PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 73, No. 3
September 2009
pp. 505–519

Transition from Poisson to circular unitary ensemble

VINAYAK and AKHILESH PANDEY*
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India
*Corresponding author
E-mail: vinayaksps2003@gmail.com; ap0700@mail.jnu.ac.in

Abstract. Transitions to universality classes of random matrix ensembles have been useful in the study of weakly-broken symmetries in quantum chaotic systems. Transitions involving Poisson as the initial ensemble have been particularly interesting. The exact two-point correlation function was derived by one of the present authors for the Poisson to circular unitary ensemble (CUE) transition with uniform initial density. This is given in terms of a rescaled symmetry breaking parameter Λ. The same result was obtained for Poisson to Gaussian unitary ensemble (GUE) transition by Kunz and Shapiro, using the contour-integral method of Brezin and Hikami. We show that their method is applicable to Poisson to CUE transition with arbitrary initial density. Their method is also applicable to the more general ℓCUE to CUE transition where ℓCUE refers to the superposition of ℓ independent CUE spectra in arbitrary ratio.

Keywords. Quantum chaos; random matrix; symmetry breaking; fluctuations; correlation functions; Brownian motion; contour integral.

PACS Nos 5.45.Mt; 24.60.Lz; 73.23.-b; 3.65.-w

1. Introduction

Random matrix theory (RMT) has been useful in the statistical study of the spectra of complex quantum systems [1–5]. Its applications cover a wide range of systems, e.g., quantum chaotic systems, mesoscopic systems, complex nuclei and atoms, etc. Universality of fluctuations is an important aspect of its applications. There are three universality classes which are described by the three invariant random-matrix ensembles, viz., orthogonal ensemble (OE), unitary ensemble (UE) and symplectic ensemble (SE). These are defined by invariance of the ensemble measure under the orthogonal, unitary and symplectic transformations respectively and are related to the time reversal and rotational symmetries of the system. Gaussian ensembles (GE) of Hermitian matrices and circular ensembles (CE) of unitary matrices are of particular interest in these studies. For GE the invariant ensembles are GOE, GUE and GSE and for CE the invariant ensembles are COE, CUE, CSE. These ensembles have been studied extensively. For large matrices, the two types of ensembles have same fluctuation properties when belonging to the same invariance class.

When the symmetry of a system is gradually broken, the spectral fluctuations undergo transition from one universality class to another. The problem of transitions between the universality classes of spectral fluctuations has been studied since the 1960's when the classic papers of Rosenzweig and Porter [6] and Dyson [7] were published. These transitions are useful in the context of complex systems with weakly broken symmetries [8,9]. For the breaking of time reversal symmetry, one considers OE–UE and SE–UE transitions [10–12]. For the breaking of a partitioning symmetry involving several quantum numbers, one considers ℓOE–OE, ℓUE–UE and ℓSE–SE transitions, where ℓ refers to the number of overlapping quantum numbers and ℓ ensembles refer to superposition of ℓ independent spectra in arbitrary ratio [6,9,13–16]. For $\ell \to \infty$ the initial ensemble becomes Poisson [9,14–19].

Typically, one considers a single symmetry breaking parameter τ, which governs the transition and is a measure of the square of the norms of symmetry breaking and symmetry preserving parts. For infinitely large matrices the transition in fluctuations occurs discontinuously at $\tau = 0$ [6–8]. However, in the same limit smooth transition in fluctuations is obtained for small τ as a function of appropriately rescaled transition parameter Λ [9–16]. Examples and applications of such transitions have been found in the spectra of complex atoms [6] and nuclei [9,13], and quantum chaotic systems [20–22]. See also [4,23,24] for applications to mesoscopic quantum transport problems.

The transition ensembles also give identical results for the Gaussian and circular cases with suitably defined parameter Λ. For example, OE–UE and SE–UE transitions in CE [12] are found to be the same as the corresponding transitions in GE [10,11]. Similarly, transition results obtained for Poisson to GUE [19] transition coincide with the results of Poisson to CUE transition [14,15] and 2CUE to CUE transition results [14,15] coincide with 2GUE to GUE results [13].

Brezin and Hikami [25] have developed the contour-integral method for deriving correlation functions for transitions to GUE. This method has been used in [19] for the two-level correlation function for Poisson to GUE transition. We have recently shown [16] that the same method can also be used for transitions to CUE and can be generalized to ℓCUE to CUE transition. In this paper we review our methods and results given in [16] and make extensions to derive the results for the Poisson to CUE transition with arbitrary initial density.

The paper is organized as follows. In §2 we review the contour integral method for transitions to CUE. In §3 we derive the two-level correlation function for Poisson to CUE transition with arbitrary initial density and give numerical illustration of our results. In §4 we briefly discuss the more general ℓCUE to CUE transition. The results are summarized in the concluding section.

2. Contour integral representations of correlation functions

Transitions in the Gaussian and circular ensembles are best described in terms of Dyson's Brownian motion model [7]. We consider N-dimensional matrices. For the GE the transition ensembles are given by

$$H(\tau + \delta\tau) = H(\tau) + \sqrt{\delta\tau} M(\tau), \tag{1}$$

where $\sqrt{\delta\tau}$ is infinitesimal, $H(0)$ is a diagonal matrix and $M(\tau)$, independent for each τ, is a member of the invariant Gaussian ensembles GOE, GUE and GSE respectively for $\beta = 1, 2$ and 4. Average of the square of the off-diagonal matrix elements is βv^2 where v^2 supplies a scale for the symmetry breaking parameter τ. An equivalent description of the transition ensembles for GEs can be given by a linear interpolation of the initial and the final matrix ensembles [6].

The transition ensembles for the CE are given by

$$U(\tau + \delta\tau) = U(\tau)\exp[i\sqrt{\delta\tau}M(\tau)], \tag{2}$$

where $U(0)$ is a diagonal matrix and $M(\tau)$ is the same as in (1). We fix the scale by $v^2 = 1$. The matrix elements of $U(0)$ is given by $U_{jk}(0) = \exp[i\phi_j]\delta_{jk}$ where ϕ_j are the eigenangles. Let θ_j be the eigenangles of $U(\tau)$. The sets $\{\phi_1, ..., \phi_N\}$ and $\{\theta_1, ..., \theta_N\}$ are written as Φ and Θ respectively. Similarly, we write the sets $\{\mathrm{e}^{i\phi_1}, ..., \mathrm{e}^{i\phi_N}\}$ and $\{\mathrm{e}^{i\theta_1}, ..., \mathrm{e}^{i\theta_N}\}$ as $\mathrm{e}^{i\Phi}$ and $\mathrm{e}^{i\Theta}$. The joint probability density (jpd) of the eigenangles, $P(\Theta; \tau)$, is given in terms of the initial jpd $P(\Phi; 0)$ by

$$P(\Theta; \tau) = \int \mathrm{d}\phi_1 ... \mathrm{d}\phi_N P(\Theta, \Phi; \tau) P(\Phi; 0). \tag{3}$$

The conditional jpd, $P(\Theta, \Phi; \tau)$, satisfies the Fokker–Planck equation [7,12]

$$\frac{\partial P}{\partial \tau} = \sum_j \frac{\partial}{\partial \theta_j}\Big[\frac{\partial P}{\partial \theta_j} - \frac{\beta}{2}\sum_{k(\neq j)} \cot\left(\frac{\theta_j - \theta_k}{2}\right) P\Big]. \tag{4}$$

For $\tau \to \infty$, (4) yields the COE, CUE and CSE densities as equilibrium densities respectively for $\beta = 1, 2$ and 4,

$$P_{\mathrm{eq}} \equiv P(\Theta; \infty) = C_{N,\beta}|Q_N(\Theta)|^\beta. \tag{5}$$

Here

$$Q_N(\Theta) = \prod_{j>k} \sin\left(\frac{\theta_j - \theta_k}{2}\right), \tag{6}$$

and $C_{N,\beta}$ is the normalization constant [1]. $Q_N(\Theta)$ is related to the Vandermonde determinant $\Delta_N(\mathrm{e}^{i\Theta})$ of the eigenangles,

$$Q_N(\Theta) = \frac{\exp\left(-i(N-1)\sum_{j=1}^N \theta_j/2\right)\Delta_N(\mathrm{e}^{i\Theta})}{(2i)^{N(N-1)/2}}. \tag{7}$$

The self-adjoint or Hamiltonian form of the diffusion equation (4) is obtained by the similarity transformation $P \to \xi = P_{\mathrm{eq}}^{-1/2} P$ and is given by

$$\frac{\partial \xi}{\partial \tau} = -\mathcal{H}\xi, \tag{8}$$

where $\mathcal{H}$ is the Sutherland Hamiltonian [12,14],

$$\mathcal{H} = -\sum_j \frac{\partial^2}{\partial\theta_j^2} - \frac{\beta^2}{48}N(N^2-1) + \frac{\beta(\beta-2)}{16}\sum_{j\neq k}\mathrm{cosec}^2\left(\frac{\theta_j-\theta_k}{2}\right). \quad (9)$$

(In the Gaussian case one obtains similarly the Calogero Hamiltonian.) For $\beta = 2$ the interaction terms in (9) drops out and a compact solution can be obtained. Thus the conditional jpd for transitions to CUE [12] is given by

$$P(\Theta,\Phi;\tau) = \frac{1}{N!}\frac{Q_N(\Theta)}{Q_N(\Phi)}\exp\left(\frac{N(N^2-1)\tau}{12}\right) \times \det[f(\theta_j-\phi_k;\tau)]_{j,k=1,\ldots,N}, \quad (10)$$

where

$$f(\psi) = \frac{1}{2\pi}\sum_{\mu=-\infty}^{\infty}\exp(-\mu^2\tau + i\mu\psi) \quad (11)$$

with integral or half-integral μ for odd and even N respectively.

For the Poisson initial ensemble, ϕ_j are statistically independent and identically distributed with density $w(\phi)$, where w is a smooth function of ϕ. Thus we have

$$P(\Phi;0) = \prod_{j=1}^{N} w(\phi_j). \quad (12)$$

In the earlier papers [14–16] $w(\phi) = 1/2\pi$ has been considered. We show in this paper that the unfolded two-level correlation function is independent of $w(\phi)$, if the parameter τ is rescaled appropriately. Equation (10) has also been used with other initial ensembles, viz., COE and CSE [12] and 2CUE [14,15]. We have recently considered the more general ℓCUE initial ensemble [16].

In the contour integral method it is convenient to deal with the Fourier expansion of the correlation functions. We compute the ensemble averages of

$$C_1(p) = \sum_{k=1}^{N}\exp(ip\theta_k), \quad (13)$$

$$C_2(p,q) = \sum_{k\neq l}^{N}\exp(ip\theta_k + iq\theta_l), \quad (14)$$

where p and q take all possible integral values. The ensemble average of a symmetric function $\mathcal{F}(\Theta)$ with respect to $P(\Theta;\tau)$ is defined in two steps. Using bars to denote average over θ_j with respect to the conditional jpd, we have

$$\bar{\mathcal{F}} \equiv \int \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N \mathcal{F}(\Theta)P(\Theta,\Phi;\tau) = \int \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N \mathcal{F}(\Theta)\frac{Q_N(\Theta)}{Q_N(\Phi)}\exp\left[\frac{N(N^2-1)\tau}{12}\right]\prod_{j=1}^{N} f(\theta_j-\phi_j;\tau), \quad (15)$$

where in the second step $\det[f(\theta_j-\phi_k)]$ has been replaced by $N!\prod_{j=1}^N f(\theta_j-\phi_j)$ using the symmetry of $\mathcal{F}$. Next we use angular brackets to represent averaging over ϕ_j with respect to the initial jpd. Thus we have finally

$$\begin{aligned}\langle\bar{\mathcal{F}}\rangle &\equiv \int \mathrm{d}\theta_1 ... \mathrm{d}\theta_N \mathcal{F}(\Theta)P(\Theta;\tau)\\ &= \int \mathrm{d}\phi_1 ... \mathrm{d}\phi_N \bar{\mathcal{F}} P(\Phi;0).\end{aligned} \tag{16}$$

Choosing $\mathcal{F}=1$ in (15), we get the identity

$$\int \mathrm{d}\theta_1 ... \mathrm{d}\theta_N Q_N(\Theta)\prod_j^N f(\theta_j-\phi_j;\tau) = \exp\left[\frac{-N(N^2-1)\tau}{12}\right] Q_N(\Phi). \tag{17}$$

From (17) we obtain the relation,

$$\begin{aligned}&\int \mathrm{d}\theta_1 ... \mathrm{d}\theta_N \mathrm{e}^{i\Sigma_{j=1}^N b_j\theta_j} Q_N(\Theta)\prod_{j=1}^N f(\theta_j-\phi_j;\tau)\\ &\quad= \exp\left[\frac{-N(N^2-1)\tau}{12}\right]\exp\left[\sum_{j=1}^N(-b_j^2\tau+ib_j\phi_j)\right] Q_N(\Phi+2ib\tau),\end{aligned} \tag{18}$$

valid for all integral values of b_j. Here $\Phi+2ib\tau$ represents the set $\{\phi_1+2ib_1\tau,...,\phi_N+2ib_N\tau\}$. In (18) we have used the identity $\sum_{l=-\infty}^{\infty} g(l)=\sum_{l=-\infty}^{\infty} g(l+b)$ for integer b. Using (6), (15) and (18) for $C_1(p)$ and $C_2(p,q)$ we get

$$\bar{C}_1(p) = \exp[-p^2\tau+(N-1)p\tau]\sum_{j=1}^N \mathrm{e}^{ip\phi_j}\prod_{k(\neq j)}\left(1+\frac{\mathrm{e}^{i\phi_j}\chi_p}{\mathrm{e}^{i\phi_j}-\mathrm{e}^{i\phi_k}}\right), \tag{19}$$

$$\begin{aligned}\bar{C}_2(p,q) &= \exp[-p^2\tau+(N-1)p\tau]\\ &\quad\times\exp[-q^2\tau+(N-1)q\tau]\sum_{j\neq k}\mathrm{e}^{ip\phi_j+iq\phi_k}\\ &\quad\times F(\mathrm{e}^{i\phi_j},\mathrm{e}^{i\phi_k})\prod_{l(\neq j)}\left(1+\frac{\mathrm{e}^{i\phi_j}\chi_p}{\mathrm{e}^{i\phi_j}-\mathrm{e}^{i\phi_l}}\right)\\ &\quad\times\prod_{l'(\neq k)}\left(1+\frac{\mathrm{e}^{i\phi_k}\chi_q}{\mathrm{e}^{i\phi_k}-\mathrm{e}^{i\phi_{l'}}}\right).\end{aligned} \tag{20}$$

Here χ_p and F are given by

$$\chi_p \equiv \chi(p,\tau) = \exp(-2p\tau)-1, \tag{21}$$

and

$$F(z_1, z_2) = \frac{(z_1 - z_2)\,(z_1 e^{-2p\tau} - z_2 e^{-2q\tau})}{(z_1 e^{-2p\tau} - z_2)\,(z_1 - z_2 e^{-2q\tau})}$$
$$= 1 + \frac{z_1 z_2 \chi_p \chi_q}{[z_1(\chi_p + 1) - z_2]\,[z_1 - z_2(\chi_q + 1)]}. \tag{22}$$

The last form of (22) is useful in the decomposition in (40). These expressions can be simplified further by replacing the summations by contour integrals. Let the contour Γ consists of two concentric circles Γ_1 and Γ_2 of radii $1 + \epsilon$ and $1 - \epsilon$ respectively, where $1 > \epsilon > 0$. Γ encloses all the initial eigenvalues. We choose Γ_1 and Γ_2 both in the anti-clockwise direction so that the Γ integral is the difference of the Γ_1 and Γ_2 integrals. We avoid singularities of F by choosing $|p|\tau > \epsilon$ and $|q|\tau > \epsilon$. Using all these, the ensemble averages of $C_1(p)$ and $C_2(p, q)$ can be written as

$$\langle \bar{C}_1(p) \rangle = K(p;\tau) \oint_\Gamma \frac{dz}{2\pi i} \frac{z^p}{z} \left\langle \prod_{k=1}^{N} \left(1 + \frac{z\chi_p}{z - e^{i\phi_k}}\right) \right\rangle, \tag{23}$$

$$\langle \bar{C}_2(p, q) \rangle = K(p;\tau)K(q;\tau) \oint_\Gamma \frac{dz_1}{2\pi i} \oint_\Gamma \frac{dz_2}{2\pi i} \frac{z_1^p}{z_1} \frac{z_2^q}{z_2} F(z_1, z_2)$$
$$\times \left\langle \prod_{l=1}^{N} \left[\left(1 + \frac{z_1\chi_p}{z_1 - e^{i\phi_l}}\right) \left(1 + \frac{z_2\chi_q}{z_2 - e^{i\phi_l}}\right) \right] \right\rangle, \tag{24}$$

where

$$K(p;\tau) = \exp[-p^2\tau + (N - 1)p\tau](\chi_p)^{-1}. \tag{25}$$

Equations (15)–(24) are analogous to the corresponding equations for transitions to GUE [19,25].

Transition in fluctuations is obtained for $\tau = O(N^{-2})$ [12,14] for large N. For $\bar{C}_1(p)$ it is adequate to consider $p = O(1)$. (For $\bar{C}_2$, p and q should both be $O(N)$, as shown in the next section.) For large N, $\chi_p = O(N^{-2})$ and $K_p = O(N^2)$. We expand the product in (23) and observe that the first non-vanishing term is linear in χ_p. Thus $\langle \bar{C}_1(p) \rangle = O(N)$. In the limit, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \langle \bar{C}_1(p) \rangle = \oint_\Gamma \frac{dz}{2\pi i} z^p \alpha(z), \tag{26}$$

where

$$\alpha(z) = \langle (z - e^{i\phi})^{-1} \rangle = \int_0^{2\pi} d\phi \rho(\phi;0)(z - e^{i\phi})^{-1} \tag{27}$$

with the level density $\rho(\phi;\tau)$ given by

$$\rho(\phi_1;\tau) = \int d\phi_2 ... d\phi_N P(\Phi;\tau). \tag{28}$$

Equation (26) implies that

$$\lim_{N\to\infty} \frac{1}{N}\langle \bar{C}_1(p)\rangle = \langle \exp(ip\phi)\rangle. \tag{29}$$

We see that, during the transition in fluctuations, the level density does not change appreciably. Moreover, $\rho(\phi) = w(\phi)$ in the Poisson case.

3. The two-level correlation function

In this section we derive the two-level correlation function for large N. We define $C(p,q)$,

$$C(p,q) = \frac{1}{N}[\langle \bar{C}_2(p,q)\rangle - \langle \bar{C}_1(p)\rangle\langle \bar{C}_1(q)\rangle], \tag{30}$$

which is related to the Fourier transform of the correlation function. With the parametrization of p and q,

$$p = \frac{m}{2} + 2\pi N\mathbf{k}\rho, \tag{31}$$

$$q = \frac{m}{2} - 2\pi N\mathbf{k}\rho, \tag{32}$$

the $N\to\infty$ limit of $C(p,q)$ is given by

$$\begin{aligned} C(p,q) &= -\int_0^{2\pi} \mathrm{d}\theta \exp(im\theta)\rho(\theta) \int_{-\infty}^{\infty} \mathrm{d}r \exp(2\pi i\mathbf{k}r) Y_2(r;\Lambda) \\ &= -\int_0^{2\pi} \mathrm{d}\theta \exp(im\theta)\rho(\theta) b_2(\mathbf{k};\Lambda). \end{aligned} \tag{33}$$

Here $Y_2(r;\Lambda)$ is the cluster correlation function [1,2], $\theta = (\theta_1+\theta_2)/2$, $r = (\theta_1 - \theta_2)N\rho$, and the transition parameter Λ is given by

$$\Lambda = \tau\rho^2 N^2. \tag{34}$$

The spectral form factor $b_2(\mathbf{k};\Lambda)$ is the Fourier transform of $Y_2(r;\Lambda)$,

$$b_2(\mathbf{k};\Lambda) = \int \mathrm{d}r \exp(2\pi i\mathbf{k}r) Y_2(r;\Lambda). \tag{35}$$

Note that $b_2(\mathbf{k};\Lambda)$ is in the integrand of the last form of (33) since ρ and Λ are in general θ-dependent. Note also that (33) does not have self-correlation term since the latter is excluded in the definition (14). We also remark that $p,q = O(N)$ but $m = O(1)$. This comes about because the density ρ is a smooth function of θ while the spectral fluctuations are defined on the $O(N^{-1})$ scale.

Using the initial jpd (12) in (23) and (24), we obtain

$$\langle \bar{C}_1(p)\rangle = K(p;\tau) \oint_\Gamma \frac{\mathrm{d}z_1}{2\pi i}\frac{z^p}{z}[\Omega(z)]^N, \tag{36}$$

and

$$\langle \bar{C}_2(p,q)\rangle = K(p;\tau)K(q;\tau)\oint_\Gamma \frac{\mathrm{d}z_1}{2\pi i}\oint_\Gamma \frac{\mathrm{d}z_2}{2\pi i}\frac{z_1^p}{z_1}\frac{z_2^q}{z_2}F(z_1,z_2)[D(z_1,z_2)]^N. \tag{37}$$

Here

$$\Omega(z) \equiv \Omega(z,p) = 1 + \chi_p z\alpha(z), \tag{38}$$

and

$$D(z_1,z_2) \equiv D(z_1,z_2,p,q) = 1 + \chi_p z_1\alpha(z_1) + \chi_q z_2\alpha(z_2) + \frac{\chi_p\chi_q z_1 z_2}{z_2 - z_1}\left[\alpha(z_1) - \alpha(z_2)\right], \tag{39}$$

with $\alpha(z)$ given in (27). For large N, (36) is consistent with (26). Using (36) and (37) in (30) we write

$$C(p,q) = \zeta_1(p,q) + \zeta_2(p,q), \tag{40}$$

where

$$\zeta_1(p,q) = \frac{K(p;\tau)K(q;\tau)}{N}\oint_\Gamma \frac{\mathrm{d}z_1}{2\pi i}\oint_\Gamma \frac{\mathrm{d}z_2}{2\pi i}\frac{z_1^p}{z_1}\frac{z_2^q}{z_2} \times \left[\{D(z_1,z_2)\}^N - \{\Omega(z_1,p)\Omega(z_2,q)\}^N\right], \tag{41}$$

$$\zeta_2(p,q) = \frac{K(p;\tau)K(q;\tau)}{N}\oint_\Gamma \frac{\mathrm{d}z_1}{2\pi i} \times \oint_\Gamma \frac{\mathrm{d}z_2}{2\pi i}\frac{z_1^p z_2^q \chi_p\chi_q\{D(z_1,z_2)\}^N}{[z_1(\chi_p+1) - z_2][z_1 - z_2(\chi_q+1)]}. \tag{42}$$

ζ_1 and ζ_2 correspond to the two terms in the last form of (22).

Now we use the change of variables,

$$z_1 = \left(1 + \frac{c\delta}{N}\right)\exp\left[i\left(\theta + \frac{y}{2N}\right)\right], \tag{43}$$

$$z_2 = \left(1 + \frac{c'\delta}{N}\right)\exp\left[i\left(\theta - \frac{y}{2N}\right)\right], \tag{44}$$

where $\delta = N\epsilon > 0$. c, c' take values ± 1 depending on the branch of Γ, being $+1$ for Γ_1 and -1 for Γ_2. It is useful to write $\alpha(z)$ as

$$\alpha(\mathrm{e}^{i\psi}) = \frac{1}{2\mathrm{e}^{i\psi}}\left(1 - 2if(\psi)\right), \tag{45}$$

where $f(\psi)$ is a transform of the density, given by

$$f(\psi) = \frac{1}{2}\int_0^{2\pi}\cot\left(\frac{\psi-\theta}{2}\right)\rho(\theta)\mathrm{d}\theta \tag{46}$$

with complex ψ [12]. For large N, $f(\psi)$ can be written in terms of density ρ as

$$f\left(\theta - i\frac{c\delta}{N}\right) = \frac{1}{2}P\int \rho(\phi)\cot\left(\frac{\theta-\phi}{2}\right)\mathrm{d}\phi + i\pi c\rho(\theta) + O\left(\frac{1}{N}\right), \quad (47)$$

where P denotes the principal value of the integral. Thus, to the leading order in N, we obtain

$$\chi_p = -\chi_q = -\frac{4\pi\Lambda\mathbf{k}}{\rho N}, \quad (48)$$

$$K(p;\tau)K(q;\tau) = -\frac{N^2\rho^2\exp(-8\pi^2\Lambda\mathbf{k}^2)}{16\pi^2\Lambda^2\mathbf{k}^2}, \quad (49)$$

$$\mathrm{d}z_1\mathrm{d}z_2 = -z_1z_2\mathrm{d}\theta\mathrm{d}y/N, \quad (50)$$

$$\begin{aligned} N(z_1 - z_2) &= iy\exp(i\theta), \quad c = c', \\ &= (iy + 2c\delta)\exp(i\theta), \quad c \neq c', \end{aligned} \quad (51)$$

$$\begin{aligned} z_1^p z_2^q &= \exp(im\theta)\exp(2\pi iy\mathbf{k}\rho), \quad c = c', \\ &= \exp(im\theta)\exp[2\pi(iy + 2c\delta)\mathbf{k}\rho], \quad c \neq c', \end{aligned} \quad (52)$$

and

$$\begin{aligned} N(z_1(\chi_p + 1) - z_2) &= N(z_1 - z_2(\chi_q + 1)) \\ &= (iy - 4\pi\mathbf{k}\Lambda/\rho)\exp(i\theta), \quad c = c', \\ &= (iy + 2c\delta - 4\pi\mathbf{k}\Lambda/\rho)\exp(i\theta), \quad c \neq c'. \end{aligned} \quad (53)$$

Similarly, we have

$$\Omega(z_1, p) = 1 - \frac{4\pi\Lambda\mathbf{k}}{N\rho}\left[-if\left(\theta - \frac{ic\delta_1}{N}\right) + \frac{1}{2}\right], \quad (54)$$

$$\Omega(z_2, q) = 1 + \frac{4\pi\Lambda\mathbf{k}}{N\rho}\left[-if\left(\theta - \frac{ic'\delta_2}{N}\right) + \frac{1}{2}\right], \quad (55)$$

where $\delta_1 = \delta + iyc/2$ and $\delta_2 = \delta - iyc'/2$. Finally, we also have

$$\begin{aligned} D &= 1 + O\left(N^{-2}\right), \quad c = c', \\ &= 1 - 8\pi^2\Lambda\mathbf{k}\,c\,N^{-1}[1 - 4\pi\Lambda\mathbf{k}/[(iy + 2c\delta)\rho]] + O\left(N^{-2}\right), \quad c \neq c'. \end{aligned} \quad (56)$$

Thus

$$[\Omega(z_1,p)\Omega(z_2,q)]^N = 1, \quad c = c', \\ = \exp(-8\pi^2\Lambda\mathbf{k}c), \quad c \neq c', \tag{57}$$

and

$$\{D(z_1,z_2)\}^N = 1, \quad c = c', \\ = \exp(-8\pi^2\Lambda\mathbf{k}c) \\ \times \exp[32\pi^3\Lambda^2\mathbf{k}^2 c/[(iy+2c\delta)\rho]], \quad c \neq c'. \tag{58}$$

We insert these large N-expressions in (41), (42) and consider limit $N \to \infty$. We obtain, after some algebra,

$$\zeta_1 = \sum_c \int_0^{2\pi} \mathrm{d}\theta \exp(im\theta)\rho(\theta)\exp[-8\pi^2\Lambda\mathbf{k}c(1+\mathbf{k}c)]L_1, \tag{59}$$

$$\zeta_2 = \sum_c \int_0^{2\pi} \mathrm{d}\theta \exp(im\theta)\rho(\theta)\exp[-8\pi^2\Lambda\mathbf{k}c(1+\mathbf{k}c)]L_2, \tag{60}$$

where, as in the Gaussian case [19],

$$L_{1,2} = \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{2\pi} \mathrm{e}^{b(iy+2\delta)} \left[\exp\left(\frac{\sigma}{iy+2\delta}\right) - 1\right] F_{1,2} \tag{61}$$

with

$$F_1 = \frac{1}{\sigma}, \tag{62}$$

$$F_2 = \frac{1}{2\pi\rho\,(y + i(4\pi\Lambda\mathbf{k}c/\rho - 2\delta))^2} \tag{63}$$

and

$$\sigma = 32\pi^3\Lambda^2\mathbf{k}^2/\rho = 2\pi\rho(4\pi\Lambda\mathbf{k}/\rho)^2, \tag{64}$$

$$b = 2\pi\rho\mathbf{k}c. \tag{65}$$

Here, in (59) and (60), only the two $c \neq c'$ terms contribute and are given as a summation over c. In (61), a change of variable $yc \to y$ has been used. Also, in (42), D^N can be replaced by $[D(z_1,z_2)]^N - [\Omega(z_1,p)\Omega(z_2,q)]^N$ without changing the value of the integral. This gives the additional (-1) term in the square bracket of L_2. This term can be dropped from further consideration as in (67) below.

Now using (40), (59) and (60) in (33), we find

$$b_2(\mathbf{k};\Lambda) = -\sum_c \exp[-8\pi^2\Lambda\mathbf{k}c(1+\mathbf{k}c)](L_1+L_2). \tag{66}$$

To solve the integrals in (61) we substitute $u = iy + 2\delta$ and close the contour by an infinite semicircle. The integrand in L_1 has pole at $u = 0$ and in L_2 has poles at $u = 0$ and $4\pi\Lambda\mathbf{k}c/\rho$. Note that $4\pi\Lambda|\mathbf{k}|/\rho > 2\delta$ because of our choice $|p|\tau > \epsilon$ in (24). The semicircle is on the left side of the line $\Re(u) = 2\delta$ if $\mathbf{k}c > 0$ and on the right side if $\mathbf{k}c < 0$. For $\mathbf{k}c > 0$ only $u = 0$ pole contributes to the integrand while for $\mathbf{k}c < 0$ no pole contributes. Thus only one of the values of c contributes to the summation and we can choose the semicircle on the left with $\mathbf{k}c$ replaced by $|\mathbf{k}|$. Replacing this contour by a circular contour of radius $< 2\pi\Lambda|\mathbf{k}|/\rho$, we find

$$\begin{aligned} b_2(\mathbf{k};\Lambda) = {} & \frac{\mathrm{e}^{-8\pi^2\Lambda|\mathbf{k}|(1+|\mathbf{k}|)}}{2\pi\rho} \\ & \times \oint_{|u|<2\pi\Lambda|\mathbf{k}|/\rho} \frac{\mathrm{d}u}{2\pi i} \exp[2\pi|\mathbf{k}|\rho u] \exp\left[\left(\frac{4\pi\Lambda|\mathbf{k}|}{\rho}\right)^2 \frac{2\pi\rho}{u}\right] \\ & \times \frac{u(8\pi\Lambda|\mathbf{k}|/\rho - u)}{(4\pi\Lambda|\mathbf{k}|/\rho)^2(4\pi\Lambda|\mathbf{k}|/\rho - u)^2}. \end{aligned} \tag{67}$$

By scaling u as $4\pi\Lambda|\mathbf{k}|u/\rho$ we get

$$\begin{aligned} b_2(\mathbf{k};\Lambda) = {} & \frac{\exp(-8\pi^2\Lambda\mathbf{k}^2 - 8\pi^2\Lambda|\mathbf{k}|)}{8\pi^2\Lambda|\mathbf{k}|} \oint_{|u|<1} \frac{\mathrm{d}u}{2\pi i} \frac{u(2-u)}{(1-u)^2} \\ & \times \exp[8\pi^2\Lambda|\mathbf{k}|(u|\mathbf{k}| + 1/u)]. \end{aligned} \tag{68}$$

Next the substitution $u = 1/z$ and a partial integration gives

$$\begin{aligned} b_2(\mathbf{k};\Lambda) = {} & \oint_{|z|>1} \frac{\mathrm{d}z}{2\pi i} \frac{1}{z(z-1)} \left(1 - \frac{|\mathbf{k}|}{z^2}\right) \\ & \times \exp[-8\pi^2\Lambda\mathbf{k}^2(1-z^{-1}) - 8\pi^2\Lambda|\mathbf{k}|(1-z)]. \end{aligned} \tag{69}$$

Now, as in [19], we choose $|z| = \sqrt{|\mathbf{k}|}$ in which case the contribution of the $z = 1$ pole has to be calculated for $|\mathbf{k}| < 1$. The latter gives $b_2(\mathbf{k};\infty)$ which is $1 - |\mathbf{k}|$ for $|\mathbf{k}| < 1$. On the other hand $b_2(\mathbf{k};\infty) = 0$ for $|\mathbf{k}| > 1$. Then the substitutions $z = \sqrt{|\mathbf{k}|}\exp(i\theta)$ along with $y = -\cos\theta$ gives the result for $b_2(\mathbf{k};\Lambda)$

$$\begin{aligned} b_2(\mathbf{k};\Lambda) = {} & b_2(\mathbf{k};\infty) - \frac{2}{\pi}\int_{-1}^{1} \mathrm{d}y \frac{\sqrt{1-y^2}(2y\sqrt{|\mathbf{k}|}+1)}{|\mathbf{k}| + 2y\sqrt{|\mathbf{k}|} + 1} \\ & \times \exp[-8\pi^2\Lambda|\mathbf{k}|(|\mathbf{k}| + 2y\sqrt{|\mathbf{k}|} + 1)]. \end{aligned} \tag{70}$$

The inverse Fourier transform of (70) gives

$$\begin{aligned} Y_2(r,\Lambda) - Y_2(r,\infty) = {} & -\frac{4}{\pi}\int_0^\infty \mathrm{d}\mathbf{k}\, \cos(2\pi r\mathbf{k}) \\ & \times \int_{-1}^{1} \mathrm{d}y\sqrt{1-y^2}\frac{(2y\sqrt{\mathbf{k}}+1)}{\mathbf{k} + 2y\sqrt{\mathbf{k}} + 1} \\ & \times \exp[-8\pi^2\Lambda\mathbf{k}(\mathbf{k} + 2y\sqrt{\mathbf{k}} + 1)], \\ = {} & -\frac{4}{\pi}\int_{-1}^{1} \mathrm{d}y\sqrt{1-y^2} \end{aligned}$$

$$\times \int_0^\infty \mathrm{d}\mathbf{k} \exp[-8\pi^2\Lambda\mathbf{k}(\mathbf{k}+2y\sqrt{\mathbf{k}}+1)]$$
$$\times[\cos(2\pi r\mathbf{k}) - \cos(2\pi r(\mathbf{k}+2y\sqrt{\mathbf{k}}+1))]. \quad (71)$$

For fixed Λ, eqs (70), (71) are independent of $\rho(\phi)$ and coincide with the results given earlier [14]. The same result is given in eq. (117) of [15] with two typing errors.

To illustrate these results we have numerically integrated (70) and (71). Also we have computed the number variance $\Sigma^2(r)$ given by

$$\Sigma^2(r;\Lambda) = r - \int_{-r}^{r} \mathrm{d}s(r-s)Y_2(s;\Lambda)$$
$$= \int_{-\infty}^{\infty} \mathrm{d}\mathbf{k}\frac{\sin^2(\pi\mathbf{k}r)}{\pi^2\mathbf{k}^2}(1-b_2(\mathbf{k};\Lambda)), \quad (72)$$

where $r > 0$. In figure 1 we show $1-b_2(\mathbf{k};\Lambda)$, $1-Y_2(r;\Lambda)$ and $\Sigma^2(r;\Lambda)$ respectively as functions of $\mathbf{k}$, r and r for several values of Λ. For Poisson spectrum $b_2(\mathbf{k}) = 0$, $Y_2(r) = 0$ and $\Sigma^2(r) = r$. As shown in figure 1a, $b_2(0;\Lambda) = 0$ for $\Lambda \neq \infty$ and 1 for $\Lambda = \infty$. $b_2(0;\Lambda)$ is a measure of spectral rigidity. Similarly $Y_2(0;\Lambda)$ is a measure of level repulsion. As we have shown in figure 1b, $Y_2(0;\Lambda) = 1$ for $\Lambda \neq 0$. Figure 1c shows how $\Sigma^2(r)$ becomes logarithmic from linear in r, as Λ increases.

4. ℓCUE to CUE transition

The contour integral method can be extended to the more general case where the initial condition is the ℓCUE. Here ℓCUE is an ensemble of block-diagonal matrices with ℓ blocks of dimensions $N_1, N_2, ..., N_\ell$ ($\sum_{j=1}^{\ell} N_j = N$), each block being an independent CUE. $\ell = 1$ corresponds to the case where the ensemble is CUE for all τ. On the other hand, $\ell = N$ corresponds to independent eigenangles, giving thereby Poisson initial spectrum for $N \to \infty$. For intermediate ℓ we have superposition of ℓ independent CUE spectra initially. This transition applies to time-reversal non-invariant systems with a weakly broken partitioning symmetry. The ℓCUE initial jpd is given by

$$P(\Phi;0) \propto [|Q_{N_1}(\phi_1,...,\phi_{N_1})Q_{N_2}(\phi_{N_1+1},...,\phi_{N_1+N_2})$$
$$...Q_{N_\ell}(\phi_{N-N_\ell+1},...,\phi_N)|^2 + \text{permutations}]. \quad (73)$$

For $\ell = 1$, (73) is the same as the CUE jpd (5). For $\ell = N$, we obtain (12) with $w(\phi) = 1/2\pi$.

For the general ℓCUE case we find [16] that

$$b_2(\mathbf{k};\Lambda) = \frac{\mathrm{e}^{-8\pi^2\Lambda|\mathbf{k}|(1+|\mathbf{k}|)}}{8\pi^2\Lambda|\mathbf{k}|} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\mathrm{d}u}{2\pi i}\mathrm{e}^{8\pi^2\Lambda\mathbf{k}^2 u}$$
$$\times\left(\frac{\prod_{j=1}^{\ell}[\mathrm{e}^{8\pi^2\Lambda|\mathbf{k}|f_j(2-u)} - (1-u)^2]}{(1-u)^2[u(2-u)]^{\ell-1}} + 1\right), \quad (74)$$

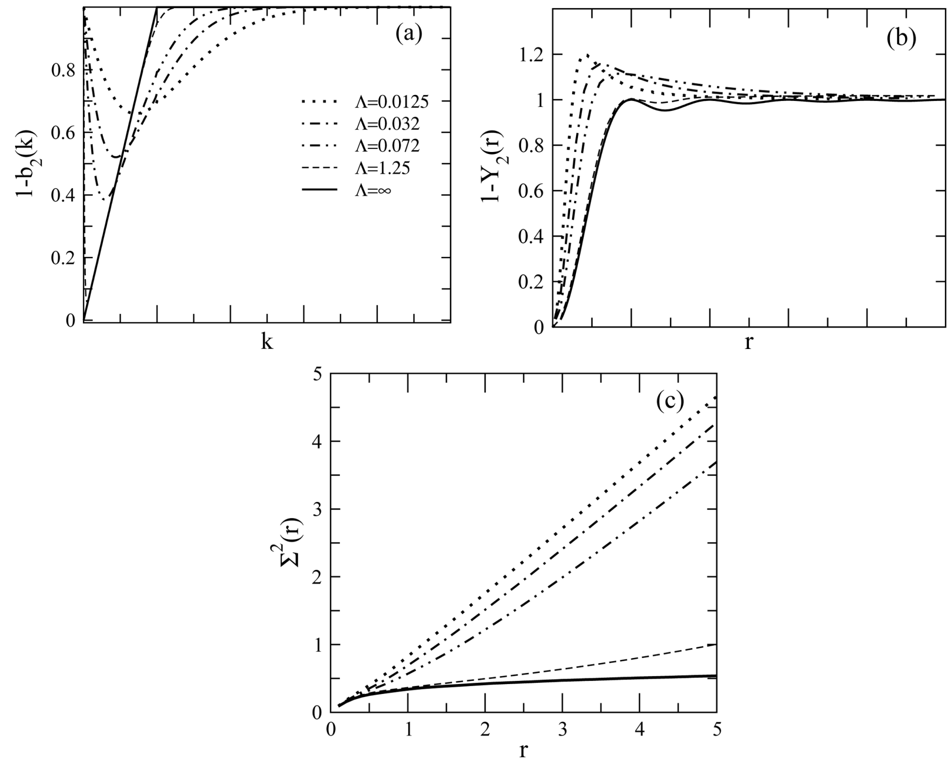

Figure 1. Plot of $1 - b_2(\mathbf{k}; \Lambda)$ vs. $\mathbf{k}$ (**a**), $1 - Y_2(r; \Lambda)$ vs. r (**b**) and $\Sigma^2(r; \Lambda)$ vs. r (**c**). These are obtained respectively by the numerical integrations of eqs (70)–(72) for different values of Λ.

where $f_j = N_j/N$ and $\gamma \to +0$. For $\ell = 1$, we obtain the CUE form factor $b_2(\mathbf{k}, \infty)$. For $\ell = 2$, we obtain

$$b_2(|\mathbf{k}|; \Lambda) = b_2(|\mathbf{k}|; \infty) - \frac{1}{2}\left[\int_{(2|\mathbf{k}|+|f_1-f_2|,1)_>}^{2|\mathbf{k}|+1} \mathrm{d}y\, g(y) - \int_{(2|\mathbf{k}|-1,1)_>}^{(2|\mathbf{k}|-|f_1-f_2|,1)_>} \mathrm{d}y\, g(y)\right], \tag{75}$$

where $g(y) = \exp[8\pi^2\Lambda|\mathbf{k}|(|\mathbf{k}| - y)]$. This result has been given earlier [14,15] along with the two-level cluster function

$$Y_2(r; \Lambda) - Y_2(r; \infty) = -\int_{|f_1-f_2|}^{1} \mathrm{d}x \int_1^\infty \mathrm{d}y\, \mathrm{e}^{2\pi^2\Lambda(x^2-y^2)} \sin(\pi r x) \sin(\pi r y). \tag{76}$$

For Poisson initial condition (viz., $\ell \to \infty$, $f_j \to 0$ such that $\sum f_j = 1$), we obtain from (74)

$$b_2(\mathbf{k};\Lambda) = \frac{e^{-8\pi^2\Lambda|\mathbf{k}|(1+|\mathbf{k}|)}}{8\pi^2\Lambda|\mathbf{k}|} \times \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{du}{2\pi i} e^{8\pi^2\Lambda\mathbf{k}^2 u} \left(e^{8\pi^2\Lambda|\mathbf{k}|/u} \frac{u(2-u)}{(1-u)^2} + 1 \right) \tag{77}$$

This is equivalent to (68). Proof of these results will be given elsewhere [16].

5. Conclusion

We have developed the contour integral method for the CUE transitions. This is analogous to the method developed for the GUE transitions [19,25]. We have used this method to derive the two-level correlation function for the Poisson to CUE transition which has been studied earlier for uniform initial density by another method [14,15]. In this paper we have used the contour integral method to generalize the result to the case where the initial density is nonuniform. We have shown that the same result is valid for all smooth initial densities when written in terms of appropriately rescaled transition parameter Λ. The result for the Poisson to GUE transition [19] also coincides with the earlier result. We remark however that the result given in [17,18] has not yet been shown to be the same.

We have reviewed briefly our recent work [16] on the ℓCUE to CUE transitions. This is a generalization of the Poisson to CUE case where the latter corrresponds to the limit $\ell \to \infty$. The finite-ℓ result may be more useful in real applications. We believe that the method is generalizable to ℓGUE to GUE transitions and also to the similar transitions in nonuniform circular ensembles [26] and in Laguerre and Jacobi ensembles [27].

Finally we mention that the original problem of Poisson to GOE [6] and the related ℓGOE to GOE transitions, as also the corresponding COE transitions, is still largely unsolved. However there are approximate results given in [14]. There are also exact results for Poisson to GOE transition given in terms of Grassmann integrals by Guhr and Kohler [28,29] and Datta and Kunz [30]. These results have neither been shown to be consistent with each other, nor with any numerical simulations of such transitions.

References

[1] M L Mehta, *Random matrices* (Academic Press, New York, 2004)
[2] T A Brody, J Flores, J B French, P A Mello, A Pandey and S S M Wong, *Rev. Mod. Phys.* **53**, 385 (1981)
[3] F Haake, *Quantum signatures of chaos* (Springer, Berlin, 1991)
[4] C W J Beenakker, *Rev. Mod. Phys.* **69**, 731 (1997)
[5] T Guhr, A M Groeling and H A Widenmüller, *Phys. Rep.* **299**, 189 (1998)
[6] N Rosenzweig and C E Porter, *Phys. Rev.* **120**, 1698 (1960)
[7] F J Dyson, *J. Math. Phys.* **3**, 1191 (1962)
[8] A Pandey, *Ann. Phys. (N.Y.)* **134**, 110 (1981)
[9] J B French, V K B Kota, A Pandey and S Tomsovic, *Ann. Phys. (N.Y.)* **181**, 198 (1988)

[10] A Pandey and M L Mehta, *Commun. Math. Phys.* **87**, 449 (1983)
[11] M L. Mehta and A Pandey, *J. Phys.* **A16**, 2655 (1983)
[12] A Pandey and P Shukla, *J. Phys.* **A24**, 3907 (1991)
[13] T Guhr and H A Weidenmüller, *Ann. Phys. (N.Y.)* **199**, 412 (1990)
[14] A Pandey, *Chaos, Solitons Fractals* **5**, 1275 (1995)
[15] A Pandey, *Phase Transitions* **77**, 835 (2004)
[16] Vinayak and A Pandey, *J. Phys. A: Math. Theor.* **42**, 315101 (2009)
[17] T Guhr, *Phys. Rev. Lett.* **76**, 2258 (1996)
[18] T Guhr, *Ann. Phys. (N.Y.)* **250**, 145 (1996)
[19] H Kunz and B Shapiro, *Phys. Rev.* **E58**, 400 (1998)
[20] D Wintgen and H Marxer, *Phys. Rev. Lett.* **60**, 971 (1988)
[21] N Dupuis and G Montambeaux, *Phys. Rev.* **B43**, 14390 (1991)
[22] P Shukla and A Pandey, *Nonlinearity* **10**, 979 (1997)
[23] K Frahm and J-L Pichard, *J. Phys. I France* **5**, 847 (1995)
[24] K Frahm and J-L Pichard, *J. Phys. I France* **5**, 877 (1995)
[25] E Brezin and S Hikami, *Nucl. Phys.* **B479**, 697 (1996)
[26] Sandeep Kumar and A Pandey, *Phys. Rev.* **E78**, 026204 (2008)
[27] Santosh Kumar and A Pandey, *Phys. Rev.* **E79**, 026211 (2009)
[28] T Guhr and H Kohler, *J. Math. Phys.* **43**, 2707 (2002)
[29] T Guhr and H Kohler, *J. Math. Phys.* **43**, 2741 (2002)
[30] N Datta and H Kunz, *J. Math. Phys.* **45**, 870 (2004)