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A new approach to two-charge fuzzball geometries

RUI-YAN YU
School of Information and Mathematics, Yangtzeu University, Jingzhou, Hubei 434023, People's Republic of China
E-mail: wtu262@yangtzeu.edu.cn

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Abstract. A few years ago, Mathur proposed a 'fuzzball' conjecture to give a microscopic description of black hole entropy. In the fuzzball scenario, the entropy in a two-charge black hole corresponds to microstates of a two-charge string (brane) system, e.g., a winding fundamental string with momentum modes. The geometry of such a two-charge system is fuzzy near the horizon, and is very difficult to get analytically in general. In this paper, we show a new method to get geometries of two-charge fuzzball. Our method is based on the multipole expansion. We find that the method is powerful enough to get a clean analytic form of metric of the fuzzball with one-momentum mode. It is expected to get multi-mode geometries using this method in the near future.

Keywords. Black holes; string theory.

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1. Introduction

In Einstein gravity, it is now widely accepted that the black hole has an entropy S determined by the area A of its horizon

$$S = \frac{A}{4G}, \tag{1}$$

where G is the Newtonian constant. This elegant relation between statistical mechanics and geometry was first conjectured by Bekenstein [1] and soon verified by Hawking [2].

However, the basic principles of statistical mechanics tell us that corresponding to an entropy S, there must be

$$N = \mathrm{e}^S \tag{2}$$

microstates of the black hole. Since the discovery of entropy (1), many efforts along this line have been made but failed. The microscopic explanation of black hole entropy becomes a longstanding problem and may relate to the deep problem of quantum gravity.

String theory is the most promising candidate for quantum gravity and provides some clues to solve the black hole entropy problem. For three-charge extremal black holes, their entropy is well-described by a three-charge brane system, as shown in the famous papers [3,4]. But it is still an open problem to give a microscopic description of two-charge black hole in string theory, even in the extremal case.

One proposal to solve this problem is provided by Lunin and Mathur [5–7]. They interpret the two-charge black hole as an oscillating fundamental string winding along an S^1 direction. One charge of the system is the winding modes, the other charge is the momentum modes carried by the string. The system is notated as an NS1-P system, where NS1 indicates a Neveu–Schwarz one-dimensional string, namely a fundamental string, and P indicates that the string carries momentum modes. If the string carries only one mode of momentum, its geometry has been obtained in [5]. But as far as we know, when we put more modes on the string, its metric is difficult to cast into a clean form.

In this paper, we show a new method to get geometries of two-charge fuzzball. Our method is based on the multipole expansion. We find that the method is powerful enough to analytically get the metric of fuzzball with one-momentum mode. It is expected to get multi-mode geometries using this method in the near future.

The organization of this paper is as follows: We collect some formulas useful for our calculation in §2. Then we briefly explain our goal and scheme in §3. In §4, as a concrete but the simplest example, we calculate the metric of two-charge fuzzball with only one-momentum mode. Our method seems to be powerful, and we subsequently in §5 outline how to apply this method to two-mode or multi-mode case in the future. We conclude our paper in §6.

2. Formulas of special functions

Here we collect some formulas of special functions. These formulas will be useful in our multipole expansion method. All of them can be found in [8].

2.1 *Hypergeometric functions* $F(\alpha,\beta;\gamma;z)$

Definition. A hypergeometric series is a series of the form

$$F(\alpha,\beta;\gamma;z) = 1 + \frac{\alpha\cdot\beta}{\gamma\cdot 1}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}z^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 1\cdot 2\cdot 3}z^3 + \cdots. \tag{3}$$

2.2 *The Gegenbauer polynomials* $C_n^\lambda(t)$

Definition. The polynomials $C_n^\lambda(t)$ of degree n are the coefficients of α^n in the power-series expansion of the function

$$(1-2t\alpha+\alpha^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^{\lambda}(t)\alpha^n. \tag{4}$$

Thus, the polynomials $C_n^{\lambda}(t)$ are a generalization of the Legendré polynomials.

One of the expressions in terms of hypergeometric functions is

$$\begin{aligned} C_n^{\lambda}(t) &= \frac{\Gamma(2\lambda+n)}{\Gamma(n+1)\Gamma(2\lambda)} F\left(2\lambda+n, -n; \lambda+\frac{1}{2}; \frac{1-t}{2}\right) \\ &= \frac{2^n\Gamma(\lambda+n)}{n!\Gamma(\lambda)} t^n F\left(-\frac{n}{2}, \frac{1-n}{2}; 1-\lambda-n; \frac{1}{t^2}\right). \end{aligned} \tag{5}$$

Actually, this expression defines the generalized functions $C_n^{\lambda}(t)$, where the subscript n can be an arbitrary number.

One of the recursion formulas is:

$$(2\lambda+n)C_n^{\lambda}(t) = 2\lambda[C_n^{\lambda+1}(t) - tC_{n-1}^{\lambda+1}(t)]. \tag{6}$$

One of the special cases and particular values is:

$$C_n^1(\cos\varphi) = \frac{\sin(n+1)\varphi}{\sin\varphi}. \tag{7}$$

2.3 *The Chebyshev polynomials of the second kind $U_n(x)$*

Definition. Chebyshev's polynomials of the second kind is:

$$\begin{aligned} U_n(x) &= \frac{\sin[(n+1)\arccos x]}{\sin[\arccos x]} \\ &= \frac{1}{2i\sqrt{1-x^2}}[(x+i\sqrt{1-x^2})^{n+1} - (x-i\sqrt{1-x^2})^{n+1}] \\ &= \binom{n+1}{1}x^n - \binom{n+1}{3}x^{n-2}(1-x^2) \\ &\quad + \binom{n+1}{5}x^{n-4}(1-x^2)^2 - \cdots. \end{aligned} \tag{8}$$

The generating function is:

$$\frac{1}{1-2tx+t^2} = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{9}$$

One of the differentiation formulas expressed as Gegenbauer polynomials is:

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}U_n(x) = 2^m m! C_{n-m}^{m+1}(x). \tag{10}$$

3. The NS1-P solution in fuzzball proposal

With the compactification $T^4 \times S^1$, the NS1-P solution in low energy type IIB string theory is

$$
\begin{aligned}
\mathrm{d}s^2_{\text{string}} &= H(-\mathrm{d}u\mathrm{d}v + K\mathrm{d}v^2 + 2A_i \mathrm{d}x^i \mathrm{d}v) + \sum_{i=1}^{4} \mathrm{d}x^i \mathrm{d}x^i + \sum_{a=1}^{4} \mathrm{d}z^a \mathrm{d}z^a, \\
B_{uv} &= -\frac{1}{2}(H-1), \quad B_{vi} = HA_i, \quad \mathrm{e}^{2\phi} = H,
\end{aligned} \tag{11}
$$

where

$$
\begin{aligned}
H^{-1} &= 1 + \frac{Q}{L}\int_0^L \frac{1}{|\vec{x} - \vec{F}(v)|^2}\mathrm{d}v, \\
K &= \frac{Q}{L}\int_0^L \frac{|\dot{\vec{F}}(v)|^2}{|\vec{x} - \vec{F}(v)|^2}\mathrm{d}v, \\
A_i &= -\frac{Q}{L}\int_0^L \frac{\dot{F}_i(v)}{|\vec{x} - \vec{F}(v)|^2}\mathrm{d}v.
\end{aligned} \tag{12}
$$

In the above expressions, a dot denotes the derivative with respect to v, $u = t + y$, $v = t - y$ are the light-cone coordinates. y is the coordinate in the S^1 direction, along which the string is wound. x^1, x^2, x^3, x^4 are coordinates in noncompact transverse directions, while z^1, z^2, z^3, z^4 are coordinates in T^4 directions. This solution is generated by bosonic exitations which bend the string in noncompact directions x^i [7]. The functions $\vec{F}(v)$ describe the vibration profile in those noncompact directions. The parameter L is the length of the string, and Q is proportional to its ADM-mass [9].

Our goal is to perform the integrals (12) with various profile functions $\vec{F}(v)$, and then get a clean analytic form of the metric (11).

Generally, integrals in such a solution are difficult to deal with, and hence the metric is complicated. But recall that in electrodynamics, a finite-size source always generates an electric field which is expressed by similar integrals. It is well-known that the multipole expansion is a powerful tool to deal with such electric fields. Based on this observation, we can treat the integrals here with multipole expansions in four dimensions [10]. As a result, the difficulty could be alleviated in favour of Chebyshev polynomials of the second kind.

We first consider the case $|\vec{x}\,| > |\vec{F}(v)|$, whose physical significance will be clear soon.

Rewriting the integrand in (12) as

$$
\frac{\rho(v)}{|\vec{x} - \vec{F}(v)|^2} = \frac{1}{r^2}\frac{\rho(v)}{1 - 2(r_0/r)\hat{x}\cdot\hat{F} + (r_0/r)^2} \tag{13}
$$

and using the generating function of Chebyshev polynomials (9), we quickly find that each integral in (13) is an infinite sum of integrals

$$\int_0^L \frac{\rho(v)}{|\vec{x}-\vec{F}(v)|^2}\mathrm{d}v = \sum_{n=0}^{\infty}\int_0^L \frac{r_0^n}{r^{n+2}}U_n(\hat{x}\cdot\hat{F})\rho(v)\mathrm{d}v, \tag{14}$$

which is nothing but the multipole expansion in four dimensions [10].

For succinctness, we have used the notations

$$r = |\vec{x}\,|, \quad r_0 = r_0(v) = |\vec{F}(v)|, \quad \hat{x} = \frac{\vec{x}}{|\vec{x}\,|}, \quad \hat{F} = \frac{\vec{F}(v)}{|\vec{F}(v)|} \tag{15}$$

in (13) and (14). Here $U_n(x)$ is the Chebyshev polynomials of the second kind, and

$$\rho(v) = \frac{Q}{L}, \quad \frac{Q|\dot{\vec{F}}(v)|^2}{L}, \quad -\frac{Q\dot{F}_i(v)}{L} \tag{16}$$

corresponding to $H^{-1}-1$, K, A_i respectively.

Now let us get the physical significance of the ratio $|\vec{x}\,|/|\vec{F}(v)|$. If we estimate the size of the NS1-P bound state by $|\vec{F}(v)|$, which is related to the fuzzball size [7], we can roughly consider it as the ratio of the radial coordinate to the fuzzball radius. Therefore, in the limit $|\vec{x}\,| \gg |\vec{F}(v)|$, the metric (11) describes the geometry far from the fuzzball, and the lowest order terms will dominate on the right-hand side of (14).

The trick for the case $|\vec{x}\,| < |\vec{F}(v)|$ is essentially similar, so like (14) one can easily get a result

$$\int_0^L \frac{\rho(v)}{|\vec{x}-\vec{F}(v)|^2}\mathrm{d}v = \sum_{n=0}^{\infty}\int_0^L \frac{r^n}{r_0^{n+2}}U_n(\hat{x}\cdot\hat{F})\rho(v)\mathrm{d}v. \tag{17}$$

In fact, up to a coordinate transformation, the profile function in (12) can always be written as [5]

$$F_i(v) = \sum_{n=1}^{\infty}\left[C_i(n)\cos\frac{2\pi nv}{L} + D_i(n)\sin\frac{2\pi nv}{L}\right]. \tag{18}$$

In the following sections, to work in details, we will concentrate on the case $F_3(v) = F_4(v) = 0$, that is, bosonic exitations which give bending in directions x^1, x^2. For the reason mentioned above, we will refer to $|\vec{x}\,| > |\vec{F}(v)|$ as the region outside the horizon, and $|\vec{x}\,| < |\vec{F}(v)|$ as the region in the horizon. However, these words are not exact when $|\vec{x}\,| \sim |\vec{F}(v)|$, because there is no clear-cut horizon in fuzzball proposal. To be a little more accurate, the size of the fuzzball is given by the size of the so-called generic state, which has typical exitations in harmonics. For further details on this issue, please refer to [7]. One should be careful when considering the physics near the region $|\vec{x}\,| \sim |\vec{F}(v)|$, but it is technically convenient to cut the geometry of spacetime (11) into two regions $|\vec{x}\,| > |\vec{F}(v)|$ and $|\vec{x}\,| < |\vec{F}(v)|$.

4. One-mode case

The most simple case is a string with all the energy put in the same mode of harmonic, generating the vibration profile

$$F_1 = a\cos\frac{2\pi n v}{L}, \quad F_2 = a\sin\frac{2\pi n v}{L}, \quad F_3 = 0, \quad F_4 = 0. \tag{19}$$

Thanks to periodicity of the integrand, for arbitrary n, it gives similar metric to that for $n = 1$. From a metric with $n = 1$, to get the one for arbitrary n, we should simply fix the value of H^{-1}, rescale K by a factor of n^2, and simultaneously rescale A_i by a factor of n. In other words, each dot in (12) contributes a factor n. So it is enough to focus on the lowest mode of vibration with profile function

$$F_1 = a\cos\frac{2\pi v}{L}, \quad F_2 = a\sin\frac{2\pi v}{L}, \quad F_3 = 0, \quad F_4 = 0. \tag{20}$$

Change the parameter $v = \xi L/2\pi$ and introduce polar coordinates in the $\vec{x}$ space

$$\begin{aligned} &x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi,\\ &x_3 = r\cos\theta\cos\psi, \quad x_4 = r\cos\theta\sin\psi, \end{aligned} \tag{21}$$

then the integrals (12) become

$$\begin{aligned} H^{-1} &= 1 + \frac{Q}{2\pi}\int_0^{2\pi}\frac{1}{f}\mathrm{d}\xi,\\ K &= \frac{Q}{2\pi}\int_0^{2\pi}\frac{|\vec{g}\,|^2}{f}\mathrm{d}\xi,\\ A_i &= -\frac{Q}{2\pi}\int_0^{2\pi}\frac{g_i}{f}\mathrm{d}\xi, \end{aligned} \tag{22}$$

in which

$$\begin{aligned} &f = (r\sin\theta\cos\phi - a\cos\xi)^2 + (r\sin\theta\sin\phi - a\sin\xi)^2 + r^2\cos^2\theta,\\ &g_1 = -\frac{2\pi a}{L}\sin\xi, \quad g_2 = \frac{2\pi a}{L}\cos\xi, \quad g_3 = 0, \quad g_4 = 0. \end{aligned} \tag{23}$$

It is easy to know in (14) the parameters

$$r_0 = a, \quad \hat{x}\cdot\hat{F} = \sin\theta\cos(\xi - \phi) \tag{24}$$

and perform the multipole expansion for non-vanishing integrals in the region $|\vec{x}\,| > |\vec{F}(v)|$

$$\begin{aligned} H^{-1} - 1 &= \frac{Q}{2\pi}\sum_{n=0}^{\infty}\frac{a^n}{r^{n+2}}\int_0^{2\pi}U_n[\sin\theta\cos(\xi - \phi)]\mathrm{d}\xi,\\ K &= (H^{-1} - 1)\left(\frac{2\pi a}{L}\right)^2, \end{aligned}$$

$$A_1 = \frac{Qa}{L} \sum_{n=0}^{\infty} \frac{a^n}{r^{n+2}} \int_0^{2\pi} U_n[\sin\theta \cos(\xi - \phi)] \sin\xi \mathrm{d}\xi,$$
$$A_2 = -\frac{Qa}{L} \sum_{n=0}^{\infty} \frac{a^n}{r^{n+2}} \int_0^{2\pi} U_n[\sin\theta \cos(\xi - \phi)] \cos\xi \mathrm{d}\xi. \tag{25}$$

Substitution of r^n/a^{n+2} for a^n/r^{n+2} in these expressions gives multipole expansion in the region $|\vec{x}\,| < |\vec{F}(v)|$. We can do the integrals after expansion and then sum up. As shown in the following, this leads to the same result as those in [5,7], which has been obtained by a different method.

Now let us reproduce the non-singular geometry derived earlier in [11,12]. As demonstrated in [5], the geometry can be obtained by starting from NS-P profile (20). We work along the same line, but use the multipole expansion, and obtain the same result as one should have expected.

Firstly, we note the useful equations

$$\int_0^{2\pi} \cos^m(\xi - \phi) \sin(\xi - \phi) \mathrm{d}\xi = 0,$$
$$\int_0^{2\pi} \cos^{2k+1}(\xi - \phi) \mathrm{d}\xi = 0,$$
$$\int_0^{2\pi} \cos^{2k}(\xi - \phi) \mathrm{d}\xi = \frac{2\pi}{2^{2k}} \frac{(2k)!}{k!k!}, \tag{26}$$

in which m, k are integers.

From (3), (4), (5), it is not hard to get

$$\left(1 + \frac{2a^2}{r^2} \cos 2\theta + \frac{a^4}{r^4}\right)^{-1/2} = \sum_{k=0}^{l} (-\sin^2\theta)^k \frac{(l+k)!}{(l-k)!k!k!}. \tag{27}$$

If we expand Chebyshev polynomials in the form of power-series

$$U_n(x) = \sum_{m=0}^{n} \mathcal{D}_n^m x^m \tag{28}$$

by making use of (10), (6), (7), we can find out the coefficients

$$\begin{aligned}\mathcal{D}_n^m &= 2^m C_{n-m}^{m+1}(0) \\ &= 2^m \frac{n+m}{2m} \frac{n+m-2}{2(m-1)} \frac{n+m-4}{2(m-2)} \cdots \frac{n-m+2}{2 \cdot 1} C_{n-m}^1(0) \\ &= \begin{cases} 0, & \text{for } n-m \text{ odd;} \\ \frac{(n+m)!!}{(n-m)!!(2m)!!} 2^m (-1)^{\frac{n-m}{2}}, & \text{for } n-m \text{ even.} \end{cases}\end{aligned} \tag{29}$$

These results facilitate the calculation of (25). Making use of them, after some careful calculations, we get finally

$$
\begin{aligned}
H^{-1}-1 &= \sum_{l=0}^{\infty} \frac{Q}{2\pi r^{2l+2}} \int_0^{2\pi} a^{2l} \sum_{k=0}^{l} \mathcal{D}_{2l}^{2k} [\sin\theta\cos(\xi-\phi)]^{2k} \mathrm{d}\xi \\
&= \frac{Q}{r^2} \sum_{l=0}^{\infty} \left(-\frac{a^2}{r^2}\right)^l \sum_{k=0}^{l} (-\sin^2\theta)^k \frac{(l+k)!}{(l-k)!k!k!} \\
&= \frac{Q}{r^2} \left(1+\frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right)^{-1/2}, \qquad (30)
\end{aligned}
$$

as well as

$$
\begin{aligned}
A_1 &= \frac{Qa}{L} \int_0^{2\pi} \sum_{l=0}^{\infty} \sin\phi\cos(\xi-\phi) \frac{a^{2l+1}}{r^{2l+3}} \\
&\quad \times \sum_{k=0}^{l} \mathcal{D}_{2l+1}^{2k+1} [\sin\theta\cos(\xi-\phi)]^{2k+1} \mathrm{d}\xi \\
&= \frac{\pi Q}{Lr} \frac{\sin\phi}{\sin\theta} \sum_{l=0}^{\infty} \left(-\frac{a^2}{r^2}\right)^{l+1} \\
&\quad \times \sum_{k=0}^{l} (-\sin^2\theta)^{k+1} \frac{(l+k+1)!}{(l-k)!(k+1)!(k+1)!} (2k+2) \\
&= \frac{\pi Q}{Lr} \frac{\sin\phi}{\sin\theta} \sum_{l=0}^{\infty} \left(-\frac{a^2}{r^2}\right)^{l+1} \\
&\quad \times \left[\sum_{k=0}^{l} (-\sin^2\theta)^{k+1} \frac{(l+k+2)!}{(l-k)!(k+1)!(k+1)!}\right. \\
&\quad \left. - \sum_{k=0}^{l-1} (-\sin^2\theta)^{k+1} \frac{(l+k+1)!}{(l-k-1)!(k+1)!(k+1)!}\right] \\
&= \frac{\pi Q}{Lr} \frac{\sin\phi}{\sin\theta} \left\{ \left[\sum_{l=-1}^{\infty} \left(-\frac{a^2}{r^2}\right)^{l+1} \sum_{k=-1}^{l} (-\sin^2\theta)^{k+1} \right.\right. \\
&\quad \left. \times \frac{(l+k+2)!}{(l-k)!(k+1)!(k+1)!} - 1 - \sum_{l=0}^{\infty} \left(-\frac{a^2}{r^2}\right)^{l+1} \right] \\
&\quad - \left[\sum_{l=0}^{\infty} \left(-\frac{a^2}{r^2}\right)^{l+1} \sum_{k=-1}^{l-1} (-\sin^2\theta)^{k+1} \right. \\
&\quad \left.\left. \times \frac{(l+k+1)!}{(l-k-1)!(k+1)!(k+1)!} - \sum_{l=0}^{\infty} \left(-\frac{a^2}{r^2}\right)^{l+1} \right] \right\}
\end{aligned}
$$

$$
= \frac{\pi Q}{Lr} \frac{\sin\phi}{\sin\theta} \left\{ \left[\sum_{l=0}^{\infty} \left(-\frac{a^2}{r^2}\right)^{l} \sum_{k=0}^{l} (-\sin^2\theta)^{k} \right.\right.
$$

$$
\begin{aligned}
&\quad \times \frac{(l+k)!}{(l-k)!k!k!} - 1 - \sum_{l=0}^{\infty}\left(-\frac{a^2}{r^2}\right)^{l+1}\Bigg] \\
&\quad + \left[\frac{a^2}{r^2}\sum_{l=0}^{\infty}\left(-\frac{a^2}{r^2}\right)^{l}\sum_{k=0}^{l}(-\sin^2\theta)^k \right.\\
&\quad \left.\times \frac{(l+k)!}{(l-k)!k!k!} + \sum_{l=0}^{\infty}\left(-\frac{a^2}{r^2}\right)^{l+1}\right]\Bigg\} \\
&= \frac{\pi Q}{Lr}\frac{\sin\phi}{\sin\theta}\left[\left(1+\frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right)^{-1/2}\left(1+\frac{a^2}{r^2}\right) - 1\right]. \qquad (31)
\end{aligned}
$$

They are not different from the results in [5,7], which can be obtained using the residue theorem in complex analysis but without expansion. The calculation for K and A_2 in (25) can be accomplished with a few modifications and give the same accordance. Until now we have only focussed on the region out of horizon. The expansion in horizon, which corresponds to (17), can also be accomplished along the same line, and also give the same results. Since out-of-horizon and in-horizon calculations produce the same values for integrals (22), by continuity we infer that these values also hold for $|\vec{x}\,| = |\vec{F}(v)|$.

At first sight, the approach showed here appears to be redundant. However, this approach might be powerful in treating multi-mode profile functions, which have not been solved in literature to our best knowledge.

5. Two-mode case

Let us add another mode to the vibrating string (19), we still constrict on the bosonic exitations which bend the string in directions x^1, x^2. A kind of such two-mode vibration profile is

$$
\begin{aligned}
&F_1 = a\cos\frac{2\pi m v}{L} + a\cos\frac{2\pi n v}{L},\\
&F_2 = a\sin\frac{2\pi m v}{L} + a\sin\frac{2\pi n v}{L},\\
&F_3 = 0, \quad F_4 = 0. \qquad (32)
\end{aligned}
$$

It is only a little more complicated than profile (20), but to our best knowledge so far, no one has performed the integrals (12) from this profile. If one obtains an analytic form of metric for two-mode or multi-mode profile, it would be an important progress in the investigation of Mathur's fuzzball conjecture [7].

In order to give a detailed illustration, we set $m = 1$, $n = 2$ and hence take the profile below.

$$
\begin{aligned}
&F_1 = a\cos\frac{2\pi v}{L} + a\cos\frac{4\pi v}{L},\\
&F_2 = a\sin\frac{2\pi v}{L} + a\sin\frac{4\pi v}{L},\\
&F_3 = 0, \quad F_4 = 0. \qquad (33)
\end{aligned}
$$

In this case, using multipole expansion, formally we can rewrite the integrals (12) as

$$
\begin{aligned}
&H^{-1}-1=\frac{Q}{2\pi}\sum_{n=0}^{\infty}\frac{r^n}{a^{n+2}}\int_0^{2\pi}\frac{1}{(2\cos\frac{\xi}{2})^{n+2}}U_n\left[\sin\theta\cos\left(\frac{3}{2}\xi-\phi\right)\right]\mathrm{d}\xi,\\
&K=\frac{Q}{2\pi}\left(\frac{2\pi a}{L}\right)^2\sum_{n=0}^{\infty}\frac{r^n}{a^{n+2}}\int_0^{2\pi}\frac{5+4\cos\xi}{(2\cos\frac{\xi}{2})^{n+2}}U_n[\sin\theta\cos(\frac{3}{2}\xi-\phi)]\mathrm{d}\xi,\\
&A_1=\frac{Qa}{L}\sum_{n=0}^{\infty}\frac{r^n}{a^{n+2}}\int_0^{2\pi}\frac{\sin\xi+2\sin 2\xi}{(2\cos\frac{\xi}{2})^{n+2}}U_n\left[\sin\theta\cos\left(\frac{3}{2}\xi-\phi\right)\right]\mathrm{d}\xi,\\
&A_2=-\frac{Qa}{L}\sum_{n=0}^{\infty}\frac{r^n}{a^{n+2}}\int_0^{2\pi}\frac{\cos\xi+2\cos 2\xi}{(2\cos\frac{\xi}{2})^{n+2}}U_n\left[\sin\theta\cos\left(\frac{3}{2}\xi-\phi\right)\right]\mathrm{d}\xi,\\
&A_3=0,\quad A_4=0.
\end{aligned}
$$

To perform the integrals (34), we follow the scheme described in the previous section. In principle, one can always deal with them order by order. Work in this direction is under progress, and we hope we can get a clean result in the near future.

6. Conclusion

Employing the multipole expansion in four dimensions [10], we developed a new method to calculate the geometry of two-charge fuzzball [7]. We find that this method is powerful in getting the fuzzball metric with one-momentum mode, and is a promising method to provide a way to simplify the metric of two-mode or multi-mode fuzzball, which remains an open problem. We hope that in the near future we can obtain and publish clean analytic results for two-mode and multi-mode cases.

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