

## Use of supersymmetric isospectral formalism to realistic quantum many-body problems

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**Abstract.** We propose a novel mathematical approach for the calculation of resonances in weakly bound systems. For any potential, families of strictly isospectral potentials (with very different shape) having desirable and adjustable features can be generated. For systems having no bound ground state, an isospectral potential with a bound state in the continuum is possible. The quasi-bound state in the original shallow potential will be effectively trapped in the deep well of the isospectral family, facilitating more accurate calculation of resonance energy. Application to  ${}^6\text{He}$ ,  ${}^6\text{Li}$  and  ${}^6\text{Be}$  yield excellent results. Another application is the calculation of Efimov states in weakly bound three-body system. We present the result of  ${}^4\text{He}$  trimer, where the first excited state is claimed to be an Efimov state.

**Keywords.** Halo nuclei; resonance; isospectral potential; Efimov.

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### 1. Introduction

Calculation of resonance energies in a potential well followed by a shallow barrier is a difficult numerical task as the width of the resonance is large. Isospectral formalism [1] is a novel technique for the calculation of such high-lying resonance states of weakly bound systems [2–5]. Starting from an effective potential  $v(r)$ , one can construct a family of isospectral potential  $\hat{v}(r, \lambda)$  (where  $\{\lambda\}$  represents a set of one or more continuously variable real parameters). The potential looks very different from the original potential but is strictly isospectral with the original potential. The quasi-bound state in the original shallow potential now will be effectively trapped in the deep narrow well of the isospectral potential [6,7]. As the resonance width is being reduced, isospectral family facilitates an easier and accurate calculation of resonance energy. The existence of isospectral family is known from a long time from inverse scattering approach [1]. However, supersymmetric approach to construct them is technically simpler. The potential  $\hat{v}(r, \lambda)$  is isospectral with the original potential  $v$  in the sense that they have identical spectra, reflection and transmission coefficients. In supersymmetric quantum mechanics, starting from a given one-dimensional potential  $v(x)$  with  $n$  bound states

and using Darboux procedure [1] one can construct one or more parameter families of isospectral potentials. For a one-parameter family, the value of  $\lambda$  lies in the permissible range  $-\infty < \lambda < -1$  and  $0 < \lambda < \infty$ . For  $\lambda \rightarrow \infty$ , one gets back the original potential. Although the set of isospectral potentials strictly have the same eigenspectra, they have different shapes depending on the parameter  $\lambda$ . For small positive values of  $\lambda$ ,  $\hat{v}$  develops a deep and narrow well followed by a high barrier near the origin. As  $\lambda \rightarrow 0^+$ , the depth of the well and the height of the adjacent barrier increase rapidly. This deep well and high barrier combination traps the system very effectively and thus gives rise to a sharp resonance. The position of resonance in  $\hat{v}(r, \lambda)$  is exactly the same as that in the original potential but more accurately calculated if one utilizes  $\hat{v}(r, \lambda)$ . The width of the resonance state is also calculated very accurately. For resonance states for which there is no bound ground state, one can construct the isospectral family with bound state in the continuum (BIC) [1]. This formalism was first successfully tested in a three-dimensional finite square well [8]. One realistic application of this technique is the calculation of resonance energy of  $A = 6$  nuclei like  ${}^6\text{He}$ ,  ${}^6\text{Be}$  and  ${}^6\text{Li}$  which are the halo nuclei [6,7]. These are considered as Borromean systems in which no binary subsystem is bound [2].  ${}^6\text{He}$  is a neutron rich halo nuclei and  ${}^6\text{Be}$  is a proton-rich halo nuclei and they are characterized by large matter radii, low binding energy and resonances at low excitation energy. With the radioactive ion beam facilities, the information about the ground and resonance states are already known experimentally. Although there are numerous theoretical studies of the ground state [3–5], the calculation of high-lying resonance states of such highly unstable nuclei is a difficult task. The shallow well can trap the system temporarily giving rise to a broad resonance. Thus the accurate calculation of resonance energy is masked by large resonance width.

Another important application of supersymmetric isospectral formalism is the calculation of Efimov states. Efimov states result when there is a zero or near zero energy two-body bound state. No clear example of Efimov state exists in any field. It is believed that such an exotic state does not exist in any atom as Coulomb interaction is long range but may exist in systems of neutral atoms. One excellent candidate is  ${}^4\text{He}$  trimer, as the dimer ( ${}^4\text{He}_2$ ) is predicted to have one weakly bound state  $\simeq 1$  mK. Although there is no experimental verification for the existence of Efimov state in the He trimer, the first excited state was claimed to be an Efimov state by several authors [9–11]. Here we revisit the problem by isospectral formalism, as a lot of numerical error may be involved in the calculation of such highly exotic state in the shallow well of original potential. But in the supersymmetric isospectral formalism, the near zero energy bound state will be more effectively bound in the deep narrow well of the isospectral potential. It will help to determine the Efimov state more accurately.

The paper is organized as follows. In §2, we present the methodology to solve the three-body problem and the calculational procedure to get isospectral potentials. Section 3 presents the results of Borromean systems. Section 4 presents the procedure to get Efimov states in He trimer and the results are discussed in the same section. Section 5 concludes the summary.

## 2. Methodology and calculational procedure

We treat the  $A = 6$  nuclei as three-body systems consisting of an  $\alpha$  core and two valance nucleons. The relative motion is described by two Jacobi coordinates  $(\vec{\zeta}, \vec{\eta})$  [12]

$$\left. \begin{aligned} \zeta &= \rho \cos \phi \\ \eta &= \rho \sin \phi \end{aligned} \right\}, \quad (1)$$

where  $\rho = \sqrt{(\zeta^2 + \eta^2)}$  is the hyper-radius. The relative angle  $\phi$  and the two polar angles, each of  $\vec{\zeta}$  and  $\vec{\eta}$  together constitute five hyperangles and they are collectively described by  $\Omega$ . Next we expand the total eigenfunction  $\psi(\rho, \Omega)$  in the complete set of hyperspherical harmonics (HH)  $\{\mathcal{Y}_{K\alpha}(\Omega)\}$  [12]

$$\psi(\rho, \Omega) = \sum_{K\alpha} \frac{U_{K\alpha}(\rho)}{\rho^{5/2}} \mathcal{Y}_{K\alpha}(\Omega) \quad (2)$$

$K$  is the hyperangular momentum quantum number.  $\alpha$  includes four other quantum numbers  $l_{\zeta}, l_{\eta}, L, M$ . Substituting the expansion (2) in the Schrödinger equation for the relative motion and the use of orthonormality of HH, gives a set of coupled differential equations

$$\left[ - \left( \frac{d^2}{d\rho^2} - \frac{\mathcal{L}_K(\mathcal{L}_K + 1)}{\rho^2} \right) - E \right] U_{K\alpha}(\rho) + \sum_{K'\alpha'} \langle K\alpha | V(\rho, \Omega) | K'\alpha' \rangle U_{K'\alpha'}(\rho) = 0, \quad (3)$$

where  $\mathcal{L}_K = K + \frac{3}{2}$  and  $\mu$  is the effective mass and

$$\langle K\alpha | V | K'\alpha' \rangle = \int \mathcal{Y}_{K\alpha}^*(\Omega) V(\rho, \Omega) \mathcal{Y}_{K'\alpha'}(\Omega) d\Omega, \quad (4)$$

where  $V$  is the sum of all pairwise potentials. For a practical calculation, the HH expansion basis is truncated to a finite set of CDE by restricting  $K$  values upto a maximum  $K_{\max}$  [13]. Next the set of CDEs are solved by the hyperspherical adiabatic approximation [14], where we assume that the hyper-radial motion is much slower compared to the hyperangular motion. Hence, the angular motion is solved adiabatically for a fixed value of  $\rho$ . Diagonalize the potential matrix (together with hypercentrifugal repulsion term) for each  $\rho$  and choose the lowest eigenpotential as the effective potential. Then the ground state energy is obtained numerically solving the uncoupled differential equation

$$\left[ - \frac{d^2}{d\rho^2} + v(\rho) - E \right] \psi(\rho) = 0 \quad (5)$$

subject to appropriate boundary conditions. Now for a state having spin-parity  $J^\pi$ , if  $v(\rho)$  can support at least a bound state with energy  $E_0$  then the normalized wave function  $\psi_0$  satisfies

$$\left[ -\frac{d^2}{d\rho^2} + v(\rho) - E \right] \psi_0(\rho) = 0. \quad (6)$$

Then by using supersymmetric quantum mechanics (SSQM) we can construct the most general superpotential [1]

$$\hat{W}(\rho, \lambda) = W(\rho) + \frac{d}{d\rho} \ln[I_0(\rho) + \lambda] \quad (7)$$

which gives one-parameter family of isospectral potentials

$$\begin{aligned} \hat{v}(\rho, \lambda) &= \hat{W}^2(\rho) - \hat{W}'(\rho) \\ &= v(\rho) - 2\frac{d^2}{d\rho^2} \ln[I_0(\rho) + \lambda], \end{aligned} \quad (8)$$

where

$$I_0(\rho) = \int_0^\rho [\psi_0(\rho')]^2 d\rho'. \quad (9)$$

Since  $\psi_0(\rho)$  is normalized and  $0 \leq I_0(\rho) \leq 1$ , the interval  $-1 \leq \lambda \leq 0$  is not allowed. It can be shown that  $\hat{v}(\rho, \lambda)$  is strictly isospectral with  $v(\rho)$  for only allowed values of  $\lambda$  excluding the above range.

In order to calculate the isospectral potential  $\hat{v}(\rho, \lambda)$  for a given  $J^\pi$ ,  $v(\rho)$  must support a bound state such that the ground state wave function  $\psi_0(\rho)$  is normalized. However, the system  ${}^6\text{Be}$  has no bound state, the above procedure fails. In ref. [15] Pappademos *et al* have generalized the above procedure starting with  $\psi_E$  with positive energy such that  $\psi_E$  is not square integrable. By following substitution,  $\psi_E$  can be converted into square integrable

$$\hat{\psi}_E(\rho, \lambda) = \frac{\psi_E(\rho)}{I_E(\rho) + \lambda}, \quad (10)$$

where

$$I_E(\rho) = \int_0^\rho [\psi_E(\rho')]^2 d\rho'. \quad (11)$$

As  $\psi_E(\rho)$  now oscillates,  $I_E(\rho)$  increases linearly with  $\rho$  and  $\hat{\psi}_E(\rho, \lambda)$  now becomes normalizable. Thus  $\hat{\psi}_E(\rho, \lambda)$  represents a bound state in the continuum. The set of isospectral is now determined by

$$\hat{v}(\rho, \lambda) = v(\rho) - \frac{4\psi_E\psi_E'}{I_E + \lambda} + \frac{2\psi_E^4}{(I_E + \lambda)^2} \quad (12)$$

$\hat{v}(\rho, \lambda)$  is again strictly isospectral with  $v(\rho)$  for all positive energies and has the same general features as the isospectral potential.

### 3. Results of $A = 6$ nuclei

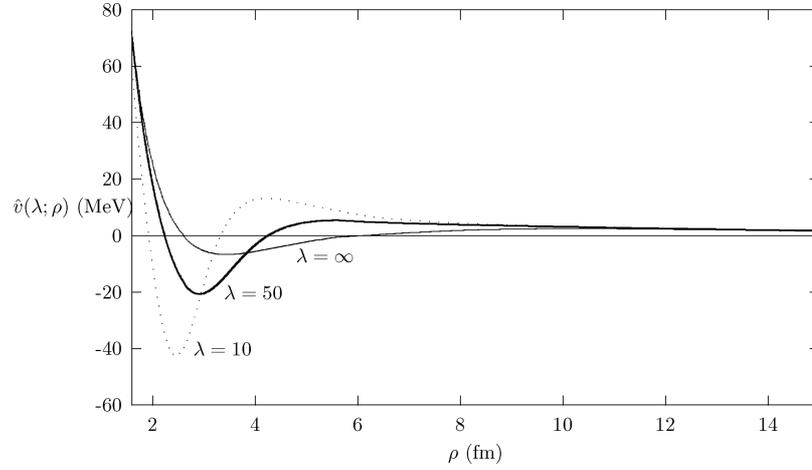
For the calculation of ground state and also the resonances in  $A = 6$  nuclei ( ${}^6\text{He}$ ,  ${}^6\text{Li}$ ,  ${}^6\text{Be}$ ) we assume  $(\alpha - n - n)$  cluster. As  $(n - n)$  potential we choose GPT potential [16], whereas for  $p - p$  potential in  ${}^6\text{Be}$  we also include Coulomb potential. For the  $(\alpha - n)$  potential we use the SBB potential [17] which fits  $(\alpha - n)$  phase shifts. The SBB potential has the form

$$V_{\alpha n}(\vec{r}) = \sum_{l=0,1,2} v_{\alpha n}^{(l)} \exp(-(r/b_{\alpha n}^{(l)})^2) + v^{(ls)}(\vec{l} \cdot \vec{s}) \exp(-(r/b_{\alpha n})^2). \quad (13)$$

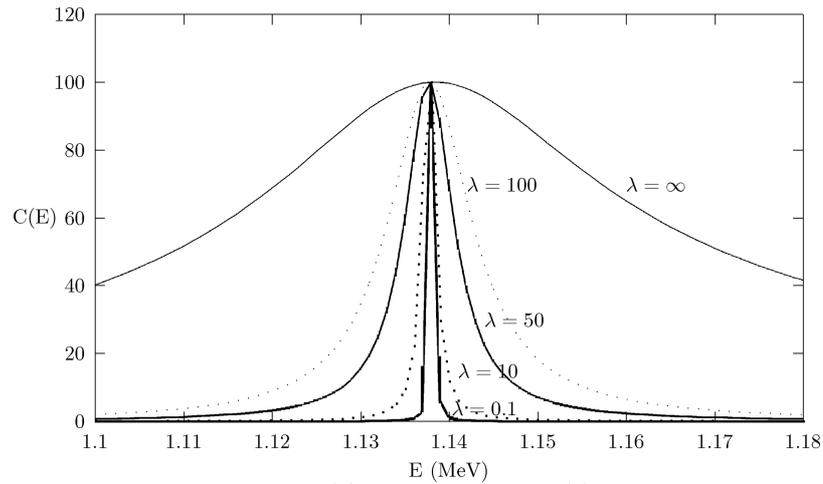
Equation (13) is the original SBB potential having a common Gaussian form but is of different strength for different  $l$  components plus a spin-orbit term. The  $s$ -wave component is repulsive with  $v_{\alpha n}^{(0)} = 50$  MeV and  $b_{\alpha n}^{(0)} = 2.3$  fm to simulate Pauli principle between the  $\alpha$  core and outer nucleons.  $v_{\alpha n}^{(1)} = -47.32$  MeV,  $v_{\alpha n}^{(2)} = -23.0$  MeV,  $v^{(ls)} = -5.855$  MeV, having the common range 2.3 fm. The spectrum of  ${}^6\text{He}$  contains one weakly bound ground state with  $J^\pi = 0^+$  and two resonance states with  $J^\pi = 2^+$ . The existence of both the ground state and the resonance states are confirmed by reactions [18]. With the above set of parameters we solve the coupled differential equations by hyperspherical adiabatic approximation where we consider that the hyper-radial motion is slow compared to hyperangular motion. With this effective potential we solve eq. (6) by the standard Runge–Kutta with predictor–corrector algorithm, subject to appropriate boundary condition imposed on ground state wave function  $\psi_0(\rho)$ . To obtain the converged binding energy we use the hyperspherical extrapolation technique [13] and the extrapolated binding energy (BE) is  $-0.675$  MeV which compares very well with the earlier calculations [2]. We next calculate the  $2^+$  resonance state of  ${}^6\text{He}$  having no bound state. We use the BIC technique. The corresponding effective three-body potential  $v(\rho)$  is shown in figure 1 ( $\lambda = \infty$ ), the shallow and wide barrier gives rise to the resonant states. Starting from positive energy ( $E > 0$ ) solution, we normalize  $\psi_E(\rho)$  and construct  $\hat{v}(\rho, \lambda)$  according to eq. (12). These are plotted in figure 1 for different choices of  $\lambda$ . Note that with appropriate choice of small values of  $\lambda$ , there is a narrow and deep well followed by high barrier which traps the system in the deep well. We calculate the probability of the system to be trapped within the well barrier combination

$$C(E) = \int_0^{\rho_B} [\psi_E(\rho')]^2 d\rho', \quad (14)$$

where  $\rho_B$  is the position of the top of the barrier. In figure 2, we plot  $C(E)$  as a function of  $E$ , which dramatically shows the enhanced trapping effect of  $\hat{v}(\rho, \lambda)$  as  $\lambda$  decreases towards  $0^+$ . The sharpness of the peak increases rapidly as  $\lambda \rightarrow 0^+$ . The resonance energy is independent of  $\lambda$ , but making  $\lambda$  too small may create large numerical error in the wave function as the well becomes extremely narrow and highly deep. Thus one has to make a judicious choice of  $\lambda$ . Alternately one can choose optimum  $\lambda$  which can correctly reproduce the measurable physical property like halo radius of these systems. The halo radius can be easily calculated by using the resonant wave function for different choices of  $\lambda$ . The width  $\Gamma$  of the resonance is obtained from the mean life of state using time–energy uncertainty relation as



**Figure 1.** Plot of one-parameter family of isospectral potentials  $\hat{v}(\lambda; \rho)$  for  $\lambda = \infty$  (original potential),  $\lambda = 50$  and  $10$  for the  $2_1^+$  state of  ${}^6\text{He}$ . Narrow and deep well followed by a high barrier effectively traps the particles to form a strong resonant state.



**Figure 2.** Probability  $C(E)$  as a function of energy ( $E$ ) to find the particle within the trapping potential for  $\lambda = \infty$  (original potential  $v(\rho)$ ),  $100$ ,  $50$ ,  $10$ ,  $0.1$ .

$$\Gamma = \frac{2 \exp \left[ -2 \int_b^c \sqrt{\hat{v}(\rho, \lambda) - E_R} d\rho \right]}{\int_a^b \frac{d\rho}{\sqrt{(E_R - \hat{v}(\rho, \lambda))}}}, \quad (15)$$

where  $a$  and  $b$  are the turning points of the well and  $b$  and  $c$  are the turning points of the barrier in  $\hat{v}(\rho, \lambda)$ , corresponding to the resonance energy  $E_R$ . With an optimum

**Table 1.** Comparison of calculated and experimental results. Resonance energy and width are in MeV.

	$J^\pi$	Energy/width	SSQM	CS [4]	Exp. [19]
${}^6\text{He}$	$0^+$	$E$	-0.675	-0.6	$-0.973 \pm 0.04$
	$2_1^+$	$E_R$	1.814	1.34	$1.797 \pm 0.025$
		$\Gamma$	0.135	0.06	$0.113 \pm 0.020$
	$2_2^+$	$E_R$	4.90	3.28	-
		$\Gamma$	1.717	4.7	-
${}^6\text{Li}$	$1^+$	$E$	-4.556	-	$-3.669 \pm 0.006$
	$1^+$	$E_R$	1.80	5.71	$1.95 \pm 0.050$
		$\Gamma$	1.23	3.89	$1.5 \pm 0.2$
	$2^+$	$E_R$	1.77	1.59	$1.696 \pm 0.015$
		$\Gamma$	0.761	0.28	$0.540 \pm 0.020$
${}^6\text{Be}$	$0^+$	$E_R$	1.37	1.52	1.371
		$\Gamma$	0.088	0.16	$0.092 \pm 0.006$
	$2^+$	$E_R$	3.00	2.81	$3.04 \pm 0.05$
		$\Gamma$	1.27	0.87	$1.16 \pm 0.06$

value of  $\lambda$  we get a sharp peak in  $C(E)$  at  $E_R = 1.138$  MeV. This corresponds to the excitation energy 1.814 MeV above the ground state, in agreement with the experimental results [19]. The width obtained is  $\Gamma = 135$  keV. The results are presented in table 1. In the same table we present the results of second  $2^+$  resonance state. Both the resonance energy and width are in excellent agreement with the experimental results [19], whereas  ${}^6\text{Li}$  has one bound ground state of  $-4.556$  MeV with spin-parity  $J^\pi = 1^+$ , in good agreement with experimental value. With the same spin-parity it has a resonance state with energy 1.80 MeV, whereas the resonance state with  $J^\pi = 2^+$  has no bound state. So for the resonance state with  $J^\pi = 1^+$  we use SUSY isospectral formalism and for the state with  $J^\pi = 2^+$  we again utilize BIC technique to formulate isospectral partners with different choices of  $\lambda$ . The supersymmetry results are presented in table 1 with other earlier results. In the same table we also present the results for resonance states of  ${}^6\text{Be}$  with  $J^\pi = 0^+$  and  $2^+$ . The results of complex scaling (CS) method [4] together with experimental results are presented in the same table. Our results compare very well with experimental results.

The fact that the resonant state is much better reproduced than the ground state may be understood from the following discussions. We are using the SBB interaction potential which are obtained from low-energy  $\alpha$ - $n$  scattering phase shifts. It implies that SBB potential represents the  $\alpha$ - $n$  interactions at large separation. The resonant states are much more spatially extended than the bound ground states. As the ground state is much more compact spatially, the SBB interaction is not so correct. It is likely that the effective  $\alpha$ - $n$  interaction should be strongly repulsive at short separations, which would reduce the ground state binding energy, bringing it closer to the experimental value. So it is not surprising that the resonant state of  ${}^6\text{Li}$  is better reproduced than the ground state.

#### 4. Calculation of Efimov state in He trimer

It was proposed by Efimov that if a system of two spinless neutral particles interact resonantly, then the addition of third particle will lead to the creation of a large number of bound states of three-body system. Efimov predicted that near the  $\pm\infty$  discontinuity in the scattering length, the number of three-body bound levels is given by

$$N \simeq \frac{1}{\pi} \ln \left( \frac{|a|}{r_c} \right), \quad (16)$$

where  $a$  is the two-body scattering length and  $r_c$  is the effective range of two-body potential. Till date there is no experimental evidence for such infinity of bound states. In nuclear physics, there is no example for the Efimov effect. At present,  $^4\text{He}$  trimer is taken as the most promising candidate. As the  $^4\text{He}$  dimer has the binding energy  $\simeq 1.0$  mK close to two-body resonance,  $^4\text{He}$  trimer is expected to exhibit Efimov state structure. For helium trimer, theory predicts one Efimov state with a weak binding energy of  $|E_b| = 2.3$  mK in addition to the ground state with  $|E_g| = 126$  mK. A lot of theoretical work has been done in order to either confirm or disprove the existence of Efimov state [10,11]. As the Efimov state is highly exotic and very crucially depends on two-body interaction strength, naturally a lot of numerical error may be involved in the existing theoretical calculation where the effective potential is very shallow. In this work we again prescribe to bypass the problem by formulating isospectral potential as the very high-lying weakly bound state in the original potential now will be converted into a true bound state in the deep narrow well of isospectral potential.

As an He-He potential we use the LM2M2 potential [20] which is the most commonly used accurate potential and it has the form

$$U(r) = \epsilon \left[ A \exp(-\alpha x - \beta x^2) - \left( \frac{C_6}{x^6} + \frac{C_8}{x^8} + \frac{C_{10}}{x^{10}} \right) F(x) + BU(x) \right], \quad (17)$$

where  $x = r/r_{\min}$ . With this potential ground state properties of  $^4\text{He}$  dimer is obtained by numerical solution of two-body Schrödinger equation by Runga-Kutta algorithm. The calculated ground state energy is  $-0.000872 \text{ cm}^{-1}$ , which is in close agreement with the DMC result of  $-0.00086 \text{ cm}^{-1}$  and it is seen that dimer wave function extends to several hundred a.u. With the same potential we next solve the trimer (three-body) problem by hyperspherical adiabatic approximation using potential harmonics basis [21].

The Hamiltonian for a system of  $A = (N + 1)$  atoms (each of mass  $m$ ) and interacting via two-body potential has the form

$$H = -\frac{\hbar^2}{m} \sum_{i=1}^{N+1} \nabla_i^2 + \sum_{i>j=1}^{N+1} V(\vec{x}_i - \vec{x}_j), \quad (18)$$

where  $V(\vec{x}_i - \vec{x}_j)$  is the He-He two-body potential. Then the relative motion of the atoms is described in terms of  $N$  Jacobi vectors where  $\{\vec{\zeta}_1, \dots, \vec{\zeta}_N\}$  as

$$\left[ -\frac{\hbar^2}{2m} \sum_{i=1}^N \nabla_{\vec{\zeta}_i}^2 + V(\vec{\zeta}_1, \dots, \vec{\zeta}_N) - E \right] \psi(\vec{\zeta}_1, \dots, \vec{\zeta}_N) = 0. \quad (19)$$

As the He atoms are spinless bosons, we decompose  $\psi$  in Faddeev components

$$\psi(\vec{x}) = \sum_{ij>i}^{N+1} \psi_{ij}(\vec{x}), \quad (20)$$

where  $\psi_{ij}$  satisfies the Schrödinger equation

$$(T - E)\psi_{ij}(\vec{x}) = -V(r_{ij}^{\vec{r}}) \sum_{k,l>k} \psi_{kl}(\vec{x}). \quad (21)$$

$T$  is the total kinetic energy. In the usual hyperspherical harmonics expansion method, the hyper-radius is defined as  $r = \sqrt{\sum_{i=1}^N \zeta_i^2}$ . The hyper-radius and  $(3N - 1)$  hyperangles together constitute  $3N$  hyperspherical variables. As the choice of Jacobi coordinate is not fixed, we choose  $\vec{r}_{ij}$  as  $\vec{\zeta}_N$  for  $(ij)$  interacting pair and  $\vec{\rho}_{ij}$  for the rest  $(N - 1)$  no. of atoms, such that  $\rho_{ij}^2 + r_{ij}^2 = r^2$ . Then the hyperspherical coordinates are  $(r, \Omega_N) = (r, \phi, \theta, \varphi, \Omega_{N-1})$ .  $(\theta, \varphi)$  are polar angles of  $\vec{r}_{ij}$  and  $\Omega_{N-1}$  involves  $(3N - 4)$  variables. In the potential harmonics expansion method, the many-body wave function is expanded in the subset called potential harmonics subset [21] from the full HH basis [12] as

$$\psi_{ij} = r^{-\left(\frac{3N-1}{2}\right)} \sum_K P_{2K+l}^{lm}(\Omega_{ij}) u_K^l(r). \quad (22)$$

The method was originally proposed by Fabre [21] for nuclear systems. The basic assumption is that only two-body correlations are important and higher-body correlations are disregarded. Consequently contribution to orbital and grand orbital quantum numbers come from the interacting pair only.  $^4\text{He}$  trimer has large spatial extent and thus it absolutely fits this requirement. Thus when  $(ij)$  pair interacts we can freeze the contribution coming from  $(N - 1)$  remaining spectators. Thus PH for  $(ij)$  partition is defined as the eigenfunctions of  $L^2(\Omega_N)$  corresponding to zero eigenvalue of  $L^2(\Omega_{N-1})$  and is given by

$$P_{2K+l}^{lm}(\Omega_{ij}) = Y_l^m(\theta, \varphi)^{(N)} P_{2K+l}^{l,0}(\phi) \mathcal{Y}_0(D - 3). \quad (23)$$

Note that it does not contain any function of  $\vec{\zeta}_i, i < N$ .  $Y_l^m(\theta, \varphi)$  is spherical harmonic.  $^{(N)} P_{2K+l}^{l,0}(\phi)$  is expressed in terms of Jacobi polynomials and  $\mathcal{Y}_0$  is the lowest order HH in  $(3N - 4)$ -dimensional hyperangular space. In this new basis set  $3N$ -dimensional Schrödinger equation reduces effectively to a four-dimensional equation and the relevant set of quantum numbers are three. These are orbital  $l$ , azimuthal  $m$  and grand orbital  $2K + l$  for any  $N$ . Next expanding Faddeev component  $\psi_{ij}$  in the complete set of potential harmonic and taking projection of the Schrödinger equation on the PH basis, a set of coupled differential equation (CDE) is obtained

$$\left[ -\frac{\hbar^2}{m} \frac{d^2}{dr^2} + \frac{\hbar^2}{mr^2} \{ \bar{\mathcal{L}}(\bar{\mathcal{L}} + 1) + 4K(K + \alpha + \beta + 1) \} - E \right] U_{Kl}(r) + \sum_{K'} \bar{V}_{KK'}(r) U_{K'l}(r) = 0, \quad (24)$$

where  $\bar{\mathcal{L}} = l + \frac{3N-3}{2}$ ,  $\alpha = \frac{3N-5}{2}$ ,  $\beta = l + \frac{1}{2}$ .  $K$  is the hyperangular momentum quantum number.  $\bar{V}_{KK'}(r)$  is the symmetric matrix element and is given by

$$\bar{V}_{KK'}(r) \propto \int_{-1}^{+1} P_K^{\alpha\beta}(z) V \left( r \sqrt{\frac{1+z}{2}} \right) P_{K'}^{\alpha\beta}(z) w_l(z) dz, \quad (25)$$

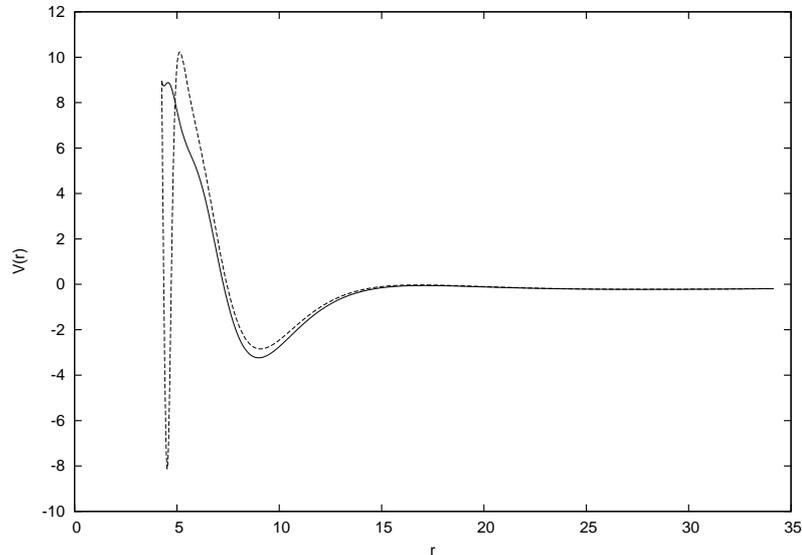
where  $w_l(z)$  is the weight function. The He-He potential becomes suddenly very strongly repulsive below a certain value of interatomic separation. This causes a very strong short-range two-body correlation in the many-body wave function. We introduce this correlation function in the expansion basis, called correlated potential harmonic (CPH) basis

$$P_{2K+l}^{lm}(\Omega_{ij}) = Y_l^m(\omega_{ij})^{(N)} P_{2K+l}^{l,0}(\phi) \mathcal{Y}_0(3N-3) \eta(r_{ij}) \quad (26)$$

$\eta(r_{ij})$  is a short-range correlation function which is obtained from zero energy two-body Schrödinger equation. Next we solve the set of coupled equation. We solve it by hyperspherical adiabatic approximation as described earlier. In figure 3 we plot the lowest eigenpotential of  $^4\text{He}$  trimer as a function of  $r$ . The calculated ground state energy is  $-0.0872 \text{ cm}^{-1} = 125 \text{ mK}$  which is in close agreement with earlier results [11]. However, we fail to get any more state which is close to two-body resonance and claimed to be an Efimov state. Next, starting from the effective potential we use the supersymmetric isospectral formalism described before. In the same figure (figure 3) we plot the isospectral potential for  $\lambda = 0.00005$ . Isospectral potential makes a narrow deep well on the left side of the original shallow well. Naturally the near zero energy bound state in the original potential now will be converted to a true bound state in the deep well of isospectral potential. Next we solve the CDE again by using the isospectral potential and we get a bound state at energy  $2.27 \text{ mK}$  [22], which is close to the predicted Efimov state of energy  $\simeq 1.0 \text{ mK}$  [11].

## 5. Conclusion

In this report we present a novel theoretical technique to study broad resonances of weakly bound systems and to study Efimov state of  $^4\text{He}$  trimer. Our method basically utilized the supersymmetric isospectral formalism in few-body systems. By using standard few-body technique we define an effective potential in hyper-radial space and calculate the family of isospectral potentials  $[\hat{v}(\rho, \lambda)]$ , having completely different shape but these are strictly isospectral to the original potential. Controlling the parameter  $\lambda$  in the isospectral potential one can have the desired shape of the potential. For small positive values of  $\lambda$ ,  $\hat{v}(\rho, \lambda)$  develops a deep and narrow well followed by a high barrier near the origin. The well-barrier combination



**Figure 3.** Lowest eigenpotential (effective potential) for  ${}^4\text{He}$  trimer with LM2M2 potential as a function of hyper-radius (solid line). Dashed line corresponds to isospectral potential with  $\lambda = 0.00005$ .

traps the system very effectively, thus reducing the resonance width and facilitating easier and accurate calculation of resonance energy. Application to weakly bound halo nuclei gives results which are in excellent agreement with experimental results. The novelty of our technique is that both the bound state and resonance states are accurately determined by the same technique.

The calculation of Efimov state in  ${}^4\text{He}$  trimer is equally difficult as it is highly exotic and it very crucially depends on two-body interaction. We again prescribe a nice complementary technique by using SUSY isospectral formalism which offers more accurate determination of Efimov effect.

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