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Scarcity of real discrete eigenvalues in non-analytic complex $\mathcal{PT}$-symmetric potentials

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Abstract. We find that a non-differentiability occurring whether in real or imaginary part of a complex $\mathcal{PT}$-symmetric potential causes a scarcity of the real discrete eigenvalues despite the real part alone possessing an infinite spectrum. We demonstrate this by perturbing the real potentials x^2 and $|x|$ by imaginary $\mathcal{PT}$-symmetric potentials $ix|x|$ and ix, respectively.

Keywords. Non-Hermitian Hamiltonians; real discrete spectrum, $\mathcal{PT}$ symmetry; complex $\mathcal{PT}$-symmetric potentials; Weber function; Airy function.

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During last ten years or so, the possibility of real discrete spectrum for the complex $\mathcal{PT}$-symmetric Hamiltonians have given rise to very interesting investigations [1–8]. Experimental observations of some distinct aspects of wave propagation in a $\mathcal{PT}$-symmetric medium have been reported more recently [9]. The Hamiltonians that do not change under joint action parity ($\mathcal{P}$: $x \to -x$) and time-reversal ($\mathcal{T}$: $i \to -i$) transformations are called $\mathcal{PT}$ symmetric. The Hamiltonian $H_{BB} = p^2 - (ix)^N$ [1] is now well-known to have infinite and real discrete spectrum if $N \geq 2$. In these calculations x was treated as complex and Dirichlet condition was demanded as $\Psi(|x| \to \infty) \to 0$.

The solvable complex harmonic potential $V_H(x) = x^2 + ix$ entails real discrete spectrum. However, being so simple, it could be dismissed as a trivial and singular instance. The potential model $V_S(x) = -V_1 \operatorname{sech}^2 x + iV_2 \tanh x \operatorname{sech} x, V_1 > 0$ [2] is the first analytically solvable model that has both real discrete spectrum (when $V_2 \leq V_1 + 1/4$) and complex conjugate pairs of eigenvalues (otherwise). In the former case the energy eigenstates are also the eigenstates of $\mathcal{PT}$ and in the latter scenario it is not the case as the $\mathcal{PT}$ symmetry is spontaneously broken. In these models the Dirichlet boundary condition is satisfied, i.e., $\Psi(x \to \pm\infty) \to 0$.

It is instructive to remark that the complex $\mathcal{PT}$-symmetric models with Dirichlet boundary condition imposed on real line essentially have the real part as zero or as a binding well and imaginary part as an antisymmetric (well and barrier) function. For real discrete spectrum, the existence of a pair of complex $\mathcal{PT}$-symmetric classical turning points of the type $\pm a + ib$ [3] is a necessary condition whether or not one is on a real line. The real discrete spectrum of

$$V_L(x) = -\frac{V_0}{a^2+x^2} + i\frac{V_0 x}{a^2+x^2} \tag{1}$$

can be checked to be null, since this potential is nothing but $V_C(x) = iV_0/(x-ia)$ and does not have a pair of $\mathcal{PT}$-symmetric complex turning points. The spectrum of $V(x) = -(ix)^N$ is null on both a real line and a complex contour for $0 \le N \le 1$; since we do not have $\mathcal{PT}$-symmetric turning points. Next, when $1 < N < 2$, the real part is a barrier. However, the condition of $\mathcal{PT}$-symmetric turning points is met and the potential starts possessing a few real discrete eigenvalues [1] on some complex contour. For $2 \le N \le 3$ there is an infinite real discrete spectrum on real line and the real part of $V_N(x)$ is either zero or a potential well (negative curvature). For $3 < N \le 5$, the real part of $V_N(x)$ is a barrier and there are pairs of complex $\mathcal{PT}$-symmetric turning points indicating real discrete spectrum on some complex contour. The real discrete spectrum of $V(x) = -x^4$ and $V(x) = -x^6$ on some complex contour has been found [4] to coincide with the reflectivity zeros of these potential barriers when the scattering problem is solved in a conventional way on real line.

Recently [5] asymptotically vanishing and purely imaginary $\mathcal{PT}$-symmetric potentials like $V(x) = i\,V_0 \operatorname{sech} x \tanh x$, $V(x) = iV_0 x\,\mathrm{e}^{-x^2}$, $V(x) = iV_0 x/(1+x^2)$ and a special rectangular potential defined as: $V_r(0 \le x \le a) = iV_0, V_r(-a \le x \le 0) = -iV_0, V_r(|x| \ge a) = 0$ are found to have null spectrum irrespective of their strength parameter V_0, when one requires $\Psi(\pm\infty) = 0$. However, igx-potential when put in a soft box namely $V(|x| > a) = 0$ was found to have two real discrete eigenvalues when $g > 1.122$.

The igx-potential when put in a hard box, namely $V(|x| > a) = \infty$ gives infinitely many real discrete eigenvalues. In fact, we claim that any arbitrary real or complex $\mathcal{PT}$-symmetric potential when put in a hard box will give infinitely many real discrete eigenvalues. These eigenvalues are generated due to Dirichlet boundary condition at $x = \pm a$. These eigenvalues for $n > n_0$ behave as n^2 and these are hardly a characteristic of the potential. We remark that a $\mathcal{PT}$-symmetric rectangular well displaying an infinite spectrum in ref. [6] is nothing but such an instance only. Notice that the hard-box/soft-box potentials are non-analytic (non-differentiable) only at the boundaries and not within the boundary. Another interesting feature of spectral problems with complex $\mathcal{PT}$-symmetric potentials is the removal (creation) of initial pairs of real (complex) discrete eigenvalues as the strength of the imaginary part passes through several critical values. The potential $V(x) = x^4 + Aix$ [7], the potential discussed in [6] and the potential in a hard box [5] are such models.

More recently, $\mathcal{PT}$-symmetric potentials having a non-analyticity within the boundaries have been found to have only a few real discrete eigenvalues. These potentials are of the type $V(x) = (ix)^a|x|^b, (0 < a, b < 2)$ [8]. It has also been conjectured that analyticity of a complex $\mathcal{PT}$-symmetric potential is an essential feature necessary for real discrete spectrum. By non-analyticity one would mean that $V(z = x + iy)$ does not satisfy the Cauchy–Reimann conditions. However, if one wishes to investigate the spectral problems on real line using the Dirichlet condition: $\Psi(x) \to 0$ as $x \to \pm\infty$ (against $|x| \to \infty$), the non-analyticity of $V(x)$ would mean that either real part or imaginary part of $V(x)$ or both are non-differentiable at some real value(s) of x. In this work, we wish to investigate the spectral problems on the real line which are more akin to the usual (Hermitian) quantum mechanics.

More interestingly, we find that non-analyticity (non-differentiability whether in real or imaginary part) of a complex $\mathcal{PT}$-symmetric potential causes only a scarcity of eigenvalues despite the real part (alone) possessing an infinite spectrum. Importantly, the scarcity may be either an increasing or a decreasing function of the strength of the imaginary part of the potential. The semiclassical methods based on classical turning points are valid only for single piece potentials which are analytic (continuous and differentiable) everywhere in the domain of the eigenvalue problem. Therefore solution of the Schrödinger equation is desired for this study. In this work, we choose two non-analytic potential models

$$V_1(x) = x^2/4 + igx|x| \quad \text{and} \quad V_2(x) = |x| + igx, \tag{2}$$

where the Schrödinger equation is exactly and analytically solvable. We propose to obtain the energy eigenvalue equations (energy discriminants) and find the number of their zeros by varying the value of the perturbation parameter g.

Letting $\hbar = 1 = 2m$, we write the Schrödinger equation for $V_1(x)$ as

$$\frac{\mathrm{d}^2\Psi(x)}{\mathrm{d}x^2} + [E - (x^2/4 + igx|x|)]\Psi(x) = 0. \tag{3}$$

This is the well-known equation whose linearly independent solutions can be written in terms of standard parabolic cylinder (Weber) functions: $D_\lambda(z), D_\lambda(-z)$ [10]. For $x < 0$ by introducing $u = x\sqrt{2}(1/4 - ig)^{1/4}$ and $\mu = \frac{1}{2}\frac{E}{\sqrt{1/4-ig}} - \frac{1}{2}$, we find that $D_\mu(-u)$ is the correct solution that vanishes at $x = -\infty$. Next, for $x > 0$ we find that the correct solution is $D_\nu(v)$ that vanishes at $x = \infty$. Here $\nu = \frac{1}{2}\frac{E}{\sqrt{1/4+ig}} - \frac{1}{2}$ and $v = x\sqrt{2}(1/4 + ig)^{1/4}$. The solution of (1) that is continuous at $x = 0$ and elsewhere can be written as

$$\Psi(x < 0) = D_\nu(0)D_\mu(-u) \quad \text{and} \quad \Psi(x > 0) = D_\mu(0)D_\nu(v). \tag{4}$$

The derivative of these two functions should also be continuous at $x = 0$. So we have

$$(1/4 - ig)^{1/4}D_\nu(0)D'_\mu(0) + (1/4 + ig)^{1/4}D_\mu(0)D'_\nu(0) = 0. \tag{5}$$

Using $D_\lambda(0) = \frac{2^{\lambda/2}}{\Gamma[(1-\lambda)/2]}, D'_\lambda(0) = \frac{2^{\nu/2}}{\Gamma[-\lambda/2]}$ we can write eigenvalue condition as

$$\begin{aligned}\mathcal{D}_1(E,g) = {} & \frac{(1/4 - ig)^{1/4}}{\Gamma[3/4 - E/(4\sqrt{1/4+ig})]\Gamma[1/4 - E/(4\sqrt{1/4-ig})]} \\ & + \frac{(1/4 + ig)^{1/4}}{\Gamma[3/4 - E/(4\sqrt{1/4-ig})]\Gamma[1/4 - E/(4\sqrt{1/4+ig})]} = 0.\end{aligned} \tag{6}$$

When $g = 0$, $\mathcal{D}_1(E,g)$ is identically equal to zero for $E = (2n + 1) + 1/2$ or $E = 2n + 1/2$ as $\Gamma[-n] = \infty$, $n = 0, 1, 2, \ldots$. So one recovers the eigenspectrum

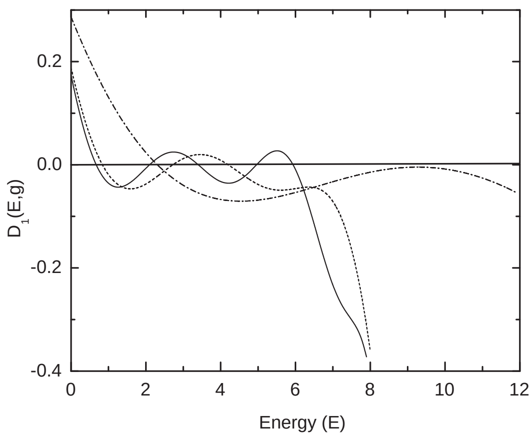

Figure 1. The values of $\mathcal{D}_1(E, g)$ as a function of E for three values of g. $g = 0.3$ (solid curve), $g = 0.5$ (dashed curve), $g = 2.0$ (dash-dot curve). The solid curve shows 5 zeros (real eigenvalues: 0.66, 2.09, 3.40, 4.95, 5.93), the dashed curve shows 3 zeros (0.83, 2.71, 4.19) and the dash-dot curve shows only one zero (1.72).

of the harmonic oscillator as $E_n = (n + 1/2)$. When $g \neq 0$ we can have more convenient forms as

$$\mathcal{D}_1(E, g) = \mathrm{Re}\left[(1/4 - ig)^{1/4}\frac{\Gamma[3/4 - E/(4\sqrt{1/4 - ig})]}{\Gamma[1/4 - E/(4\sqrt{1/4 - ig})}\right] = 0 \qquad (7)$$

or

$$\mathcal{D}_1(E, g) = \mathrm{Re}\left[(1/4 - ig)^{1/4}\frac{\Gamma[1/4 - E/(4\sqrt{1/4 + ig})]}{\Gamma[3/4 - E/(4\sqrt{1/4 + ig})}\right] = 0.$$

A simultaneous use of both the discriminants for plotting or computing zeros of $D_1(E, g)$ may be preferred. Both the formulae would essentially yield coincident results. We find that for $g = 0$ the spectrum is infinite. However, for $g = 0.1, 0.2, 0.3, 0.4, ..., 2.3, 2.4, ..., 10.0, ..., 100.0, ...$ the number of real discrete eigenvalues are $19, 7, 5, 3, ..., 3, 1, ..., 1, ..., 1, ..., 1$. So the number of real discrete eigenvalues reduces from 19 to just one! We propose to call this phenomenon scarcity of real discrete eigenvalues in non-analytic complex $\mathcal{PT}$-symmetric potential e.g., $V_1(x)$. In figure 1, we give the plot of $\mathcal{D}_1(E, g)$ for $g = 0.2, 0.4, 1.0$. These curves, after having a finite number of zeros, keep going downwards monotonically testifying to the non-occurrence of any more real eigenvalue.

Now we take up the second model potential, $V_2(x)$, and write the Schrödinger equation

$$\frac{\mathrm{d}^2\Psi(x)}{\mathrm{d}x^2} + [E - (|x| + igx)]\Psi(x) = 0. \qquad (8)$$

Using $\nu_1 = -1 + ig$, $\nu_2 = 1 + ig$, $\mu_1 = -((\nu_1)^2)^{1/3}$ and $\mu_2 = -((\nu_2)^2)^{1/3}$, we can

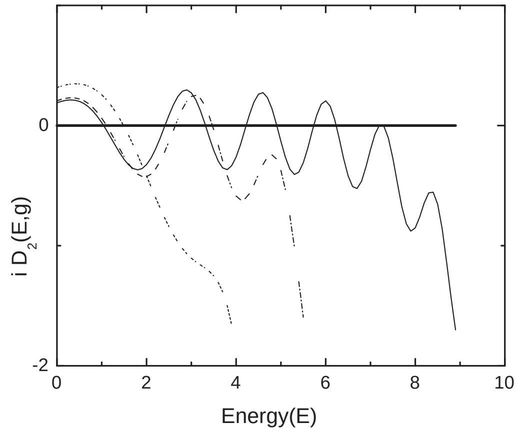

Figure 2. The values of $i\mathcal{D}_2(E,g)$ as a function of E for three values of g. $g=0.2$ (solid curve), $g=0.4$ (dashed curve), $g=1.0$ (dash-dot curve). The solid curve shows 9 zeros (real eigenvalues: 1.04, 2.41, 3.31, 4.22, 4.90, 5.73, 6.23, 7.19, 7.28), the dashed curve shows 3 zeros (1.10, 2.64, 3.47) and the dash-dot curve shows only one zero (1.48).

transform eq. (1) to the standard Airy equation. We can then write the solutions in two regions using Airy function $\mathrm{Ai}(z)$ [10] as

$$\Psi(x<0)=\mathrm{Ai}(E/\mu_2)\mathrm{Ai}[(E-\nu_1 x)/\mu_1]$$

and

$$\Psi(x>0)=\mathrm{Ai}(E/\mu_1)\mathrm{Ai}[(E-\nu_2 x)/\mu_2]. \tag{9}$$

The other linearly independent solution of (8), namely, $\mathrm{Bi}(z)$ has been dropped due to its asymptotic divergence. The solution (9) follows the Dirichlet boundary condition by vanishing at $x=\mp\infty$. The continuous solution (9) should also be differentiable at $x=0$ and this gives the eigenvalue condition as

$$\mathcal{D}_2(E,g)=\nu_1\mu_2\mathrm{Ai}(E/\mu_2)\mathrm{Ai}'(E/\mu_1)-\mu_1\nu_2\mathrm{Ai}(E/\mu_1)\mathrm{Ai}'(E/\mu_2)=0. \tag{10}$$

We compute and find that the number of real discrete eigenvalues as we vary $g = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, ..., 10.0...$; the respective number of eigenvalues are $19, 9, 5, 3, 3, 3, 1, 1, ..., 1, ..., 1, ..., 1...$ (see figure 2) testifying to the occurrence of only a finite number of zeros of $i\mathcal{D}_2(E,g)$ for $g=0.2, 0.4, 1.0$. Like the first model, in this one too, there exists at least one real eigenvalue irrespective of the strength of the imaginary part and there are no negative eigenvalues.

In Hermitian quantum mechanics non-differentiable potentials like $V(x)=|x|^\nu$ and $V(x)=-V_0\mathrm{e}^{-|x|}$ can have infinitely many or finitely many discrete eigenvalues. However, when a complex $\mathcal{PT}$-symmetric potential entails a non-analyticity (non-differentiability in either real part or imaginary part or in both), the spectral problem on real line presents a contrast in this regard. We conclude that all but one real discrete eigenvalues are scared away as the strength of the imaginary part of a non-analytic complex $\mathcal{PT}$-symmetric is varied slowly and smoothly in an adiabatic way. Alternatively, we get a class of complex $\mathcal{PT}$-symmetric potentials which have only one discrete eigenvalue. However, it would require a study of more number of such potential models and a model-independent theory for the confirmation of this phenomenon. We feel that the recent interesting results on $V(x) = (ix)^a|x|^b$ ($0 < a$, $b < 2$) [8] would rather support the phenomenon of scarcity of real discrete spectrum discussed here than the conjecture [8] that analyticity is an essential feature of a complex $\mathcal{PT}$-symmetric potential necessary for real discrete spectrum.

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