  and  $SU(\Omega)$  interchangeably. As  $H$  preserves  $SU(4)$ , it is a scalar in the  $SU(4)$  space. However, with respect to  $SU(\Omega)$ , the tensorial characters, in Young tableaux notation, for  $f_2 = \{2\}$  are  $\mathbf{F}_\nu = \{0\}$ ,  $\{21^{\Omega-2}\}$  and  $\{42^{\Omega-2}\}$  with  $\nu = 0, 1$  and 2

respectively. Similarly, for  $f_2 = \{1^2\}$  they are  $\mathbf{F}_\nu = \{0\}, \{21^{\Omega-2}\}$  and  $\{2^21^{\Omega-4}\}$  with  $\nu = 0, 1, 2$  respectively. Note that  $\mathbf{F}_\nu = f_2 \times \bar{f}_2$  where  $\bar{f}_2$  is the irrep conjugate to  $f_2$  and the  $\times$  denotes Kronecker product. Then we can define unitary tensors  $B$ s that are scalars in  $SU(4)$  space,

$$B(f_2\mathbf{F}_\nu\omega_\nu) = \sum_{v_2^i, v_2^f, \beta_2} A^\dagger(f_2v_2^f\beta_2)A(f_2v_2^i\beta_2) \langle f_2v_2^f\bar{f}_2\bar{v}_2^i | \mathbf{F}_\nu\omega_\nu \rangle \times \langle F_2\beta_2 \bar{F}_2\bar{\beta}_2 | 00 \rangle. \quad (8)$$

In eq. (8),  $\langle f_2 - - - \rangle$  are  $SU(\Omega)$  Wigner coefficients and  $\langle F_2 - - - \rangle$  are  $SU(4)$  Wigner coefficients. The expansion of  $H$  in terms of  $B$ s is,

$$H = \sum_{f_2, \mathbf{F}_\nu, \omega_\nu} W(f_2\mathbf{F}_\nu\omega_\nu)B(f_2\mathbf{F}_\nu\omega_\nu). \quad (9)$$

The expansion coefficients  $W$ s follow from the orthogonality of the tensors  $B$ s with respect to the traces over fixed  $f_2$  spaces. Then we have the most important relation needed for all the results given ahead,

$$\overline{W(f_2\mathbf{F}_\nu\omega_\nu)W(f_2'\mathbf{F}'_\nu\omega'_\nu)} = \delta_{f_2f_2'}\delta_{\mathbf{F}_\nu\mathbf{F}'_\nu}\delta_{\omega_\nu\omega'_\nu}(\lambda_{f_2})^2d(F_2). \quad (10)$$

This is derived starting with eq. (7) and substituting, in two-particle matrix elements  $V$ , for  $H$  the expansion given by eq. (9). Also used are the sum rules for Wigner coefficients appearing in eq. (8).

Turning to  $m$  particle  $H$  matrix elements, first we denote the  $U(\Omega)$  and  $U(4)$  irreps by  $f_m$  and  $F_m$  respectively. Correlations generated by EGUE(2)- $SU(4)$  between states with  $(m, f_m)$  and  $(m', f_{m'})$  follow from the covariance between the  $m$ -particle matrix elements of  $H$ . Now using eqs (9) and (10) along with the Wigner–Eckart theorem applied using  $SU(\Omega) \otimes SU(4)$  Wigner–Racah algebra (see for example [16]) will give

$$\begin{aligned} & \overline{H_{f_m v_m^i v_m^f} H_{f_{m'} v_{m'}^i v_{m'}^f}} \\ &= \overline{\langle f_m F_m v_m^f \beta | H | f_m F_m v_m^i \beta \rangle \langle f_{m'} F_{m'} v_{m'}^f \beta' | H | f_{m'} F_{m'} v_{m'}^i \beta' \rangle} \\ &= \sum_{f_2, \mathbf{F}_\nu, \omega_\nu} \frac{(\lambda_{f_2})^2}{d(f_2)} \sum_{\rho, \rho'} \langle f_m ||| B(f_2\mathbf{F}_\nu) ||| f_m \rangle_\rho \langle f_{m'} ||| B(f_2\mathbf{F}_\nu) ||| f_{m'} \rangle_{\rho'} \\ & \quad \times \langle f_m v_m^i \mathbf{F}_\nu \omega_\nu | f_m v_m^f \rangle_\rho \langle f_{m'} v_{m'}^i \mathbf{F}_\nu \omega_\nu | f_{m'} v_{m'}^f \rangle_{\rho'}; \\ & \quad \langle f_m ||| B(f_2\mathbf{F}_\nu) ||| f_m \rangle_\rho = \sum_{f_{m-2}} F(m) \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{U(f_m \bar{f}_2 f_m f_2; f_{m-2} \mathbf{F}_\nu)_\rho}{U(f_m \bar{f}_2 f_m f_2; f_{m-2} \{0\})}. \end{aligned} \quad (11)$$

Here the summation in the last equality is over the multiplicity index  $\rho$  and this arises as  $f_m \otimes \mathbf{F}_\nu$  gives in general more than once the irrep  $f_m$ . In eq. (11),  $F(m) = -m(m-1)/2$ ,  $d(f_m)$  is dimension with respect to  $U(\Omega)$  and  $\mathcal{N}_{f_m}$  is dimension with respect to the  $S_m$  group; formulas for these dimensions are given in [17]. Similarly,  $\langle \dots \rangle$  and  $U(\dots)$  are  $SU(\Omega)$  Wigner and Racah coefficients respectively.

**Table 1.**  $P^{f_2}(m, f_m)$  for  $f_m = \{4^r, p\}$ ;  $p = 0, 1, 2$  and  $3$  and  $\{f_2\} = \{2\}, \{1^2\}$ .

$f_m$	$P^{f_2}(m, f_m)$	
	$f_2 = \{2\}$	$f_2 = \{1^2\}$
$\{4^r\}$	$-3r(r+1)$	$-5r(r-1)$
$\{4^r, 1\}$	$-\frac{3r}{2}(2r+3)$	$-\frac{5r}{2}(2r-1)$
$\{4^r, 2\}$	$-(3r^2+6r+1)$	$-5r^2$
$\{4^r, 3\}$	$-\frac{3}{2}(r+2)(2r+1)$	$-\frac{5r}{2}(2r+1)$

#### 4. Lower-order cross-correlations in EGUE(2)-SU(4)

Lower-order cross-correlations between states with different  $(m, f_m)$  are given by the normalized bivariate moments  $\Sigma_{rr}(m, f_m : m', f_{m'})$ ,  $r = 1, 2$  of the two-point function  $S^r$  where, with  $\rho^{m, f_m}(E)$  defining fixed- $(m, f_m)$  density of states,

$$S^{m f_m : m' f_{m'}}(E, E') = \overline{\rho^{m, f_m}(E) \rho^{m', f_{m'}}(E')} - \overline{\rho^{m, f_m}(E)} \overline{\rho^{m', f_{m'}}(E')},$$

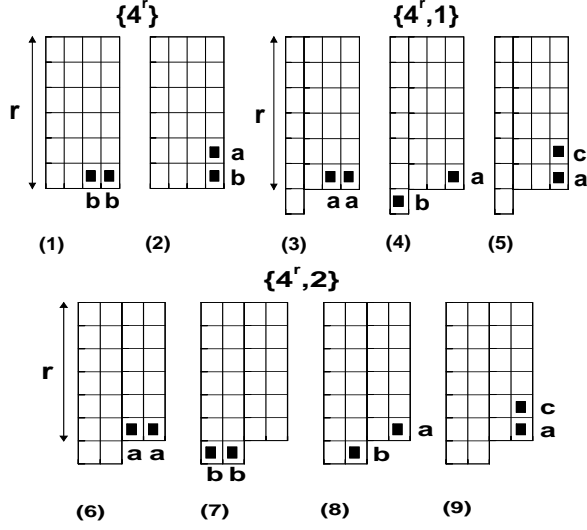
$$\begin{aligned} \Sigma_{11}(m, f_m : m', f_{m'}) &= \frac{\overline{\langle H \rangle^{m, f_m} \langle H \rangle^{m', f_{m'}}}}{\sqrt{\overline{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}}}}, \\ \Sigma_{22}(m, f_m : m', f_{m'}) &= \frac{\overline{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}}}{[\overline{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}}] - 1}. \end{aligned} \tag{12}$$

In eq. (12),  $\overline{\langle H^2 \rangle^{m, f_m}}$  is the second moment (or variance) of  $\overline{\rho^{m, f_m}(E)}$  and its centroid  $\overline{\langle H \rangle^{m, f_m}} = 0$  by definition. We begin with  $\overline{\langle H \rangle^{m, f_m} \langle H \rangle^{m', f_{m'}}$ . As  $\langle H \rangle^{m, f_m}$  is the trace of  $H$  (divided by dimensionality) in  $(m, f_m)$  space, only  $\mathbf{F}_\nu = \{0\}$  will generate this. Then trivially,

$$\overline{\langle H \rangle^{m, f_m} \langle H \rangle^{m', f_{m'}}} = \sum_{f_2} \frac{(\lambda_{f_2})^2}{d(f_2)} P^{f_2}(m, f_m) P^{f_2}(m', f_{m'}). \tag{13}$$

Note that  $P^{f_2}(m, f_m) = F(m) \sum_{f_{m-2}} [\mathcal{N}_{f_{m-2}} / \mathcal{N}_{f_m}]$ . The formulas for  $P^{f_2}(m, f_m)$  are given in table 1. Writing  $\overline{\langle H^2 \rangle^{m, f_m}}$  explicitly in terms of  $m$  particle  $H$  matrix elements,  $\overline{\langle H^2 \rangle^{m, f_m}} = [d(f_m)]^{-1} \sum_{v_m^1, v_m^2} \overline{H_{f_m v_m^1 v_m^2} H_{f_m v_m^2 v_m^1}}$ , and applying eq. (11) and the orthonormal properties of the  $SU(\Omega)$  Wigner coefficients lead to

$$\overline{\langle H^2 \rangle^{m, f_m}} = \sum_{f_2} \frac{(\lambda_{f_2})^2}{d(f_2)} \sum_{\nu=0,1,2} \mathcal{Q}^\nu(f_2 : m, f_m). \tag{14}$$



**Figure 1.** Schematic representation of the Young tableaux  $f_m = \{4^r, p\}$  with  $p = 0, 1$  and  $2$ . (1)  $f_m = \{4^r\}$ ,  $f_2 = \{2\}$ ,  $f_{m-2} = \{4^{r-1}, 2\}$  and here  $a = b$ ; (2)  $f_m = \{4^r\}$ ,  $f_2 = \{1^2\}$ ,  $f_{m-2} = \{4^{r-2}, 3^2\}$  and  $\tau_{ab} = 1$ ; (3)  $f_m = \{4^r, 1\}$ ,  $f_2 = \{2\}$ ,  $f_{m-2} = \{4^{r-1}, 2, 1\}$  and here  $a = b$ ; (4)  $f_m = \{4^r, 1\}$ ,  $f_2 = \{2\}, \{1^2\}$ ,  $f_{m-2} = \{4^{r-1}, 3\}$  and  $\tau_{ab} = 4$ ; (5)  $f_m = \{4^r, 1\}$ ,  $f_2 = \{1^2\}$ ,  $f_{m-2} = \{4^{r-2}, 3^2, 1\}$  and  $\tau_{ac} = -1$ ; (6)  $f_m = \{4^r, 2\}$ ,  $f_2 = \{2\}$ ,  $f_{m-2} = \{4^{r-1}, 2^2\}$  and here  $a = b$ ; (7)  $f_m = \{4^r, 2\}$ ,  $f_2 = \{2\}$ ,  $f_{m-2} = \{4^r\}$  and here  $a = b$ ; (8)  $f_m = \{4^r, 2\}$ ,  $f_2 = \{2\}, \{1^2\}$ ,  $f_{m-2} = \{4^{r-1}, 3, 1\}$ ,  $\tau_{ab} = 3$ ; (9)  $f_m = \{4^r, 2\}$ ,  $f_2 = \{1^2\}$ ,  $f_{m-2} = \{4^{r-2}, 3^2, 2\}$  and  $\tau_{ac} = -1$ .

The functions  $\mathcal{Q}^\nu(f_2; m, f_m)$  involve  $SU(\Omega)$  Racah coefficients and they are available in various tables in a complex form involving functions of  $\tau_{ab}$  [18]. Here  $\tau_{ab}$  are the axial distances for a given Young tableaux (see figure 1 for example). Evaluating all the functions, we have derived analytical formulas for  $\mathcal{Q}^\nu(f_2; m, f_m)$  and also for  $\langle H^2 \rangle^{m, f_m}$ . Some of these results are given in table 2. It is easily seen that  $\mathcal{Q}^{\nu=0}(f_2; m, f_m) = [P^{f_2}(m, f_m)]^2$ . Results in tables 1 and 2 will give formulas for the covariances  $\Sigma_{11}$  in energy centroids. Similarly, analytical results for covariances  $\Sigma_{22}$  in spectral variances are derived using eqs (11) and (12) and then,

$$\Sigma_{22}(m, f_m; m', f_{m'}) = \frac{X_{\{2\}} + X_{\{1^2\}} + 4X_{\{1^2\}\{2\}}}{\langle H^2 \rangle^{m, f_m} \langle H^2 \rangle^{m', f_{m'}}},$$

$$X_{f_2} = \frac{2(\lambda_{f_2})^4}{[d(f_2)]^2} \sum_{\nu=0,1,2} [d(\mathbf{F}_\nu)]^{-1} \mathcal{Q}^\nu(f_2; m, f_m) \mathcal{Q}^\nu(f_2; m', f_{m'}),$$

$$X_{\{1^2\}\{2\}} = \frac{\lambda_{\{2\}}^2 \lambda_{\{1^2\}}^2}{d(\{2\})d(\{1^2\})} \sum_{\nu=0,1} [d(\mathbf{F}_\nu)]^{-1} R^\nu(m, f_m) R^\nu(m', f_{m'}). \quad (15)$$

Here  $d(\mathbf{F}_\nu)$  is the dimension of the irrep  $\mathbf{F}_\nu$ , and we have  $d(\{0\}) = 1$ ,  $d(\{2, 1^{\Omega-2}\}) = \Omega^2 - 1$ ,  $d(\{4, 2^{\Omega-2}\}) = \Omega^2(\Omega + 3)(\Omega - 1)/4$ , and  $d(\{2^2, 1^{\Omega-4}\}) = \Omega^2(\Omega - 3)(\Omega +$

**Table 2.**  $\overline{\langle H^2 \rangle}^{m, f_m}$ ,  $Q^{\nu=1,2}(f_2: m, f_m)$  and  $R^{\nu=1}(m, f_m)$  for some examples.

$f_m$	$\overline{\langle H^2 \rangle}^{m, f_m}$
$\{4^r\}$	$\frac{r(\Omega-r+4)}{2} [\lambda_{\{2\}}^2 3(r+1)(\Omega-r+3) + \lambda_{\{1^2\}}^2 5(r-1)(\Omega-r+5)]$
$\{4^r, 1\}$	$\frac{r(\Omega-r+4)}{4} [\lambda_{\{2\}}^2 \{6r(\Omega-r+1) + 9\Omega + 15\} + \lambda_{\{1^2\}}^2 5\{2r(\Omega-r+5) - \Omega - 9\}]$
$\{4^r, 2\}$	$\lambda_{\{2\}}^2 \frac{1}{2} [3r^4 - 6(\Omega+2)r^3 + (3\Omega^2 + 6\Omega - 5)r^2 + (\Omega+2)(6\Omega+17)r + \Omega(\Omega+1)] + \lambda_{\{1^2\}}^2 \frac{5r}{2} (\Omega-r+4) \{(\Omega+4)r - r^2 - 3\}$
$\{4^r, 3\}$	$\frac{1}{4} [\lambda_{\{2\}}^2 3(r+2)(\Omega-r+2)(2r\Omega - 2r^2 + 6r + \Omega + 1) + \lambda_{\{1^2\}}^2 5r(\Omega-r+4)(2r\Omega - 2r^2 + 6r + \Omega - 1)]$

$f_m$	$f_2$	$\nu$	$Q^\nu(f_2: m, f_m)$
$\{4^r\}$	$\{2\}$	1	$\frac{9r(r+1)^2(\Omega-r)(\Omega+1)(\Omega+4)}{2(\Omega+2)}$
		2	$\frac{3r\Omega(r+1)(\Omega-r+1)(\Omega-r)(\Omega+4)(\Omega+5)}{4(\Omega+2)}$
	$\{1^2\}$	1	$\frac{25r(r-1)^2(\Omega-r)(\Omega-1)(\Omega+4)}{2(\Omega-2)}$
		2	$\frac{5r\Omega(r-1)(\Omega+3)(\Omega+4)(\Omega-r)(\Omega-r-1)}{4(\Omega-2)}$

$f_m$	$R^{\nu=1}(m, f_m)$
$\{4^r\}$	$-\frac{15r}{2} \sqrt{\frac{\Omega^2-1}{\Omega^2-4}} (r^2-1)(\Omega-r)(\Omega+4)$

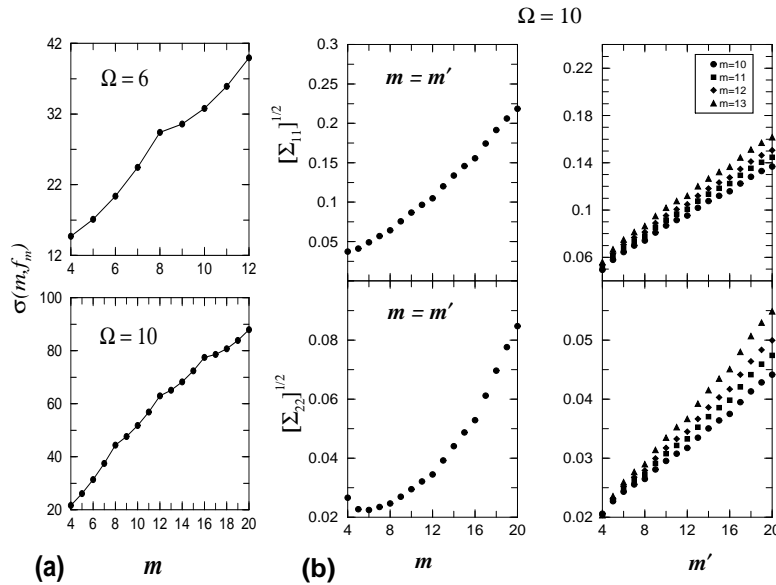
1)/4. Again the functions  $R^\nu$  involve  $SU(\Omega)$  Racah coefficients and  $R^{\nu=0}(m, f_m) = P^{\{2\}}(m, f_m)P^{\{1^2\}}(m, f_m)$ . Formulas for  $R^{\nu=1}(m, f_m)$  for  $f_m = \{4^r, p\}$ ,  $p = 0-3$  are derived and the result for  $\{4^r\}$  is given in table 2 as an example. Complete tabulations for  $Q^{\nu=1,2}$  and  $R^{\nu=1}$  will be reported elsewhere. Equations (13), (14) and (15) are similar in structure to the corresponding equations for EGUE(2)-s [9]. However, the functions  $P$ ,  $Q$  and  $R$  are more complicated for EGUE(2)- $SU(4)$ .

### 5. Results and discussion

Numerical calculations are carried out for  $\overline{\langle H^2 \rangle}^{m, f_m}$ ,  $\Sigma_{11}$  and  $\Sigma_{22}$  for some  $\Omega = 6$  [(2s1d)-shell] and  $\Omega = 10$  [(2p1f)-shell] examples. Here we have



employed  $\lambda_{\{1^2\}}^2 = \lambda_{\{2\}}^2 = 1$ . Figure 2a shows the variation in the spectral widths  $\sigma(m, f_m) = [\langle H^2 \rangle_{m, f_m}]^{1/2}$  with particle number  $m$ . Notice the peaks at  $m = 4r$ ;  $r = 2, 3, \dots$ . Except for this structure, there are no other differences between  $\{4^r\}$  and  $\{4^r, 2\}$  systems, i.e. for ground states of even-even and odd-odd  $N = Z$  nuclei. Results for the cross-correlations  $\Sigma_{11}$  and  $\Sigma_{22}$  are shown in figure 2b. It is seen that  $[\Sigma_{11}]^{1/2}$  and  $[\Sigma_{22}]^{1/2}$  increases almost linearly with  $m$ . At  $m = 4r$ ,  $r = 2, 3, \dots$  there is a slight dip in  $[\Sigma_{11}]^{1/2}$  as well as in  $[\Sigma_{22}]^{1/2}$ . For  $\Omega = 6$  with  $m = m'$ ,  $[\Sigma_{11}]^{1/2} \sim 10\text{--}28\%$  and  $[\Sigma_{22}]^{1/2} \sim 6\text{--}16\%$  as  $m$  changes from 4 to 12. Similarly for  $m \neq m'$ ,  $[\Sigma_{11}]^{1/2} \sim 10\text{--}24\%$  and  $[\Sigma_{22}]^{1/2} \sim 6\text{--}12\%$ . The values are somewhat smaller for  $\Omega = 10$  (see figure 2b) which is in agreement with the results obtained for EGOE(2) for spinless fermions and EGOE(2)-s. For further understanding we compare, for fixed  $N$ , these covariances with those for EGUE(2) and EGUE(2)-s. Using the analytical formulas given in [7] for EGUE(2), [9] for EGUE(2)-s and the present paper for EGUE(2)- $SU(4)$ , it is found that the magnitude of the covariances in energy centroids and spectral variances increases with increasing symmetry. For example, with  $N = 24$  (so that  $\Omega = 12$  for EGUE(2)-s and  $\Omega = 6$  for EGUE(2)- $SU(4)$ ) the results are as follows. For  $m = m' = 6$  ( $m = m' = 8$ ) we have: (i)  $[\Sigma_{11}]^{1/2} = 0.017(0.026)$  and  $[\Sigma_{22}]^{1/2} = 0.006(0.006)$  for EGUE(2); (ii) for EGUE(2)-s with  $S = S' = 0$ ,  $[\Sigma_{11}]^{1/2} = 0.043(0.066)$  and  $[\Sigma_{22}]^{1/2} = 0.017(0.021)$ ; (iii) for EGUE(2)- $SU(4)$ ,  $[\Sigma_{11}]^{1/2} = 0.124(0.16)$  and  $[\Sigma_{22}]^{1/2} = 0.069(0.082)$ . As fluctuations are growing with increasing symmetry, it is plausible to conclude that symmetries play a significant role in generating chaos. From a different perspective a similar conclusion was reached in [19] by Papenbrock and Weidenmüller. As they



**Figure 2.** (a) Widths  $\sigma(m, f_m) = [\langle H^2 \rangle_{m, f_m}]^{1/2}$  for  $\Omega = 6$  and  $\Omega = 10$  examples. (b) Cross-correlations for  $\Omega = 10$  examples. Note that  $f_m = \{4^r, p\}$ ;  $m = 4r + p$  and  $f_{m'} = \{4^s, q\}$ ;  $m' = 4s + q$ . See text for details.

state: “While the number of independent random variables decreases drastically as we follow this sequence, the complexity of the (fixed) matrices which support the random variables, increases even more. In that sense, we can say that in the TBRE, chaos is largely due to the existence of (an incomplete set of) symmetries.”

## 6. Summary and future outlook

In summary, we have introduced the embedded ensemble EGUE(2)- $SU(4)$  in this paper and our main emphasis has been in presenting analytical results. Our study is restricted to  $U(\Omega)$  irreps of the type  $\{4^r, p\}$ ,  $p = 0, 1, 2$  and  $3$ . Using eqs (13)–(15) and the formulas for the functions  $P$ ,  $\langle H^2 \rangle^{m, f_m}$ ,  $Q$  and  $R$  given in tables 1 and 2, cross-correlations in spectra with different  $(m, f_m)$  irreps are studied with results presented in figure 2 and §5 (see [2,5,9] for further discussion on the significance of cross-correlations generated by embedded ensembles (they will vanish for GEs). Elsewhere we will discuss the results for EGOE(2)- $SU(4)$  and in the limit  $\Omega \rightarrow \infty$  the results for these two ensembles are expected to coincide except for a difference in scale factors.

In future we also plan to investigate EGUE(2)- $SU(4)$  for general  $U(\Omega)$  irreps for any  $m$  and this is indeed feasible with the tabulations for sums of Racah coefficients given in [18]. Then it is possible to examine the extent to which EGUE(2)- $SU(4)$ , i.e. random interactions with  $SU(4)$  symmetry, carry the properties of Majorana or the  $C_2[SU(4)]$  operator. This study is being carried out and the results will be presented elsewhere. With this, it is possible to understand the role of random interactions in generating the differences in the ground state structure of even–even and odd–odd  $N = Z$  nuclei (see [20] for a numerical random matrix study of  $N = Z$  nuclei). In addition, just as the pairing correlations in EGOE(1+2)-s have been investigated recently [6], it is possible to consider  $SU(\Omega) \supset SU(3)$ , where  $SU(3)$  is Elliott’s  $SU(3)$  algebra [21], and examine rotational collectivity with random interactions. To this end we plan to analyse in future expectation values of the quadratic Casimir invariant of  $SU(3)$  or equivalently that of quadrupole–quadrupole ( $Q \cdot Q$ ) operator over the EGOE(1+2)- $SU(4)$  ensemble. Finally, going beyond EGUE(2)- $SU(4)$ , it is both interesting and possible (by extending and applying the  $SU(4) \supset SU_S(2) \otimes SU_T(2)$  Wigner–Racah algebra developed by Hecht and Pang [22]) to define and investigate, analytically, the ensemble with full  $SU(4)$ – $ST$  symmetry. In principle, it is also possible to construct the  $m$  particle  $H$  matrix, which is  $SU(4)$  or  $SU(4)$ – $ST$  scalar, on a computer and analyse its properties numerically but this is for future.

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