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The auxiliary elliptic-like equation and the exp-function method

HONG LI[1] and JIN-LIANG ZHANG[2,*]
[1]College of Science, Zhongyuan University of Technology, Zhengzhou, 450007, China
[2]College of Science, Henan University of Science and Technology, Luoyang, 471003, China
[*]Corresponding author. E-mail: zhangjin6602@sohu.com; zhangjin6602@163.com

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Abstract. The auxiliary equation method is very useful for finding the exact solutions of the nonlinear evolution equations. In this paper, a new idea of finding the exact solutions of the nonlinear evolution equations is introduced. The idea is that the exact solutions of the auxiliary elliptic-like equation are derived using exp-function method, and then the exact solutions of the nonlinear evolution equations are derived with the aid of auxiliary elliptic-like equation. As examples, the RKL models, the high-order nonlinear Schrödinger equation, the Hamilton amplitude equation, the generalized Hirota–Satsuma coupled KdV system and the generalized ZK–BBM equation are investigated and the exact solutions are presented using this method.

Keywords. Auxiliary equation method; exp-function method; exact solution; RKL models; high-order nonlinear Schrödinger equation; Hamilton amplitude equation; generalized Hirota–Satsuma coupled KdV system; generalized ZK–BBM equation.

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1. Introduction

Due to the wide applications in various fields [1–4], such as physics, fluid mechanics, biomathematics, optical fibres, chemical physics, etc., nonlinear evolution equations (NEE) have been paid attention by many researchers, especially the investigations of exact solutions for the NEEs is a hot topic. There are so many methods for finding exact solutions of NEEs, such as the homogeneous balance principle [5], F-expansion method [6], tanh method [7], auxiliary equation method (the auxiliary equations include Riccati equation [7,8], the coupled Riccati equations [9], the projective Riccati equation [10], the Jacobi elliptic equations [11], elliptic-like equation [12,13], etc. Because of the complexity and variety of the NEEs, these methods are insufficient and new methods should be explored.

In this paper, the elliptic-like equation is considered as

$$A\varphi''(\xi) + B\varphi(\xi) + D\varphi^3(\xi) = 0, \tag{1}$$

where A, B, D are arbitrary constants. Equation (1) is one of the most important auxiliary equations, because many nonlinear evolution equations can be converted to eq. (1) using the travelling wave reduction (it is shown in §2). In ref. [12], Abdou used the extended F-expansion method to derive the exact solutions including the single and the combined nondegenerative Jacobi elliptic function solutions and their degenerative solutions. Zhang *et al* [13] derived the exact solutions of eq. (1) with the aid of coupled projective Riccati equations.

Recently, He and Wu [14] proposed a straightforward and concise method called exp-function method to explore the exact solutions of modified KdV equation, the Dodd–Bullough–Mikhailov equation. Many researchers showed interest in this method. Up to now, the exp-function method has been applied to find the solutions of a class of the nonlinear evolution equations, such as the nonlinear Schrödinger equations with cubic and power law nonlinearity [15], combined KdV–mKdV equation [16], KdV equation with variable coefficients [17], the discrete (2+1)-dimensional Toda lattice equation [18] and the Maccari's system [19]. So it is easy to see that the exp-function method is very powerful and can be used to study the exact solutions of the high-dimensional system, the discrete system and the system with variable coefficients.

According to the introduction above, the method presented in this paper is described as follows: in order to obtain the exact solutions of the auxiliary elliptic-like equation (1), we first apply exp-function method to derive the exact solutions of Riccati equation, and then the exact solutions of the auxiliary elliptic-like equation (1) are given with the help of the Riccati equation. The exact solutions of a class of nonlinear evolution equations which can be converted to eq. (1) using travelling wave reduction are presented with the aid of auxiliary elliptic-like equation.

This paper is organized as follows: in §2, a class of the nonlinear evolution equations, such as RKL models, the high-order nonlinear Schrödinger equation, the Hamilton amplitude equation, the generalized Hirota–Satsuma coupled KdV system and the generalized ZK–BBM equation, which can be converted to auxiliary elliptic-like equation with the aid of the travelling wave reduction are introduced. The exact solutions of Riccati equation are derived by exp-function method in §3 and the exact solutions of the auxiliary elliptic-like equation are derived using Riccati equation in §4. Thus the exact solutions of nonlinear evolution equations which can be converted to auxiliary elliptic-like equation with the aid of travelling wave reduction can be obtained. Some conclusions and discussions are given in §5.

2. The travelling wave reduction of some nonlinear evolution equations

2.1 *RKL models*

We consider RKL models

$$iu_z + u_{tt} + \tau |u|^2 u + i\gamma_1 u_{ttt} + i\gamma_2 (|u|^2 u)_t = 0. \tag{2}$$

Since $u(z,t)$ in eq. (2) is a complex function, we suppose that

$$u(z,t) = \varphi(\xi)\exp(i\theta), \quad \xi = \alpha z - \omega t, \quad \theta = kz - ct. \tag{3}$$

Here $\varphi(\xi)$ is a real function to be determined later, and q, ω, k and c are the undetermined constants.

Substituting (3) in eq. (2) yields

$$(\alpha + 2c\omega + 3\gamma_1 c^2\omega)\varphi' - 3\gamma_2\omega\varphi^2\varphi' - \gamma_1\omega^3\varphi''' = 0, \tag{4}$$

$$-(k + c^2 + \gamma_1 c^3)\varphi + (\tau + \gamma_2 c)\varphi^3 + \omega^2(1 + 3\gamma_1 c)\varphi'' = 0. \tag{5}$$

(a) When $1 + 3\gamma_1 c = 0$, i.e.,

$$c = -\frac{1}{3\gamma_1}. \tag{6}$$

Substituting (6) in eq. (5) yields

$$-\left(k + \frac{2}{27\gamma_1^2}\right)\varphi + \left(\tau - \frac{1}{3\gamma_1}\gamma_2\right)\varphi^3 = 0.$$

Setting the coefficients of φ and φ^3 to zero respectively, we have

$$k = -\frac{2}{27\gamma_1^2}, \quad \tau = \frac{\gamma_2}{3\gamma_1}. \tag{7}$$

Integrating eq. (4) once and setting integral constant to zero and using (6) and (7), eq. (4) is converted to eq. (1), where

$$A = \gamma_1\omega^3, \quad B = -\left(\alpha - \frac{\omega}{3\gamma_1}\right), \quad D = \gamma_2\omega.$$

(b) When $1 + 3\gamma_1 c \neq 0$.

Equation (5) is rewritten as

$$\varphi'' - \frac{(k + c^2 + \gamma_1 c^3)}{\omega^2(1 + 3\gamma_1 c)}\varphi + \frac{(\tau + \gamma_2 c)}{\omega^2(1 + 3\gamma_1 c)}\varphi^3 = 0. \tag{8}$$

Substituting (8) in eq. (4) and setting the coefficients of φ' and $\varphi^2\varphi'$ to zero yields

$$\alpha + 2c\omega + 3\gamma_1 c^2\omega - \frac{\gamma_1\omega^3(k + c^2 + \gamma_1 c^3)}{\omega^2(1 + 3\gamma_1 c)} = 0,$$

$$\frac{3\gamma_1\omega^3(\tau + \gamma_2 c)}{\omega^2(1 + 3\gamma_1 c)} - 3\gamma_2\omega = 0.$$

Then we have

$$\alpha = \frac{k\gamma_1\omega - 2c\omega - 8\gamma_1\omega c^2 - 8\gamma_1^2\omega c^3}{1 + 3\gamma_1 c}, \quad c = \frac{\tau}{2\gamma_2} - \frac{1}{2\gamma_1}.$$

It is easy to see that eq. (8) becomes eq. (1) when

$$A = 1, \quad B = -\frac{k + c^2 + \gamma_1 c^3}{\omega^2(1 + 3\gamma_1 c)}, \quad D = \frac{\gamma_2 + 3\gamma_1\gamma_2 c}{\omega^2\gamma_1(1 + 3\gamma_1 c)}. \tag{9}$$

2.2 *The high-order nonlinear Schrödinger equation*

In this section, the high-order nonlinear Schrödinger equation is considered as

$$\frac{\partial\psi}{\partial z} = i\alpha_1\frac{\partial^2\psi}{\partial t^2} + i\alpha_2\psi\left|\psi\right|^2 + \alpha_3\frac{\partial^3\psi}{\partial t^3} + \alpha_4\frac{\partial\psi\left|\psi\right|^2}{\partial t} + \alpha_5\psi\frac{\partial\left|\psi^2\right|}{\partial t}. \tag{10}$$

We suppose that

$$\psi(z,t) = \varphi(\xi)\exp\left[i(kz-\omega t)\right], \quad \xi = t - \lambda t + \xi_0, \tag{11}$$

where k, ω, λ are constants to be determined, ξ_0 is an arbitrary constant and $\varphi(\xi)$ satisfies eq. (1), and in this case, the coefficients of eq. (1) are of the form as

$$A = 1, \quad B = \frac{2\omega\alpha_1 + \lambda - 3\omega^3\alpha_3}{\alpha_3}, \quad D = \frac{3\alpha_4 + 2\alpha_5}{3\alpha_1},$$

$$\omega = \frac{\alpha_1(3\alpha_4 + 2\alpha_5) - 3\alpha_2\alpha_3}{6\alpha_3(\alpha_4 + \alpha_5)},$$

$$k = -\frac{1}{\alpha_3}\left[(\alpha_1 - 3\omega\alpha_3)(2\omega\alpha_1 - 3\omega^2\alpha_5 + \lambda)\right] - \omega^2\alpha_1 + \omega^3\alpha_3.$$

2.3 *The Hamilton amplitude equation*

The Hamilton amplitude equation is considered as

$$iu_x + u_{tt} + 2\sigma\left|u\right|^2 u - \varepsilon u_{xt} = 0. \tag{12}$$

Similar to §2.1, we suppose

$$u(x,t) = \varphi(\xi)\exp(i\eta), \tag{13}$$

where

$$\xi = (2\beta + \alpha\varepsilon)x + (1 + \beta\varepsilon)t, \quad \eta = \alpha x - \beta t. \tag{14}$$

α, β are constants and $\varphi(\xi)$ satisfies eq. (1) and in this case, the coefficients of eq. (1) are of the form as

$$A = 1 - \beta^2\varepsilon^2 - (\varepsilon^2 + \beta\varepsilon^3)\alpha, \quad B = -[\beta^2 + (1 + \beta\varepsilon)\alpha], \quad D = 2\sigma. \tag{15}$$

2.4 *The generalized Hirota–Satsuma coupled KdV system*

The generalized Hirota–Satsuma coupled KdV system is considered as

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x + 3(w - v^2)_x, \tag{16}$$

$$v_t = -\tfrac{1}{2}v_{xxx} - 3uv_x, \tag{17}$$

$$w_t = -\tfrac{1}{2}w_{xxx} - 3uw_x. \tag{18}$$

From ref. [31], we obtain

$$u(x,t) = \alpha\left[a\varphi(\xi) - \frac{\beta}{2}\right]^2 + \gamma,$$

$$v(x,t) = \left[a\varphi(\xi) - \frac{\beta}{2}\right],$$

$$w(x,t) = A_0\left[a\varphi(\xi) - \frac{\beta}{2}\right] + B_0,$$

where $\varphi(\xi)$ is a solution of eq. (1), and in this case, the coefficients of eq. (1) are of the form as

$$A = k^2, \quad B = -a\left(\frac{3\beta^2}{2\alpha} + 2c - 6\gamma\right), \quad D = 2\alpha a^3, \quad \xi = k(x - ct),$$

and $a, \alpha, \beta, \gamma, k, c, A_0, B_0$ are constants.

2.5 *The generalized ZK–BBM equation*

The generalized ZK–BBM equation is considered as

$$u_t + u_x + a(u^3)_x + b(u_{xt} + u_{yy})_x = 0. \tag{19}$$

From ref. [20], we obtain

$$u(x,t) = \varphi(\xi), \quad \xi = x + y - ct,$$

where $\varphi(\xi)$ is a solution of eq. (1), and in this case, the coefficients of eq. (1) are of the form as

$$A = b(1-c), \quad B = (1-c), \quad D = a. \tag{20}$$

Remark 1. If the exact solutions of eq. (1) are derived, it is easy to obtain the exact solutions of eqs (2), (10), (12), (16)–(19).

Remark 2. In fact, there are other nonlinear evolution equations which can be converted to eq. (1) with the aid of the travelling wave reduction.

3. The exact solutions of Riccati equation

In this section, some exact solutions of the Riccati equation are derived by exp-function method.

$$F'(\eta) = h_0 + h_2 F^2(\eta), \tag{21}$$

where $\eta = \lambda(x + \omega t)$, h_0, h_2, λ and ω are constants.

According to exp-function method, we suppose that the exact solution of eq. (21) is of the form as

$$F(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_f \exp(f\eta) + \cdots + b_{-g} \exp(-g\eta)}, \tag{22}$$

where f, c, d and g are positive integers, and a_n $(n = c, \ldots, 0, \ldots, -d)$ and b_n $(n = f, \ldots, 0, \ldots, -g)$ are unknown constants to be determined.

The balancing between $F'(\eta)$ and $F^2(\eta)$ in eq. (21) yields

$$f = c, \quad d = g. \tag{23}$$

Case 3.1. $f = c = 1, d = g = 1$.

Equation (22) is converted to

$$F(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{24}$$

Substituting (24) into (21) and using *Mathematica*, and in the case of $h_0 = -1/4h_2$, we obtain exact solution of eq. (21) as

$$F(\eta) = \frac{1}{2h_2} \frac{b_1 \exp(\eta) - \mathrm{sign}(h_2)\sqrt{b_0^2 - 4b_{-1}b_1} + b_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}, \tag{25}$$

where $b_0^2 - 4b_{-1}b_1 \geq 0$, b_1, b_0, b_{-1} are constants, $\eta = \frac{1}{h_2}(x - \frac{t}{4h_2^2})$.

Case 3.1.1. When $b_1 = b_0, b_{-1} = -b_0$.

It is obvious that eq. (25) is simplified as

$$F(\eta) = -\frac{1}{2h_2} \left(\frac{2\cosh(\eta) \pm \sqrt{5}}{1 + 2\sinh(\eta)} \right). \tag{26}$$

Case 3.1.2. When $b_0 = -2b_1, b_{-1} = b_1$.

Equation (25) is simplified as

$$F(\eta) = -\frac{1}{2h_2} \coth\left(\frac{\eta}{2}\right). \tag{27}$$

Case 3.1.3. When $h_2 = 1/iK$.

The exact solution of eq. (21) is derived as

$$F(\zeta) = \frac{1}{2} \frac{(-b_1 + b_{-1})Ki\cos\zeta - |K|\sqrt{4b_{-1}b_1 - b_0^2} + (b_1 + b_{-1})K\sin\zeta}{(b_1 - b_{-1})i\sin\zeta + b_0 + (b_1 + b_{-1})\cos\zeta}, \tag{28}$$

where $\zeta = K(x + \frac{1}{4}K^2 t)$, $b_0^2 - 4b_{-1}b_1 \geq 0$, b_1, b_0, b_{-1} are constants.

Case 3.1.3.1. When $b_{-1} = b_1$.
Equation (28) is simplified as

$$F(\zeta) = \frac{1}{2}\frac{|K|\sqrt{4b_{-1}^2 - b_0^2} - 2b_{-1}K\sin\zeta}{b_0 + 2b_{-1}\cos\zeta}. \tag{29}$$

Case 3.2. $f = c = 2, d = g = 2$.
Equation (22) is converted as

$$F(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{b_2\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta)}. \tag{30}$$

We set $b_{-1} = b_1 = 0$ for simplicity and eq. (30) is converted as

$$F(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{b_2\exp(2\eta) + b_0 + b_{-2}\exp(-2\eta)}. \tag{31}$$

Similar to Case 3.1, we obtain

$$\begin{aligned} F(\eta) = \Bigg\{ & \frac{1}{2h_2}b_{-2}\exp(-2\eta) + \frac{\sqrt{-\frac{1}{h_2^2}b_{-2}(b_0 + \frac{h_2}{|h_2|}\sqrt{b_0^2 - 4b_2b_{-2}})}}{\sqrt{2}} \\ & \times\exp(-\eta) + \frac{(b_0 - \frac{h_2}{|h_2|}\sqrt{b_0^2 - 4b_2b_{-2}})\sqrt{-\frac{1}{h_2^2}b_{-2}(b_0 + \frac{h_2}{|h_2|}\sqrt{b_0^2 - 4b_2b_{-2}})}}{2\sqrt{2}b_{-2}} \\ & \times\exp(\eta) - \frac{1}{2|h_2|}\sqrt{b_0^2 - 4b_2b_{-2}} - \frac{1}{2h_2}b_2\exp(2\eta)\Bigg\} \\ & \times\{b_2\exp(2\eta) + b_0 + b_{-2}\exp(-2\eta)\}^{-1}, \end{aligned} \tag{32}$$

where

$$h_0 = -\frac{1}{4h_2}, \quad \eta = \frac{1}{h_2}\left(x - \frac{t}{4h_2^2}\right),$$

Case 3.2.1. When $b_{-2} = b_2, b_0 = -2b_2$.
Equation (32) is simplified as

$$F(\eta) = -\frac{1}{2h_2}\left[\coth(\eta) \pm \text{csch}(\eta)\right]. \tag{33}$$

Case 3.2.2. When $b_{-2} = b_2, b_0 = 2b_2$.
Equation (32) is simplified as

$$F(\eta) = \frac{1}{2h_2}\left[\pm i\mathrm{sech}(\eta) - \tanh(\eta)\right]. \tag{34}$$

Case 3.2.3.
We set $a_{-1} = b_1 = 0, b_2 = 0$ for simplicity. Equation (30) is converted as

$$F(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-2}\exp(-2\eta)}{b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta)}. \tag{35}$$

Similar to Case 3.1, we obtain

$$F(\eta) = \frac{1}{h_2}\left(-1 + \frac{2b_{-2}}{b_0\exp(2\eta) + b_{-2}}\right), \tag{36}$$

where

$$\eta = \frac{1}{h_2}\left(x - \frac{t}{h_2^2}\right), \quad b_{-2}, b_0 \text{ are free parameters.}$$

Case 3.3. $f = c = 2, d = g = 1$.
Equation (22) is converted as

$$F(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_2\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{37}$$

Setting $b_2 = 1$ for simplicity, and using the same method in Case 3.1, the exact solution of eq. (21) can be derived as

$$F(\eta) = \frac{1}{2h_2}\frac{-\exp(2\eta) + \mathrm{sign}(h_2)\sqrt{b_1^2 - 4b_0}\exp(\eta) + b_0}{\exp(2\eta) + b_1\exp(\eta) + b_0}, \tag{38}$$

where

$$\eta = \frac{1}{h_2}\left(x - \frac{t}{4h_2^2}\right), \quad b_1, b_0 \text{ are free parameters.}$$

Case 3.4 $f = c = 3, d = g = 3$.

$$\begin{aligned} F(\eta) = {} & [a_3\exp(3\eta) + a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) \\ & + a_{-2}\exp(-2\eta) + a_{-3}\exp(-3\eta)] \\ & \times [b_3\exp(3\eta) + b_2\exp(2\eta) + b_1\exp(\eta) \\ & + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta) + b_{-3}\exp(-3\eta)]^{-1}. \end{aligned} \tag{39}$$

Setting $b_{-1} = b_1 = 0$ for simplicity, and using the same method in Case 3.1, the exact solution of eq. (21) can be derived as

$$F(\eta) = \left\{ -\frac{1}{2h_2} b_3 \exp(3\eta) + \frac{a_{-1}b_3}{2b_0} \exp(2\eta) - \frac{1}{2h_2} b_0 + a_{-1} \exp(-\eta) \right.$$
$$\left. - \frac{b_{-2}b_0 + 2h_2^2 a_{-1}^2}{2h_2 b_0} \exp(-2\eta) + \frac{a_{-1}(b_0 b_{-2} + h_2^2 a_{-1}^2)}{2b_0^2} \exp(-3\eta) \right\}$$
$$\times \left\{ b_3 \exp(3\eta) + \frac{h_2 a_{-1} b_3}{b_0} \exp(2\eta) + b_0 \right.$$
$$\left. + b_{-2} \exp(-2\eta) + \frac{a_{-1}(h_2 b_0 b_{-2} + h_2^3 a_{-1}^2)}{b_0^2} \exp(-3\eta) \right\}^{-1}, \qquad (40)$$

where $b_3, b_0 \neq 0, a_1$ are free parameters.

Remark 3. We can obtain other solutions of eq. (21) such as we set $f = c = 4, d = g = 4$.

4. The exact solution of auxiliary ordinary differential equation (1)

Case 4.1

According to the homogeneous balance principle [5], we suppose that the exact solution of eq. (1) is in the form of

$$u = a_0 + a_1 F, \qquad (41)$$

where a_0, a_1 are constants to be determined later and F satisfies eq. (21). Substituting (41) into eq. (1) and using eq. (21) simultaneously, we can obtain the exact solution of eq. (1) which can be expressed as

$$u = \pm\sqrt{\frac{2}{B} h_2^2} F, \qquad (42)$$

where F satisfies eq. (21).

Case 4.2

We suppose that the exact solution of eq. (1) is in the form of

$$u = a_{-1} F^{-1} + a_0 + a_1 F, \qquad (43)$$

where a_0, a_1 are constants to be determined later and F satisfies eq. (21).

Case 4.2.1

As in §4.1, we can obtain the exact solution of eq. (1) which can be expressed as

$$u = \pm\sqrt{\frac{2}{B} h_0^2} F^{-1} \pm \sqrt{\frac{2}{B} h_2^2} F, \qquad (44)$$

where F satisfies eq. (21).

Case 4.2.2

As in §4.1, we can obtain the exact solution of eq. (1) which can be expressed as

$$u = \pm\sqrt{\frac{2}{B}h_0^2 F^{-1}} \quad , \tag{45}$$

where F satisfies eq. (21).

Remark 4. It is easy to obtain the exact solution of eq. (1) with the aid of eq. (21). Here we omit it for simplicity.

Remark 5. It is easy to obtain the exact solutions of eqs (2), (10), (12) and (16)–(19) with the aid of eq. (1). Here we omit it for simplicity.

5. Conclusions and discussions

The auxiliary equation method is very important in finding the exact solutions of nonlinear evolution equations, and the auxiliary elliptic-like equation is one of the most important auxiliary equations because many nonlinear evolution equations, such as RKL models, the high-order nonlinear Schrödinger equation, the Hamilton amplitude equation, the generalized Hirota–Satsuma coupled KdV system and the generalized ZK–BBM equation, can be converted to this equation using the travelling wave reduction.

In this paper, we apply exp-function method to derive the exact solutions of the auxiliary elliptic-like equation. The exact solutions of RKL models, the high-order nonlinear Schrödinger equation, the Hamilton amplitude equation, the generalized Hirota–Satsuma coupled KdV system and the generalized ZK–BBM equation are derived. The idea introduced in this paper can be applied to other nonlinear evolution equations.

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