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Non-Hermitian Hamiltonians with a real spectrum and their physical applications

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Abstract. We present an evaluation of some recent attempts to understand the role of pseudo-Hermitian and $\mathcal{PT}$-symmetric Hamiltonians in modelling unitary quantum systems and elaborate on a particular physical phenomenon whose discovery originated in the study of complex scattering potentials.

Keywords. $\mathcal{PT}$ symmetry; pseudo-Hermiticity; metric operator; unitary equivalence; scattering potential; spectral singularity; resonance.

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1. Introduction

The use of non-Hermitian operators in theoretical physics has a long history [1]. These operators are traditionally employed in the effective description of physical systems displaying decay or dissipative behaviour. The main quality of non-Hermitian operators that motivated these applications is that a generic non-Hermitian operator has complex eigenvalues whose imaginary part may be associated with decay rates. This property is not however common to all non-Hermitian operators; there is a class of non-Hermitian operators that, similar to Hermitian operators, have a real spectrum. During the past ten years or so, these operators have been the focus of intensive research activity particularly following the work of Bender and Boettcher [2] on $\mathcal{PT}$-symmetric Hamiltonian operators. The possibility that these operators can have a purely real spectrum has led to the conjecture that one can actually use them to describe unitary quantum systems. In the present article, I elaborate on the physical significance of the above conjecture and draw the attention of the reader to a physical phenomenon that has been recently discovered in an attempt to understand a class of non-Hermitian Hamiltonian operators with a continuous spectrum. To the best of my knowledge, this is the first example of a physical effect whose discovery has its origin in the study of non-Hermitian Hamiltonians possessing a real spectrum.

2. Basic ingredients of the formalism

For a system having the real line $\mathbb{R}$ as the configuration space, the parity and time reversal operators are respectively defined by $\mathcal{P}\psi(x) := \psi(-x)$ and $\mathcal{T}\psi(x) := \psi(x)^*$, where ψ is an arbitrary complex-valued function typically belonging to $L^2(\mathbb{R}) := \{\psi\colon \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x < \infty\}$. A linear operator H acting on $L^2(\mathbb{R})$ is said to posses $\mathcal{PT}$ symmetry if $[H, \mathcal{PT}] = 0$. If there is a complete set of eigenvectors ψ_n of H that are $\mathcal{PT}$-invariant, i.e., $\mathcal{PT}\psi_n = \psi_n$, then H is said to have an exact $\mathcal{PT}$ symmetry. The latter is a very strong and difficult-to-check condition on H.

Because $\mathcal{PT}$ is an antilinear operator, $\mathcal{PT}$ symmetry implies that the spectrum of H is symmetric about the real axis. In particular, complex eigenvalues come in complex-conjugate pairs. This is actually very easy to show. Similarly, it is easy to show that exact $\mathcal{PT}$ symmetry implies the reality of all the eigenvalues. What is by no means easy to show is whether a given operator possesses exact $\mathcal{PT}$ symmetry. To do this one must first establish the existence of a $\mathcal{PT}$-invariant set of eigenvectors of H and prove their completeness.

In order to introduce a notion of completeness, one must adopt a particular notion of convergence on the function space where H acts. If this is $L^2(\mathbb{R})$, one usually takes the standard L^2-inner product: $\langle\psi|\phi\rangle := \int_{-\infty}^{\infty} \psi(x)^*\phi(x)\,\mathrm{d}x$ to define a norm, namely $||\psi|| := \sqrt{\langle\psi|\psi\rangle}$, and use the latter to determine the convergence of sequences. We shall denote by $\mathcal{H}$ the Hilbert space obtained by endowing $L^2(\mathbb{R})$ with this inner product.

If $\{\xi_n\}$ is a sequence in $\mathcal{H}$, and for every $\psi \in \mathcal{H}$ there are complex numbers c_n such that the series $\sum_{n=1}^{\infty} c_n\xi_n$ converges to ψ, i.e., $\lim_{N\to\infty} ||\psi - \sum_{n=1}^{N} c_n\psi_n|| = 0$, we say that $\{\xi_n\}$ is a basis of $\mathcal{H}$. It turns out that the notion of 'completeness' is a stronger condition on a sequence $\{\xi_n\}$. It is equivalent to the existence of a complementary sequence $\{\zeta_n\}$ in $\mathcal{H}$ that satisfies $\langle\zeta_m|\xi_n\rangle = \delta_{mn}$. $\{\zeta_n\}$ is then also a basis and $\{(\xi_n, \zeta_n)\}$ is called a biorthonormal system [1]. The bases $\{\xi_n\}$ that have this property are called Riesz bases. Every orthonormal basis $\{\varepsilon_n\}$ is clearly a Riesz basis, because $\{(\varepsilon_n, \varepsilon_n)\}$ is a biorthonormal system. We say that a linear operator $H\colon \mathcal{H} \to \mathcal{H}$ is diagonalizable if H has a set of eigenvectors ψ_n that form a Riesz basis, i.e., it has a complete set of eigenvectors. In this case the complementary (biorthonormal) basis $\{\phi_n\}$ associated with $\{\psi_n\}$ are eigenvectors of the adjoint of H that we denote by $H^\dagger$. A precise definition of $H^\dagger$ is given in [1]. Here it would be suffice to mention that for any ψ and ϕ taken respectively from the domains of H and $H^\dagger$, we have $\langle\psi|H\phi\rangle = \langle H^\dagger\psi|\phi\rangle$. We will also refer to $\{(\psi_n, \phi_n)\}$ as a biorthonormal eigensystem for H.

An operator $H\colon \mathcal{H} \to \mathcal{H}$ is said to be Hermitian or self-adjoint if $H^\dagger = H$. This means that for every ψ, ϕ in the domain of H, $\langle\psi|H\phi\rangle = \langle H\psi|\phi\rangle$. Much of the confusion in the study of $\mathcal{PT}$-symmetric Hamiltonians may be traced to the misconception that the notion of 'Hermiticity' can be defined independently of the choice of the inner product of $\mathcal{H}$. Many authors follow the naive and unjustified practice of choosing a preferred basis (such as the position basis) in $\mathcal{H}$, represent the operator H using a matrix $\underline{H}$ in this basis, and define the Hermiticity condition as the requirement that the transpose of $\underline{H}$ be equal to its complex-conjugate, $\underline{H}^t = \underline{H}^*$. This is OK only if the basis one works with is an orthonormal basis.

But to determine the orthonormality of a basis one needs to use the inner product of $\mathcal{H}$. The term 'Hermitian operator' is meaningless unless one specifies this inner product. Clearly, different choices for the inner product lead to different notions of 'Hermiticity'.

The mystery underlying the reality of the spectrum of the $\mathcal{PT}$-symmetric Hamiltonian operators such as $p^2 + ix^3$ is unraveled once one recognizes that these operators are actually Hermitian with respect to a nonstandard inner product [4]. It is misleading to claim that Hermiticity is an unphysical condition, and hence it must be replaced by the physical condition of $\mathcal{PT}$ symmetry which represents space-time reflection symmetry [3]. It is in fact easy to show that one can never avoid the requirement of the Hermiticity of observables, because this is a necessary condition for the reality of expectation values [1]. What the recent developments have revealed is the possibility of employing nonstandard inner products in quantum mechanics. This summarizes the main conceptual outcome of more than ten years of intensive research on this subject.

This result has a number of important implications [1].

1. Contrary to initial expectations, $\mathcal{PT}$ symmetry does not play any distinctively important role. Any $\mathcal{PT}$-symmetric or non-$\mathcal{PT}$-symmetric operator that has a real spectrum and a complete set of eigenvectors can serve the same purpose. These operators (with proper extension of the notion of completeness to the cases that the spectrum possesses a continuous part) can be related to Hermitian operators by a similarity transformation [4]. Hence they are quasi-Hermitian [5]. Furthermore, every quasi-Hermitian operator H has an exact symmetry generated by an antilinear operator $\mathfrak{S}$ that is an involution, i.e., H has a $\mathfrak{S}$-invariant complete set of eigenvectors and $\mathfrak{S}^2 = 1$ [6,7]. Clearly, $\mathcal{PT}$ is just a particular example of $\mathfrak{S}$. Other examples of antilinear symmetries have been considered in [8].
2. In contrast to the initial expectations [3], the use of $\mathcal{PT}$-symmetric Hamiltonians does not lead to a genuine extension of quantum mechanics. Rather, it provides a new representation of the same theory where the physical Hilbert space is defined using the new inner product. The latter can be most straightforwardly constructed as follows. First, one recalls that every inner product has the form $\langle\psi,\phi\rangle_{\eta_+} := \langle\psi|\eta_+\phi\rangle$ for some positive automorphism (a bounded invertible linear operator) η_+ called a metric operator. The inner products $\langle\cdot,\cdot\rangle_{\eta_+}$ with respect to which a quasi-Hermitian operator H is Hermitian are given by metric operators satisfying the pseudo-Hermiticity condition [9]:

$$H^\dagger = \eta_+ H \eta_+^{-1}. \tag{1}$$

   The physical Hilbert space is then defined using the inner product $\langle\psi,\phi\rangle_{\eta_+}$ [10,11]. We will denote the resulting space by $\mathcal{H}_{\eta_+}$.
3. To determine the physical meaning of a given quasi-Hermitian operator H, one needs to choose an admissible inner product (that renders H Hermitian) and map the latter to an equivalent Hermitian operator. A canonical choice is $h_{\eta_+} := \eta_+^{1/2} H\, \eta_+^{-1/2}$. One can show that as a linear operator mapping $\mathcal{H}_{\eta_+}$ to $\mathcal{H}$ the operator $\eta_+^{1/2}$ is a unitary operator [10]. Therefore the Hilbert

space-Hamiltonian pairs $(\mathcal{H}_{\eta_+}, H)$ and $(\mathcal{H}, h_{\eta_+})$ are unitary-equivalent; they describe the same physical system [12]. We will refer to them as the pseudo-Hermitian and Hermitian representations of the system, respectively.

4. Some of the notions developed in the study of $\mathcal{PT}$-symmetric Hamiltonians do not actually play a fundamental role. The primary example is the $\mathcal{C}$ operator that is used as a tool for specifying a particular example of the inner products $\langle\cdot,\cdot\rangle_{\eta_+}$ called the $\mathcal{CPT}$-inner product [3]. As shown in [1,7], this inner product corresponds to the choice $\eta_+ = \mathcal{PC}$, where $\mathcal{C}$ is required to fulfil [3]

$$\mathcal{C}^2 = 1, \quad [\mathcal{C}, H] = 0, \quad [\mathcal{C}, \mathcal{PT}] = 0. \tag{2}$$

According to the prescription given in [3,13], one must first solve the operator equations (2) and

$$\mathcal{C} = \mathrm{e}^{Q}\mathcal{P}, \tag{3}$$

for a Hermitian operator Q, substitute the result in (3) to determine $\mathcal{C}$, and then construct the $\mathcal{CPT}$-inner product which actually coincides with $\langle\cdot,\cdot\rangle_{\mathrm{e}^{-Q}}$. Therefore, this procedure provides means for computing a metric operator of the form $\eta_+ = \mathrm{e}^{-Q}$ [13a]. In fact, all the quantities of interest, for example the equivalent Hermitian Hamiltonian, physical observables, and expectation values, only involve the metric operator. Therefore, the $\mathcal{C}$ operator has a secondary role as far as the physical aspects are concerned.

An alternative procedure to the one based on the $\mathcal{C}$ operator, that actually gives the most general admissible inner product, is to solve a single operator equation, namely (1), for η_+. Different methods of solving this equation are discussed in [1]. In particular, there is a highly effective method of dealing with this equation that involves expressing it as a differential equation for the kernel $\langle x|\eta_+|y\rangle$ [14]. The approach based on the pseudo-Hermiticity relation (1) avoids dealing with a $\mathcal{C}$ operator and its defining equations (2) and (3) [1]. Therefore, it is more direct.

3. From formalism to applications

Pseudo-Hermitian representation of quantum mechanics and the techniques developed in the course of its investigation have found applications in a variety of subjects [1]. Here we wish to discuss a rare example of a physical phenomenon whose discovery originated in trying to address the problem of the existence of a metric operator for a class of non-Hermitian operators with a real and continuous spectrum [15].

As we explained in §2, a linear operator H that satisfies the pseudo-Hermiticity relation (1) acts as a Hermitian operator in the Hilbert space $\mathcal{H}_{\eta_+}$. This implies that H has a real spectrum and a complete set of eigenvectors. Clearly, these two properties are independent; there are operators with a complete set of eigenvectors (diagonalizable operators) that lack a real spectrum, and there are operators with a real spectrum that are not diagonalizable.

Usually the lack of diagonalizability is associated with the presence of exceptional points. These are degeneracies at which both the eigenvalues and eigenvectors coalesce. Exceptional points have interesting physical implications [16]. They may appear for operators acting in finite or infinite-dimensional Hilbert spaces. There is also another less-known obstruction to the diagonalizability of non-Hermitian operators called spectral singularities. These may occur for non-Hermitian operators whose spectrum has a continuous part (hence the space in which they act is necessarily infinite-dimensional).

Spectral singularities were discovered in the mid-1950s by Naimark [17] and studied thoroughly by mathematicians [18]. In the context of recent study of $\mathcal{PT}$-symmetric Hamiltonians, the possibility of the presence of a spectral singularity was initially noted by Samsonov [20] who following the work of Naimark [17,19] only considered models defined on the half-line. In ref. [21], I have worked out the computation of a metric operator and the corresponding Hermitian Hamiltonian for $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + z\delta(x)$ where z is a complex coupling constant, x takes values in the whole real line, and $\delta(x)$ is the Dirac delta function. For this model a spectral singularity manifests itself as an obstruction for the construction of a biorthonormal eigensystem for the case that z is purely imaginary. In ref. [15] we explore the mechanism by which spectral singularities spoil the completeness of the eigenfunctions for a general complex scattering potential. We also offer a detailed investigation of the spectral singularities for the Hamiltonians of the form $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + z_-\delta(x+a) + z_+\delta(x-a)$, with $a \in \mathbb{R}^+$ and $z_\pm \in \mathbb{C}$, which include as a special case the $\mathcal{PT}$-symmetric Hamiltonians corresponding to the choice $z_- = z_+^*$ [22].

Reference [23] provides a physical interpretation for spectral singularities. It turns out that spectral singularities correspond to the energies where both the left and right transmission and reflection coefficients diverge. In other words, they are associated with resonances having a zero width. This resonance phenomenon may be realized in an electromagnetic waveguide modelled using the $\mathcal{PT}$-symmetric barrier potential:

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v_{a,\zeta}(x),$$
$$v_{a,z}(x) := \begin{cases} i\zeta & \text{for} \quad -a < x < 0, \\ -i\zeta & \text{for} \quad 0 < x < a, \\ 0 & \text{for} \quad x = 0 \text{ or } |x| > a, \end{cases} \qquad \zeta \in \mathbb{R}, \quad a \in \mathbb{R}^+.$$

It implies that at the energies (frequencies) of spectral singularities, such a waveguide may be used as a resonator. This is a new physical effect that awaits an experimental confirmation.

In the next section the spectral singularities of an imaginary delta-function potential is explored. This is one of the simplest exactly solvable complex potentials that one can consider. Yet the possibility that this potential might involve spectral singularities was noted quite recently [21]. This is mainly because spectral singularities do not play an important role unless one attempts at constructing a metric operator for the system. The latter could be realized only after the development of a particular method of computing metric operators [4,10,11] that is called 'spectral method' in §4 of [1].

4. Spectral singularities

Consider a complex potential v: $\mathbb{R} \to \mathbb{C}$ such that $\int_{-\infty}^{\infty}(1+|x|)|v(x)|\mathrm{d}x < \infty$. Then the spectrum of the Schrödinger operator $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x)$ that is defined over the whole real line has a continuous part. Suppose for simplicity that the spectrum is just $[0,\infty)$. The (generalized or scattering) eigenvalues E are doubly degenerate and the corresponding eigenfunctions have the following asymptotic behaviour.

$$\psi_k^{\mathfrak{g}}(x) \to A_\pm^{\mathfrak{g}} \mathrm{e}^{ikx} + B_\pm^{\mathfrak{g}} \mathrm{e}^{-ikx}, \quad \text{for } x \to \pm\infty. \tag{4}$$

Here $\mathfrak{g}$ is a degeneracy label taking values 1 and 2, $k\!: = \sqrt{E}$ and $A_\pm^{\mathfrak{g}}$, $B_\pm^{\mathfrak{g}}$ are possibly k-dependent complex coefficients. One can use the eigenvalue equation for H to relate $A_+^{\mathfrak{g}}$ and $B_+^{\mathfrak{g}}$ to $A_-^{\mathfrak{g}}$ and $B_-^{\mathfrak{g}}$. The result can be expressed in terms of a transfer matrix $\mathbf{M}$ that by definition fulfills

$$\begin{pmatrix} A_+^{\mathfrak{g}} \\ B_+^{\mathfrak{g}} \end{pmatrix} = \mathbf{M} \begin{pmatrix} A_-^{\mathfrak{g}} \\ B_-^{\mathfrak{g}} \end{pmatrix}. \tag{5}$$

It is easy to show that $\det \mathbf{M} = 1$ [15].

A particularly useful choice for a pair of eigenfunctions in each energy level are the Jost solutions $\psi_{k\pm}$. These are determined by their asymptotic behaviour that is given by

$$\psi_{k\pm}(x) \to \mathrm{e}^{\pm ikx} \quad \text{for } x \to \pm\infty. \tag{6}$$

If we respectively denote the coefficients $A_\pm^{\mathfrak{g}}$ and $B_\pm^{\mathfrak{g}}$ for the Jost solutions $\psi_{k\pm}$ as $A_\pm^\pm$ and $B_\pm^\pm$, then in view of (4)–(6) we find [15]

$$\begin{aligned} &A_+^+ = B_-^- = 1, \quad A_-^- = B_+^+ = 0, \quad A_-^+ = B_+^- = M_{22}, \\ &A_+^- = M_{12}, \quad B_-^+ = -M_{21}, \end{aligned} \tag{7}$$

where M_{ij} are the entries of $\mathbf{M}$. Equations (7) show that $\psi_{k\pm}$ are nothing but the left- and right-going scattering solutions [24], and that the left and right transmission $T^{l,r}$ and reflection $R^{l,r}$ amplitudes are given by [23]

$$T^l = T^r = 1/M_{22}, \quad R^l = -M_{21}/M_{22}, \quad R^r = M_{12}/M_{22}. \tag{8}$$

Spectral singularities of H are eigenvalues $E_\star = k_\star^2$ at which the Jost solutions become linearly dependent [17,18]. This happens if and only if $M_{22} = 0$ [15]. It is easy to see from (8) that both the transmission and reflection coefficients diverge at a spectral singularity [23]. The latter condition is a characteristic property of resonances. So a spectral singularity may be identified with a peculiar type of resonances that have a vanishing width. This is because unlike ordinary resonances, the eigenvalue associated with these resonances is real. As mentioned earlier, the resonance effect related with spectral singularities can be realized in certain electromagnetic waveguides. These waveguides can be used to amplify incoming waves with frequencies close to that of a spectral singularity. Therefore, they operate as resonators at these frequencies [23].

In the remainder of this section we examine the possibility of the realization of the above resonance effect for the complex delta function potential:

$$v(x) = z\,\delta(x), \quad z \in \mathbb{C}. \tag{9}$$

The solution of the time-independent Schrödinger equation, $H\psi = k^2\psi$, for this potential is elementary [21]. We can use this solution, to determine the transfer matrix $\mathbf{M}$. This yields

$$\mathbf{M} = \begin{pmatrix} 1 - \frac{iz}{2k} & -\frac{iz}{2k} \\ \frac{iz}{2k} & 1 + \frac{iz}{2k} \end{pmatrix}. \tag{10}$$

Therefore, $M_{22} = 0$ if and only if $z = 2ik$. This condition can be satisfied for a real k only if z is imaginary and $k = -iz/2$. Therefore, as noted in [21], a spectral singularity arises only for imaginary coupling constants. Furthermore, in view of (8) and (10), we have [24a]

$$T := T^l = T^r = \frac{2k}{2k + iz}, \quad R := R^l = R^r = \frac{-iz}{2k + iz}. \tag{11}$$

In particular,

$$|T|^2 + |R|^2 = \left(1 - \frac{4k\Im(z)}{4k^2 + |z|^2}\right)^{-1}, \tag{12}$$

where $\Im$ stands for the imaginary part of its argument.

Clearly, for the case that z is real the right-hand side of (12) is equal to unity. As is well-known, this is a manifestation of the unitarity of the time evolution with respect to the L^2-inner product. For the cases that z is not real, $|T|^2 + |R|^2$ deviates from unity. A spectral singularity corresponds to the extreme situation where this quantity diverges.

Next, suppose that z is imaginary, i.e., $z = i\lambda$ for some $\lambda \in \mathbb{R} - \{0\}$, and $\epsilon := 1 - \lambda/(2k)$ so that the spectral singularity corresponds to $\epsilon = 0$. Then (11) and (12) take the form

$$T = R + 1 = \frac{1}{\epsilon}, \quad |T|^2 + |R|^2 = \frac{2(1-\epsilon)}{\epsilon^2} + 1. \tag{13}$$

In particular, the spectral singularity of this potential corresponds to a quadratic divergence of $|T|^2 + |R|^2$.

5. Concluding remarks

The pioneering work of Bender and Boettcher on the reality of the spectrum of $\mathcal{PT}$-symmetric potentials such as $v(x) = ix^3$ generated a great deal of enthusiasm among theoretical physicists who had mostly distanced themselves from non-Hermitian Hamiltonians and complex potentials. This enthusiasm led to an extensive research activity on the subject and produced a very large number of publications. Most of these involve the study of various toy models sharing the spectral properties of the imaginary cubic potential. It soon became clear that the non-Hermitian Hamiltonians H defined by these potentials could not be used to model fundamental (non-effective) physical processes unless one defined an inner product that restored their Hermiticity. The existence of such inner products and a basic method for their construction were obtained as byproducts of a study of the mathematical structure behind the appealing spectral properties of these operators [4,6,9]. In fact, all that is needed is a metric operator η_+ that satisfies the pseudo-Hermiticity relation $H^\dagger = \eta_+ H \eta_+^{-1}$. All the ingredients of the formalism are determined in terms of η_+ and independently of the choice of a $\mathcal{C}$ operator.

For the cases that H has an exact $\mathcal{PT}$ symmetry, one can use the prescription based on the $\mathcal{C}$ operator [13]. This involves substituting the ansatz $\mathcal{C} = \mathrm{e}^Q\mathcal{C}$ in $\mathcal{C}^2 = 1$, $[\mathcal{C}, H] = 0$ and $[\mathcal{C}, \mathcal{PT}] = 0$, solving the resulting operator equations for Q, and defining the physical Hilbert space using the $\mathcal{CPT}$-inner product that is identical with $\langle\cdot,\cdot\rangle_{\mathrm{e}^{-Q}}$.

This prescription has the disadvantage of relying on the construction of a $\mathcal{C}$ operator that does not enter the calculation of the physical quantities. Furthermore, to employ it one needs to deal with three operator equations. A more important drawback is that this approach cannot be applied for systems that lack a manifest antilinear symmetry. A typical example is the complex delta function potential $v(x) = z\delta(x)$. For the case that z has a positive real part, the spectrum of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + z\delta(x)$ is purely real and continuous. It is also free of spectral singularities. Hence one can apply the methods of pseudo-Hermitian quantum mechanics [1] to 'Hermitize' the Hamiltonian $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + z\delta(x)$ and explore the physical aspects of the quantum system it describes [21]. The approach based on the $\mathcal{C}$ operator cannot be applied to this system, because *a priori* the nature of the underlying nonlinear symmetry of this system (the generalized $\mathcal{PT}$ symmetry [7]) is not known.

The study of quasi-Hermitian Hamiltonians that lack $\mathcal{PT}$ symmetry has been crucial in understanding the role and meaning of spectral singularities. In this article, the essential features of spectral singularities are reviewed and the complex delta function potential is used to demonstrate how spectral singularities appear as degeneracies of the reflection and transmission coefficients. They are naturally interpreted as resonances with a vanishing width. An experimental realization of the ensuing resonance effect will be one of the rare instances of a physical discovery that owes its existence to the recent study of non-Hermitian Hamiltonian operators with a real spectrum.

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