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The non-equivalence of pseudo-Hermiticity and presence of antilinear symmetry

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Abstract. The non-equivalence of the presence of antilinear symmetry and pseudo-Hermiticity is shown for bounded operators. Two appropriate examples are operators with non-empty residual spectrum. The class of operators for which the equivalence holds is extended to the spectral operators of scalar type. The importance of J-self-adjointness is stressed and new proofs using this property are provided.

Keywords. Pseudo-Hermiticity; antilinear symmetry; J-self-adjoint operators.

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1. Introduction

In the framework of $\mathcal{PT}$-symmetric quantum mechanics (PTSQM), there arises a natural question as to whether $\mathcal{PT}$-symmetric operators are pseudo-Hermitian at the same time and vice versa. Mostafazadeh, Scolarici and Solombrino [1–3] investigated the equivalence of pseudo-Hermiticity and the presence of antilinear symmetry for very special classes of operators (diagonalizable or with finite Jordan blocks). Although it may seem that only technical generalization of proofs presented there brings a complete solution for all operators, and thus also often stressed equivalence of PTSQM and pseudo-Hermitian QM, the deeper analysis shows that the two properties are not equivalent even for bounded operators.

Moreover, two examples presented in this paper have an 'alarming' property – non-empty residual spectrum. This fact strongly affects all possible physical applications involving these types of non-Hermitian operators. It is necessary to explain the physical meaning of the residual spectrum, or to restrict the considered operators to some suitable class.

The presence of antilinear symmetry together with pseudo-Hermiticity already excludes the non-empty residual spectrum. Furthermore, such operators may be J-self-adjoint (Definition 7, more in [4]). This class of operators, with empty residual spectrum, has been already suggested as 'adequate for a rigorous formulation of

$\mathcal{PT}$-symmetric problems' by Borisov and Krejčiřík [5]. J-self-adjoint operators are intensively studied in the mathematical literature [6–9] as well. Typical examples considered in PTSQM are both $\mathcal{PT}$-symmetric and $\mathcal{P}$-pseudo-Hermitian operators which are then also $\mathcal{T}$-self-adjoint. The parity $\mathcal{P}$ and the time reversal operators $\mathcal{T}$ are defined on $L_2(\mathbb{R})$ by $(\mathcal{P}\psi)(x) = \psi(-x)$ and $(\mathcal{T}\psi)(x) = \bar{\psi}(x)$, respectively.

At first, we briefly, but precisely, formulate basic definitions and propositions. Then we show a new proof of the equivalence for finite-dimensional spaces using J-self-adjoint operators. In fact, this approach can be generalized to spectral operators of scalar types as we show later. The main result is the proof of non-equivalence of antilinear symmetry and pseudo-Hermiticity for bounded operators on Hilbert spaces of infinite dimension. To this end, we present two examples of operators contradicting the equivalence. We try to explain why this situation may occur – operators are not spectral. We conclude with possible application of the equivalence on the construction of the metric for quasi-Hermitian operators.

We give the proofs of the main statements only. Nonetheless, the rest of the proofs, particularly for the following section, can be found in [10].

2. Basic definitions and properties

To be able to work with '$\mathcal{PT}$ symmetry-like models' in a general Hilbert space, the $\mathcal{PT}$ symmetry is generalized to the presence of antilinear symmetry. $\mathcal{B}(\mathcal{H})$ denotes the set of bounded operators on a Hilbert space $\mathcal{H}$.

DEFINITION 1
Let A be a densely defined closed operator in $\mathcal{H}$. We say that A has an antilinear symmetry if there exists an antilinear bijective operator C and the relation

$$AC\psi = CA\psi \tag{1}$$

holds for all $\psi \in \mathrm{Dom}(A)$.

We may easily show the consequence of antilinear symmetry.

PROPOSITION 2
Let A be an operator having an antilinear symmetry C. Then $\lambda \in \mathbb{C}$ is in the spectrum of A if and only if $\bar{\lambda}$ is in the spectrum of A. Moreover, this equivalence is also valid for the disjoint parts of spectrum, i.e. $\lambda \in \sigma_{\mathrm{p,c,r}}(A) \Longleftrightarrow \bar{\lambda} \in \sigma_{\mathrm{p,c,r}}(A)$.

Remark 3. We use the definition of spectrum and its point, continuous, and residual part presented e.g. in [11].

DEFINITION 4
Let A be a densely defined operator. A is called weakly pseudo-Hermitian if there exists an operator η with properties

(i) $\eta, \eta^{-1} \in \mathcal{B}(\mathcal{H})$,
(ii) $A = \eta^{-1} A^* \eta$.

If η is self-adjoint, then A is called pseudo-Hermitian. If we want to specify particular η, we say that A is η-(weakly)-pseudo-Hermitian. If η is positive, then A is called quasi-Hermitian.

PROPOSITION 5
Let A be a weakly pseudo-Hermitian operator. Then A is closed.

Weak pseudo-Hermiticity implies spectral properties which are in general not the same as in the case of the presence of an antilinear symmetry.

PROPOSITION 6
Let A be a weakly pseudo-Hermitian operator. Then point, continuous, and residual spectrum $\sigma_{\mathrm{p,c,r}}(A)$ of A and $\sigma_{\mathrm{p,c,r}}(A^*)$ of A^* are equal.

We recall also a notion of J-self-adjoint operators used in next sections.

DEFINITION 7 [4]
Let A be densely defined operator. Let J be an antilinear isometric involution, i.e. $J^2 = I$ and $\langle Jx, Jy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. A is called J-symmetric if $A \subset JA^*J$. A is called J-self-adjoint if $A = JA^*J$.

3. Equivalence relations

3.1 *Finite dimension*

$\mathscr{L}(V_n)$ denotes the space of linear operators acting in a Hilbert space V_n of finite dimension n, A_{J} is a Jordan form of the matrix A and X is the similarity transformation.

PROPOSITION 8
Let $A \in \mathscr{L}(V_n)$. Then A is pseudo-Hermitian if and only if it possesses an antilinear symmetry.

Proof. Every matrix is similar to a J-self-adjoint operator, i.e. there exists invertible $X \in \mathscr{L}(V_n)$, such that $A_{\mathrm{J}} = XAX^{-1}$ is J-self-adjoint. Proof of this fact is elementary and can be found in [7].

Let A be pseudo-Hermitian, i.e. $A = \eta^{-1}A^*\eta$. Using the similarity to the J-self-adjoint operator, we obtain

$$XAX^{-1} = A_{\mathrm{J}} = JA_{\mathrm{J}}^*J = J(X^{-1})^*A^*X^*J = J(X^{-1})^*\eta A\eta^{-1}X^*J. \quad (2)$$

We denote $C = \eta^{-1}X^*JX$ and it follows from (2) that $CA = AC$ and C is antilinear due to the antilinearity of J.

Let C be the antilinear symmetry of A. We find easily, using again the Jordan form A_{J} and its J-self-adjointness, that

$$A = C^{-1}X^{-1}J(X^{-1})^*A^*X^*JXC. \quad (3)$$

Hence A is η-weakly-pseudo-Hermitian with $\eta = X^*JXC$. Weak pseudo-Hermiticity and pseudo-Hermiticity are equivalent properties on V_n [12]. □

3.2 *Two counterexamples*

The first operator is the standard example of operator with non-empty residual spectrum [13] and in our framework it represents an operator with antilinear symmetry, however, without pseudo-Hermiticity. We constructed the second example for showing pseudo-Hermitian operator without antilinear symmetry.

Example 9. Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthonormal basis of $\mathcal{H} = l_2(\mathbb{N})$, i.e. $e_n(m) = \delta_{nm}$. Let T be an operator on $\mathcal{H}$ acting as

$$Te_n := e_{n-1}, \quad n \in \mathbb{N}, \quad e_0 := 0. \tag{4}$$

T is bounded and it possesses the antilinear symmetry $\mathcal{T}$-complex conjugation. Adjoint operator T^* acts as

$$T^*e_n := e_{n+1}, \quad n \in \mathbb{N}. \tag{5}$$

Every complex number λ with absolute value $|\lambda| < 1$ is in the point spectrum $\sigma_{\mathrm{p}}(T)$ of T, corresponding eigenvector x_λ reads $x_\lambda = \sum_{n=1}^{\infty} \lambda^{n-1} e_n$. Spectrum $\sigma_{\mathrm{p}}(T^*)$ of T^* is different. The point spectrum of T^* is empty and the set $\{\lambda \in \mathbb{C} \colon |\lambda| < 1\}$ belongs to the residual spectrum $\sigma_{\mathrm{r}}(T^*)$.

Hence operator T is not weakly pseudo-Hermitian because the necessary condition for the weak pseudo-Hermiticity is the equality of the spectra of T and T^* (Proposition 6).

Example 10. Let $\mathcal{H} = l^2(\mathbb{Z})$ and let $\{e_i\}_{-\infty}^{\infty}$ be the orthonormal basis, $e_n(m) = \delta_{nm}$. Operator T acts as

$$Te_i := \begin{cases} \lambda_0 e_i + e_{i+1}, & i \geq 1, \\ 0, & i = 0, \\ \bar{\lambda}_0 e_{-1}, & i = -1, \\ \bar{\lambda}_0 e_i + e_{i+1}, & i < -1, \end{cases} \tag{6}$$

where $\lambda_0 \in \mathbb{C}$, $\operatorname{Im} \lambda_0 > \frac{1}{2}$. We find T^* easily from the definition of the adjoint operator,

$$\langle e_i, Te_j \rangle = \begin{cases} \lambda_0 \delta_{i,j} + \delta_{i,j+1}, & i,j \geq 1, \\ \bar{\lambda}_0 \delta_{i,j} + \delta_{i,j+1}, & i,j \leq -1, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

Hence

$$T^*e_i = \begin{cases} \bar{\lambda}_0 e_i + e_{i-1} & i > 1, \\ \bar{\lambda}_0 e_1, & i = 1, \\ 0, & i = 0, \\ \lambda_0 e_i + e_{i-1}, & i \leq -1. \end{cases} \tag{8}$$

Let $\mathcal{P}$ be a parity, i.e.

$$\mathcal{P}e_i := e_{-i}. \tag{9}$$

We may show immediately from the definitions that T is $\mathcal{P}$-pseudo-Hermitian,

$$T = \mathcal{P}T^*\mathcal{P}. \tag{10}$$

It is obvious that $\bar{\lambda}_0 \in \sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{p}}(T^*)$. We show that $\lambda_0 \in \sigma_{\mathrm{r}}(T) = \sigma_{\mathrm{r}}(T^*)$. We express the equation

$$(T - \lambda_0) \sum_{i=-\infty}^{\infty} \alpha_i e_i = 0, \tag{11}$$

and determine coefficients α_i.

$$\begin{aligned}(T - \lambda_0) \sum_{i=-\infty}^{\infty} \alpha_i e_i &= \sum_{i=-\infty}^{-2} \alpha_i \left[(\bar{\lambda}_0 - \lambda_0)e_i + e_{i+1}\right] \\ &\quad + \alpha_{-1}(\bar{\lambda}_0 - \lambda_0)e_{-1} - \alpha_0\lambda_0 e_0 + \sum_{i=1}^{\infty} \alpha_i e_{i+1}.\end{aligned} \tag{12}$$

Hence

$$\begin{aligned}&\alpha_i = 0 \quad \text{for } i \geq 0, \\ &\alpha_i(\bar{\lambda}_0 - \lambda_0) + \alpha_{i-1} = 0 \quad \text{for } i < 0.\end{aligned} \tag{13}$$

α_{-1}, and hence all α_i, must be 0. If $\alpha_{-1} \neq 0$ then for $i < -1$, $\alpha_i = \alpha_{-1}\left(2i \text{ Im } \lambda_0\right)^{-i-1}$. Since $\operatorname{Im} \lambda_0 > \frac{1}{2}$ by the definition of T,

$$\sum_{i=-\infty}^{\infty} |\alpha_i|^2 = +\infty. \tag{14}$$

Hence λ_0 is not an eigenvalue and $\overline{\operatorname{Ran}(T - \lambda_0)} \neq \mathcal{H}$ for

$$e_1 \notin \overline{\operatorname{span}(e_i)_{i\neq 1}} = \overline{\operatorname{Ran}(T - \lambda_0)}. \tag{15}$$

This proves that $\lambda_0 \in \sigma_{\mathrm{r}}(T)$ and there holds also $\lambda_0 \in \sigma_{\mathrm{r}}(T^*)$ by Proposition 6.

The operator T cannot have any antilinear symmetry because the necessary condition is that $\lambda_0 \in \sigma_{\mathrm{p,c,r}}(T) \Leftrightarrow \bar{\lambda}_0 \in \sigma_{\mathrm{p,c,r}}(T)$, by Proposition 2.

COROLLARY 11

Weak pseudo-Hermiticity and antilinear symmetry are non-equivalent properties even for bounded operators on $\mathcal{H}$.

Nonetheless, it is necessary to remark that operators from previous examples are not spectral (the definition of spectral operators can be found in [14]). To justify this, consider that in a separable Hilbert space the point and residual spectra of a spectral operator are countable [14]. Spaces $l^2(\mathbb{N})$ and $l^2(\mathbb{Z})$ are separable and we have already shown that the set $\{\lambda \in \mathbb{C} \colon |\lambda| < 1\}$ is included in the point spectrum of the operator from Example 9. Similarly, the set $\omega = \left\{\lambda \in \mathbb{C} \colon |\lambda - \bar{\lambda}_0| < 1\right\}$ is included in the point spectrum of the adjoint operator T^* from Example 10. Therefore, ω is included in the union of the point and residual spectrum of T.

3.3 *The equivalence for spectral operators of scalar type*

Despite the presented counterexamples, the equivalence holds at least for the class of spectral operators of scalar type – Definition in [14]. The proof of the following proposition is a direct generalization of the proof for finite dimensional spaces.

PROPOSITION 12

Let $S \in \mathcal{B}(\mathcal{H})$ be a spectral operator of scalar type. S is weakly pseudo-Hermitian if and only if it possesses an antilinear symmetry.

Proof. Every spectral operator of scalar type is similar to a normal operator N [14], $S = X^{-1}NX$. Every normal operator is J-self-adjoint [7] (proof is based on the spectral theorem for normal operators). The resting part is exactly the same procedure as for matrices.

Remark 13. We say that $A \in \mathscr{L}(\mathcal{H})$ is similar to $B \in \mathscr{L}(\mathcal{H})$ if there exists $X \in \mathcal{B}(\mathcal{H})$ with a bounded everywhere defined inverse such that $A = X^{-1}BX$.

It is obvious that the presented statement is not limited to operators with discrete spectrum, usually considered in the previous works, and hence it is the generalization for the operators with continuous spectra. The case of operators with finite Jordan blocks [2] corresponds to the spectral operator of finite type [14] (the quasi-nilpotent part is nilpotent).

4. Applications

The equivalence of the presence of antilinear symmetry and pseudo-Hermiticity could be very helpful in the construction of the metric Θ. Actually, the presence of both the properties is only the first step. We need also other suitable properties of C and η as it can be seen in the following proposition.

It is well-known [15] that the metric Θ for the quasi-Hermitian operator A with purely discrete spectrum can be calculated as

$$\Theta = \underset{N\to\infty}{\text{s–lim}} \sum_{j=1}^{N} \langle \phi_j, \cdot \rangle \phi_j, \tag{16}$$

where $\{\phi_n\}_{n=1}^{\infty}$ are eigenvectors of A^*.

A converse question, whether an operator A with a purely discrete real spectrum is quasi-Hermitian and also the related problem of the construction of metric with appropriate properties (Definition 4), is much more delicate. First of all, neither antilinear ($\mathcal{PT}$) symmetry nor pseudo-Hermiticity alone, or even together, do not guarantee the completeness of eigenvectors (see special cases in the model [16]). Then, although the metric Θ exists, i.e. the strong limit (16) is a bounded operator, and it is invertible, the boundedness of Θ^{-1} is not obvious and it is usually difficult to prove. The much earlier paper on quasi-Hermitian operators [17] (in broader sense, $\Theta^{-1} \in \mathcal{B}(\mathcal{H})$ is not required in the definition of quasi-Hermitian operator) illustrates how essential the boundedness of Θ^{-1} for the reality of the spectrum is – an example with complex spectrum is constructed there.

Nevertheless, we can show that the problem for at least J-self-adjoint operators, which is a case of $\mathcal{PT}$-symmetric and $\mathcal{P}$-pseudo-Hermitian operators at the same time, becomes easier because complete J-orthonormal systems are Riezs bases if they satisfy certain technical properties [6]. These conditions are guaranteed by the assumptions in the following statement. The detailed proofs and discussion can be found in [10].

PROPOSITION 14

Let A be an η-pseudo-Hermitian operator with antilinear symmetry C, where η and C are both involutive, commuting, and ηC is isometric. Let the resolvent of A be compact for some $\mu \in \mathbb{C}$ and the spectrum of A be real. If

$$\Theta = \operatorname*{s-lim}_{N\to\infty} \sum_{j=1}^{N} c_j \langle \phi_j, \cdot \rangle \phi_j, \tag{17}$$

where $\|\phi_j\| = 1$ are eigenvectors of A^* and c_j are positive constants satisfying

$$\exists\, m, M > 0, \quad m \le c_j \le M \quad \text{for all} \quad j \in \mathbb{N}, \tag{18}$$

exists and it is an invertible bounded operator, i.e. $0 \notin \sigma_{\mathrm{p}}(\Theta)$, then A is quasi-Hermitian with the metric Θ, i.e. $0 \notin \sigma_{\mathrm{c}}(\Theta)$ (or $\Theta^{-1} \in \mathcal{B}(\mathcal{H})$).

5. Concluding remarks

Spectral operators seem to be suitable for considering the equivalence of antilinear symmetry and pseudo-Hermiticity. However, the proof of the equivalence for general bounded spectral operator is not known.

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